Lecture 3: Fundamentals of Differential Equations

MATH 303 ODE and Dynamical Systems

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Definitions and terminology

What are differential equations?

An **ordinary differential equation** involves an unknown function $y(x)$ and its derivatives

$$y'(x) = \frac{dy}{dx}, \quad y''(x) = \frac{d^2y}{dx^2}, \qquad \dots$$

Therefore, a differential equation has the general form

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0.$$

where F is an arbitrary expression. It is convenient, both from a theoretical and practical point of view, to express $y^{(n)}$ in terms of the other derivatives, that is, to isolate $y^{(n)}$ and write

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}).$$

The unknown function $y(x)$ is called the **dependent variable** while x is called the **independent variable**.

The **order of a differential equation** is the order of the highest order derivative appearing in the equation.

Example 1. Newton's second law of motion for a particle moving in one dimension under the influence of a force $F(t, x, x')$ is the second order differential equation

$$mx'' = F(t, x, x').$$

Example 2. The Bessel equation is a second order differential equation given by

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

Example 3. The logistic equation in population dynamics is a first order equation given by

$$P' = a\,P\,(b - P).$$

Solutions

Definition. A **solution** (explicit solution) to the equation $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$ or $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$ is any function $y(x)$ that satisfies these equations.

Example. The function $P_1 = e^t$ is a solution of the differential equation $P' = P$. However, so is $P_2 = 2e^t$ and any function of the form $P_a = ae^t$ where a is a constant.

Remark. It is very easy to check if a given function is a solution for a differential equation. However, it can be incredibly hard, or even impossible, to find such solutions.

Example. Consider the equation $y'' + y = 0$. To check that $y_1 = \cos x$ is a solution we compute $y_1' = -\sin x$ and $y_1'' = -\cos x$ so we have

$$y_1'' + y_1 = -\cos x + \cos x = 0.$$

We can similarly show that $y_2 = \sin x$ is also a solution. A differential equation will typically have more than one solutions.

General solution

Definition. The **general solution** to a differential equation is the collection of all its solutions.

Example. The general solution of $P' = P$ is $P = c_1 e^t$ where c_1 is an arbitrary constant.

Example. The general solution of $y'' + y = 0$ is

$$y = c_1 \cos x + c_2 \sin x$$

where c_1, c_2 are arbitrary constants.

Remark. In the previous examples it is easy to check that the given general solutions are indeed solutions. We will see soon in the course methods for obtaining such general solutions.

A more difficult question is how do we know that these are *all possible solutions*? This is related to the existence and uniqueness theorem for solutions that we discuss later.

Initial Value Problems

The general solution contains the information about all solutions to a differential equation. However, in a practical application we need to know a specific solution.

For example, when we study a question related to population we want to be able to predict the population after some time. For that we need to know what is the **current population**.

If we want to find where a mass on a spring following Newton's second law is going to be after 5 seconds then we need to know its **current position and velocity**.

The values of the quantities that we need for determining the future evolution of a system are called **initial conditions**. Given specific initial conditions we will see later that there is (under certain technical assumptions) a unique solution to the differential equation that satisfies these assumptions.

Examples

Population

Equation $P' = P$; General solution $P(t) = c_1 e^t$; Initial condition $P(0) = 100$.

To find the specific solution we see that we have $P(0) = c_1 e^0 = c_1$ and $P(0) = 100$. Therefore, $c_1 = 100$ and the solution is $P(t) = 100e^t$.

Mass on spring

Equation $x'' = -x$; General solution $x(t) = c_1 \cos t + c_2 \sin t$; Initial condition $x(0) = 1,\ x'(0) = 0$.

To find the specific solution we note that $x(0) = c_1$ and $x'(t) = -c_1 \sin t + c_2 \cos t$ so $x'(0) = c_2$. This gives $c_1 = 1,\ c_2 = 0$. Therefore the motion of the mass is given by $x(t) = \cos t$.

