Lecture 4: First-Order Differential Equations

MATH 303 ODE and Dynamical Systems

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First-order differential equations

We will focus today on first-order differential equations of the form

$$y' = f(x, y),$$

and we will discuss solution methods for two types of such equations, separable and linear.

Separable Equations

How to recognize separable equations

A **separable differential equation** has the form

$$y' = g(x)\,p(y).$$

This means that an equation is separable if the right-hand side can be written as a product of two factors. One depends only on the dependent variable (here, y) and the other one depends only on the independent variable (here, x).

How to solve separable equations

First, we rewrite the equation

$$y' = g(x)\,p(y)$$

as

$$y' = \frac{g(x)}{h(y)}$$

where $h(y) = 1/p(y)$, and also write the derivative using d-notation as

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

For the next step, we **separate** the dependent and independent variables to get

$$h(y)\,dy = g(x)\,dx.$$

Then we integrate both sides of $h(y)\,dy = g(x)\,dx$ to get

$$\int h(y)\,dy = \int g(x)\,dx.$$

If $H(y)$ is an anti-derivative of $h(y)$ and $G(x)$ is an anti-derivative of $g(x)$, then we have

$$H(y) = G(x) + c,$$

where c is an arbitrary constant.

If we can solve the last equation for y then we get an **(explicit) solution**.

If not, then we leave the relation $H(y) = G(x) + c$ as it is. Such a relation between the dependent and independent variables is often called an **implicit solution**.

Example

We solve $y' = x\,y$.

First, write

$$\frac{dy}{dx} = x\,y.$$

Then separate variables to get

$$\frac{dy}{y} = x\,dx.$$

Integrating the two sides we find

$$\ln|y| = \frac{1}{2}x^2 + c,$$

where c is an arbitrary constant. This is an implicit solution. To get an explicit solution we solve for y.

We first get

$$|y| = e^c e^{x^2/2}$$

and then

$$y = (\pm e^c)\, e^{x^2/2}.$$

One thing to pay attention to here is that when we wrote dy/y we implicitly assumed that $y \neq 0$. However, clearly $y = 0$ is also a solution to the equation. To take this into account we rewrite the general solution as

$$y = Ae^{x^2/2},$$

where A can be any real number and replaces $\pm e^c$ which can be any real number except 0.

Remark

All equations of the form $y' = y\,g(x)$ can be treated in the same way. If $G(x)$ is an anti-derivative of $g(x)$ then we find the general solution

$$y = Ae^{G(x)},$$

where A can be any real number.

In particular, if the equation is $y' = ky$ so that $g(x) = k$ then $G(x) = kx$ and the general solution is

$$y = Ae^{kx}.$$

Example

We solve $y' = 2y^2x$.

First, we write

$$\frac{dy}{dx} = 2y^2x,$$

and then we separate variables to get

$$\frac{dy}{y^2} = 2x\,dx.$$

Integrating we find

$$-\frac{1}{y} = x^2 + c,$$

where c is an arbitrary constant, and solving for y we finally find

$$y = -\frac{1}{x^2 + c}.$$

Note that $y = 0$ is also a solution.

Therefore, the general solution consists of the solution $y = 0$ together with the one-parameter family of solutions $y = -\dfrac{1}{x^2 + c}$.

Poll

Solve $y' = \cos(x)\, y^2$ **using separation of variables** and choose the correct answer at pollev.com/ke1.

A. $\dfrac{1}{c - \sin(x)}$

B. $\dfrac{2}{c - \sin(x)}$

C. $\dfrac{1}{c + \sin(x)}$

D. $\dfrac{1}{c + \cos(x)}$

Justification

Rewrite the equation $y' = g(x)/h(y)$ as

$$h(y)\, y' = g(x).$$

For any solution $y(x)$ we have

$$h(y(x))\, y'(x) = g(x),$$

that is,

$$H'(y(x))\, y'(x) = G'(x).$$

At the left-hand side we recognize the chain rule for $H(y(x))$. That is,

$$(H(y(x)))' = G'(x).$$

Since these derivatives are equal we conclude that the two functions differ by a constant. That is,

$$H(y(x)) = G(x) + c.$$

Linear Equations

How to recognize linear equations

A first-order differential equation is called **linear** if it has the form

$$a_1(x)\, y' + a_0(x)\, y = b(x),$$

where $a_1,\ a_0,\ b$ are continuous functions for x in some interval $I \subseteq \mathbb{R}$ and $a_1(x) \neq 0$ for all $x \in I$.

The first step in solving a linear equation is to bring it into **standard form** by dividing by $a_1(x)$. Then we have

$$y' + P(x)\, y = Q(x),$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$. From now on we will always assume that the linear equation is in standard form.

How to solve linear equations (integrating factor)

If we look at the left-hand side of the standard form

$$y' + Py = Q$$

we can see that it has some similarities with the product rule for differentiation which says that for any function $\mu(x)$ we have

$$(\mu y)' = \mu y' + \mu' y.$$

Multiplying both sides of the standard form by μ we find

$$\mu y' + \mu P y = \mu Q.$$

Comparing $\mu y' + \mu P y$ and $(\mu y)' = \mu\, y' + \mu' y$ we see that they can be made equal if we can find **any** function μ such that

$$\mu' = \mu\, P(x).$$

This is a separable differential equation for μ and according to a previous remark, its general solution is $Ae^{R(x)}$ where A is an arbitrary constant and

$$R(x) = \int P(x)\, dx$$

is any anti-derivative of $P(x)$. Here we do not need the general solution. It is enough to find any μ that satisfies $\mu' = \mu\, P$ and thus we can choose $A = 1$ so that

$$\mu = e^{R(x)}.$$

Then, by the requirement that $\mu' = \mu\, P$ we have that

$$\mu\, y' + \mu\, P\, y = \mu\, y' + \mu'\, y = (\mu\, y)' = \mu\, Q.$$

The equation $(\mu\, y)' = \mu\, Q$ can be integrated to obtain

$$\mu\, y = \int \mu\, Q\, dx + c.$$

Solving for y we finally obtain the general solution

$$y = \frac{1}{\mu}\left(\int \mu\, Q\, dx + c\right) = e^{-R(x)}\left(\int e^{R(x)}\, Q(x)\, dx + c\right).$$

Even though we have found a closed form expression for y, in practice we rarely directly use this expression. Instead, we repeat the step-by-step procedure that leads to the final solution.

Example

We solve $y' = x - y$.

We first bring the equation into the standard form $y' + y = x$.

We recognize $P(x) = 1,\ Q(x) = x$.

Then the integrating factor is $\mu = e^x$ where $R(x) = x$ is an anti-derivative of $P(x) = 1$.

Multiplying both sides of the equation by the integrating factor we get

$$e^x y' + e^x y = e^x x.$$

The left-hand side equals $(e^x y)'$ and therefore the equation becomes $(e^x y)' = e^x x$.

Integrating $(e^x y)' = e^x x$ we find $e^x y = \int e^x x \, dx + c$. The indefinite integral can be done using integration by parts:

$$\int e^x x \, dx = e^x x - \int e^x dx = e^x x - e^x = e^x(x-1).$$

Therefore,

$$e^x y = e^x(x-1) + c.$$

Finally, solving for y we find

$$y = x - 1 + ce^{-x}.$$

Poll

Solve $x^2y' + xy = 1$ for $x > 0$ and choose the correct answer on pollev.com/ke1.

A. $y = \dfrac{\ln x + c}{x^2}$

B. $y = \dfrac{\ln x + c}{x}$

C. $y = \dfrac{\ln x}{x + c}$

D. $y = \dfrac{x}{\ln x} + c$