Initial Value Problems

Definition. By an **initial value problem** for an n-th order differential equation $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$ we mean the problem of finding a solution to the differential equation on an interval I that satisfies at $x_0 \in I$ the n **initial conditions**

$$y(x_0) = a_0,\ y'(x_0) = a_1,\ \ \ldots,\ y^{(n-1)}(x_0) = a_{n-1},$$

where $a_0, a_1, \ldots, a_{n-1}$ are given constants.

Poll

Which of the following solves the IVP $y' = xy$ with $y(1) = 2$?

A. $y(x) = e^{x^2/2}$

B. $y(x) = 2e^{(x^2-1)/2}$

C. $y(x) = 2e^{x^2-1}$

D. $y(x) = e^{x^2-1}$

Direction fields

Direction fields

Here we focus on first-order differential equations of the form

$$y' = f(x, y).$$

Consider a point (x_0, y_0) on the xy-plane and let y_s be the solution to the equation with $y_s(x_0) = y_0$.

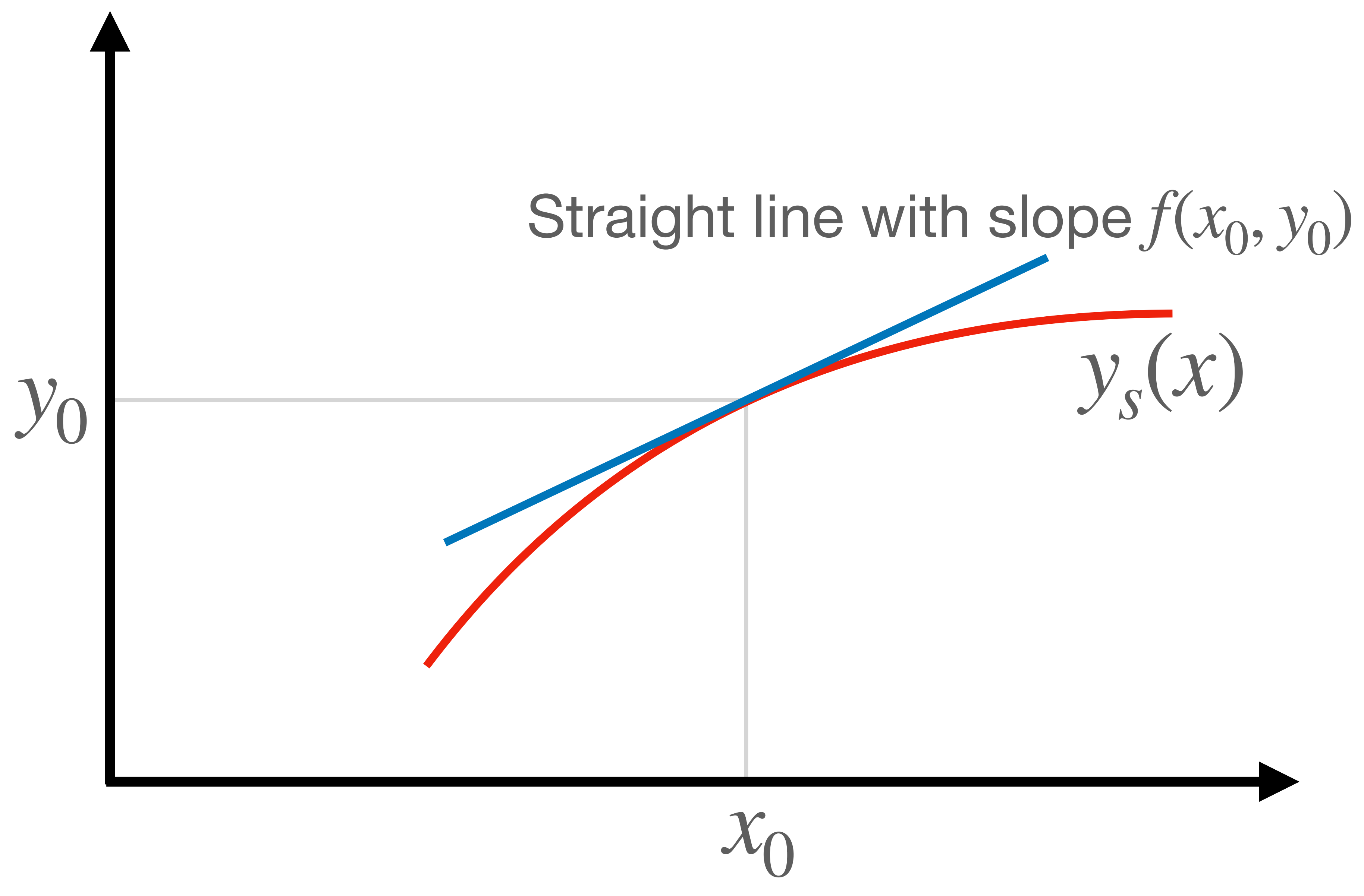
Geometrically, we are considering the solution whose graph is a curve on the xy-plane that passes through the point (x_0, y_0).

We do not assume that we know an expression for $y_s(x)$ — only that it exists.

Even though we may not know the shape of the graph of $y_s(x)$ we do know that at the point $(x_0, y_0 = y_s(x_0))$ its derivative is given by

$$y_s'(x_0) = f(x_0, y_s(x_0)) = f(x_0, y_0).$$

Therefore, the graph of the solution that passes through a point (x_0, y_0) must have slope $f(x_0, y_0)$ at that point.



Then we can consider each point on the xy-plane and draw the corresponding slope at that point. This is the **direction field** corresponding to the equation $y' = f(x, y)$.

In practice, we cannot do this for all points but only for a subset of points on the plane.

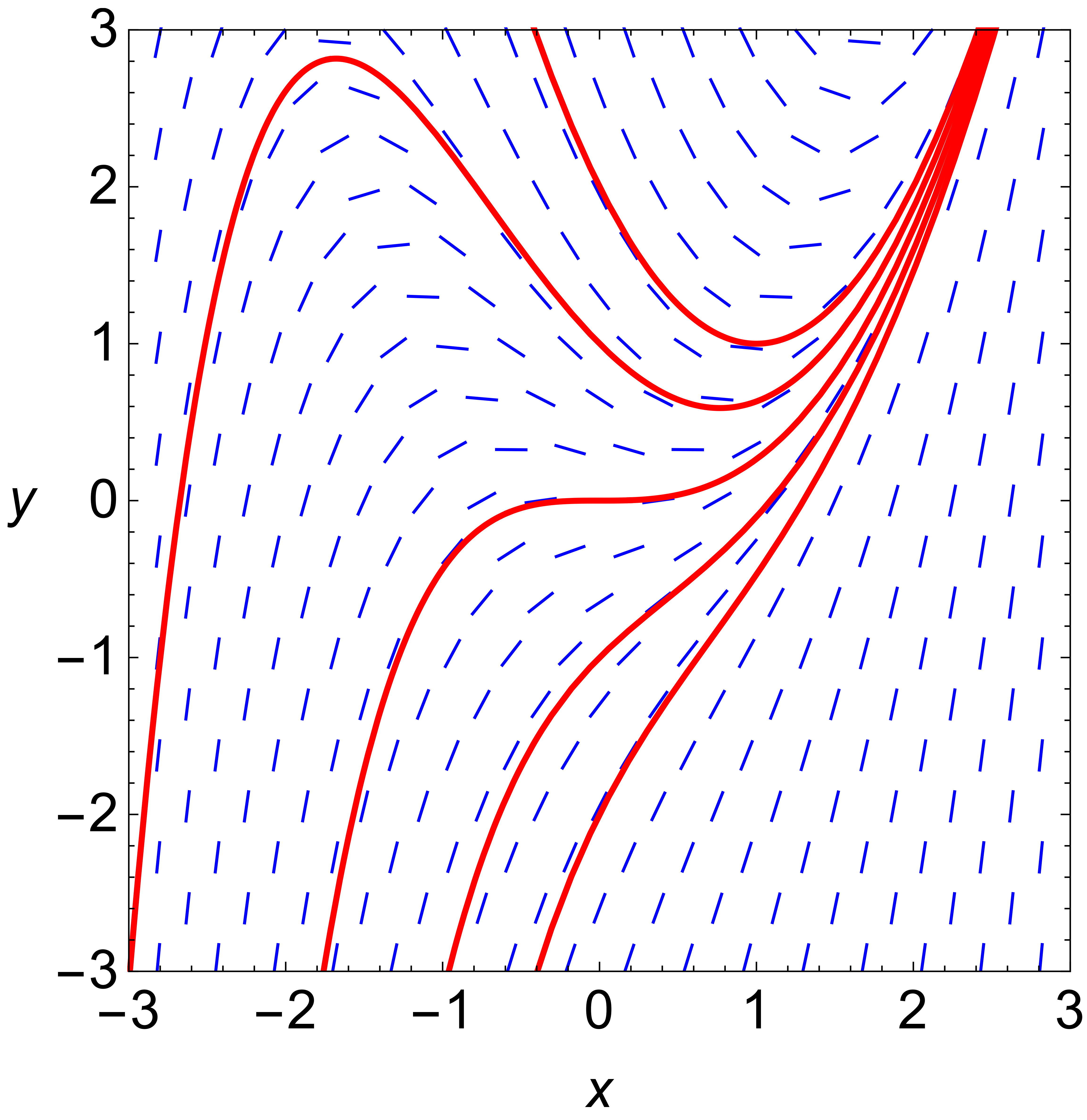
From this geometric point of view, solving the IVP $y' = f(x, y), \; y(x_0) = y_0$ means to find the curve $y_s(x)$ that passes through (x_0, y_0) and that its slope at each point is given by the direction field.

Example

The plot at the right shows the direction field for the equation

$$y' = x^2 - y.$$

The red curves represent solutions to the same equation with different initial condition $y(x_0) = y_0$.



Poll

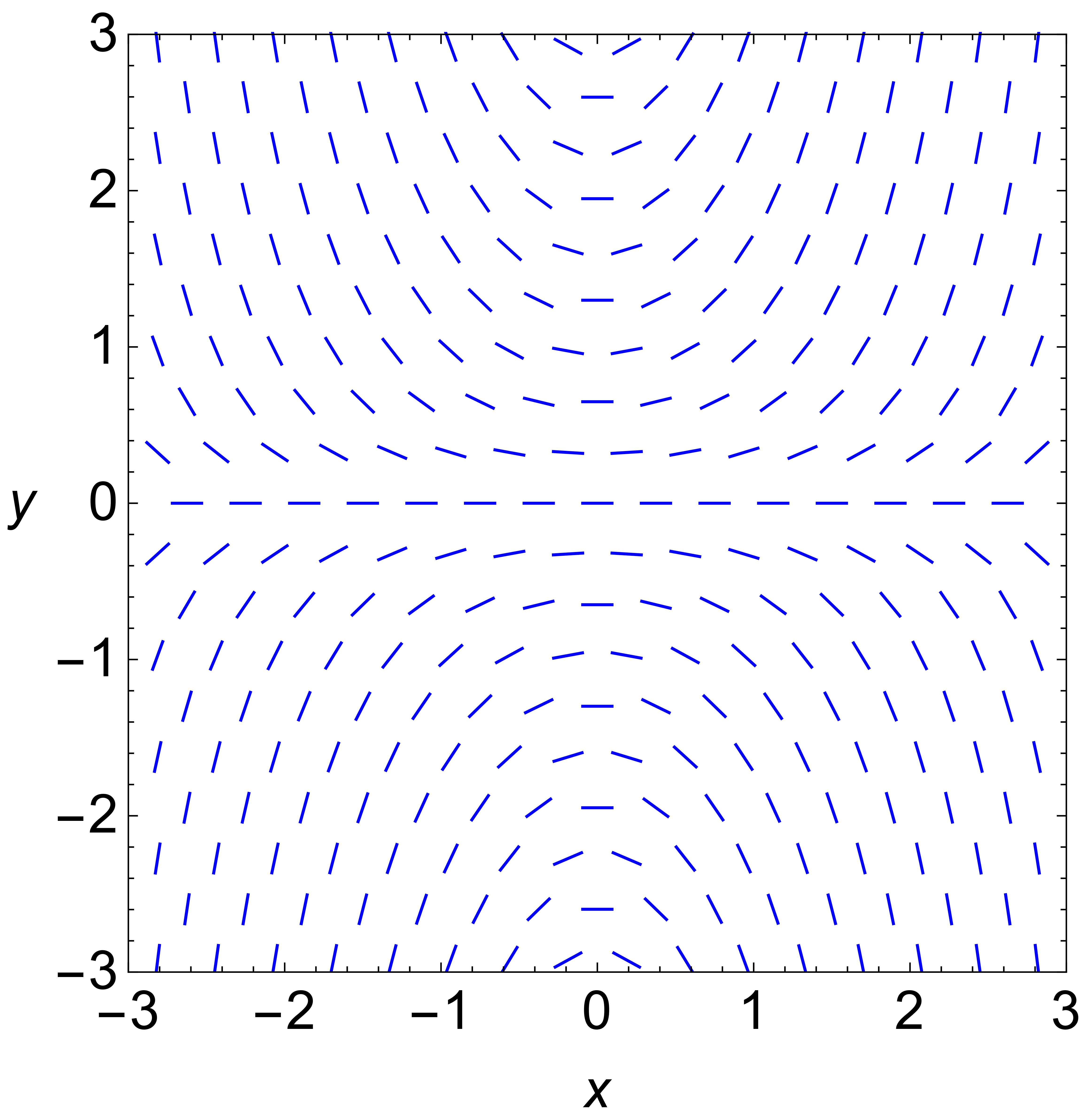
The direction field plot at the right corresponds to which of the following differential equations?

A. $y' = xy$

B. $y' = y$

C. $y' = x^2 y$

D. $y' = \dfrac{y}{x}$



Numerical approximation

Approximate solutions

Consider the equation $y' = f(x, y)$. We want to compute an approximation to the solution to the IVP with $y(x_0) = y_0$ in the interval $[x_0, x_1]$.

The main idea to compute an approximation is to choose a small step-size h and then find an approximate value y_1 of the solution at $x_0 + h$. Then we can repeat this process and compute an approximate value y_2 at $x_0 + 2h$ and after k steps an approximate value y_k at $x_0 + kh$. We stop this procedure when $x_0 + kh = x_1$.

The question now is how to define the approximation for each step?

Exploration

Let's define the problem as follows. Suppose that $y(x)$ is the solution of $y' = f(x, y)$ that satisfies $y(x_0) = y_0$. **What is the easiest approximation to $y(x_0 + h)$ that you can come up with?**

Implement this approximation in Mathematica for the equation $y' = xy$ with $x_0 = 1$, $y_0 = 1$, $h = 0.01$ as follows.

1. Define a function `step[{x0_,y0_}]` that takes as input a list with the values of x_0 and y_0 and returns a list with the values of $x_1 = x_0 + h$ and the approximation y_1.

2. Then use Mathematica's function Nest (check its documentation!) to compute the approximation y_{100} at $x_{100} = x_0 + 100h = 2$. What value do you find?

Euler's method

Euler's method is based on the Taylor expansion

$$y(x_0 + h) \approxeq y(x_0) + hy'(x_0) = y_0 + hf(x_0, y_0)$$

and defines the approximation y_1 as

$$y_1 = y_0 + hf(x_0, y_0).$$

Example

The following code will compute an approximation to the exploration problem using Euler's method.

```
f[x0_, y0_] := x0 y0

step[{x0_, y0_}] := {x0 + 0.01, y0 + 0.01 f[x0, y0]}

{x100, y100} = Nest[step, {1, 1}, 100]
```

The approximate result is $y_{100} = 4.40848$ while the exact value is $y(2) = e^{3/2} = 4.48169\ldots$

Comparison

At the right you can see a plot of the exact solution (red) given by

$$y(x) = e^{(x^2-1)/2}$$

and the approximation (blue points) obtained using Euler's method, the function NestList and the plotting function ListPlot.