Lecture 5: Applications; Substitutions

MATH 303 ODE and Dynamical Systems

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Population Dynamics

Exponential growth

The first, rather naïve, population model assumes that the rate of increase of the population is proportional to the population level. This is described by the **differential equation**

$$\frac{dP}{dt} = kP.$$

We have seen that the **general solution** to this equation is

$$P = Ae^{kt},$$

where A is an arbitrary constant. Usually, we need to predict the population at time t, if we know the population P_0 at $t = 0$. Then the **initial value problem** $P(0) = P_0$ gives $A = P_0$ and the solution is

$$P = P_0 e^{kt}.$$

Logistic model

The logistic population model was proposed by Pierre-François Verhulst around 1840. It tries to take into account that in a specific environment the population cannot grow without bound but there is a maximal population that can be supported by the environment, the **carrying capacity** M.

Verhulst came up with a model described by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

We will solve this equation in a moment but before doing that let's see some of the properties of the solutions.

The solutions of the equation $P' = kP\left(1 - P/M\right)$ have the following properties:

A. $P = 0$ and $P = M$ are solutions. Such solutions that are constant in time are called **equilibrium solutions.**

B. When $0 < P < M$ the solution increases with time (since $P' > 0$).

C. When $P > M$ the solution decreases with time (since $P' < 0$).

To solve the logistic equation we start with a simplifying substitution. Define

$$x = \frac{P}{M}.$$

Then

$$\frac{dx}{dt} = \frac{1}{M}\frac{dP}{dt} = k\frac{P}{M}\left(1 - \frac{P}{M}\right) = kx(1 - x)\,.$$

The equation for x is a separable equation which can be written as

$$\frac{dx}{x\,(1 - x)} = k\,dt\,.$$

The integral at the left hand side can be done using partial fractions as

$$\int \frac{dx}{x(1 - x)} = \int \frac{dx}{x} - \int \frac{dx}{x - 1} = \ln|x| - \ln|x - 1| = \ln\left|\frac{x}{x - 1}\right|.$$

Therefore, $\ln\left|\dfrac{x}{x-1}\right| = kt + c$ and from here $\dfrac{x}{x-1} = (\pm e^c)\, e^{kt} = Ae^{kt}$,

where to allow for the solution $x = 0$ we have replaced $\pm e^c$ by $A \in \mathbb{R}$.

Then, solving for x gives the solution

$$x = \frac{A}{A - e^{-kt}} \quad (\,*\,).$$

Recall that to solve the equation we divided by $x\,(x-1)$. The solution $x = 0$ is expressed by setting $A = 0$. However, the solution $x = 1$ formally corresponds to $A = \infty$ and it is not included in the last expression. Therefore, we can say that the general solution is the solution $(\,*\,)$ together with $x = 1$.

Consider now the IVP $x(0) = x_0$. We have for $t = 0$ the equation $x_0 = \dfrac{A}{A-1}$ which gives

$$A = \frac{x_0}{x_0 - 1}.$$

Substituting back into the solution ($*$), and multiplying numerator and denominator by $x_0 - 1$, we find

$$x = \frac{x_0}{x_0 - (x_0 - 1)e^{-kt}} \quad (\dagger).$$

Note that we do not need to separately consider the solution $x = 1$ since for $x_0 = 1$ we get $x = 1$.

This means that the solution ($\dagger$) to the IVP can also be considered as the general solution if x_0 is considered as an arbitrary constant.

Translating this solution back in terms of P we have

$$P = Mx = \frac{Mx_0}{x_0 - (x_0 - 1)e^{-kt}} = \frac{M\,(Mx_0)}{Mx_0 - (Mx_0 - M)e^{-kt}}$$

and finally

$$P = \frac{MP_0}{P_0 - (P_0 - M)e^{-kt}}.$$

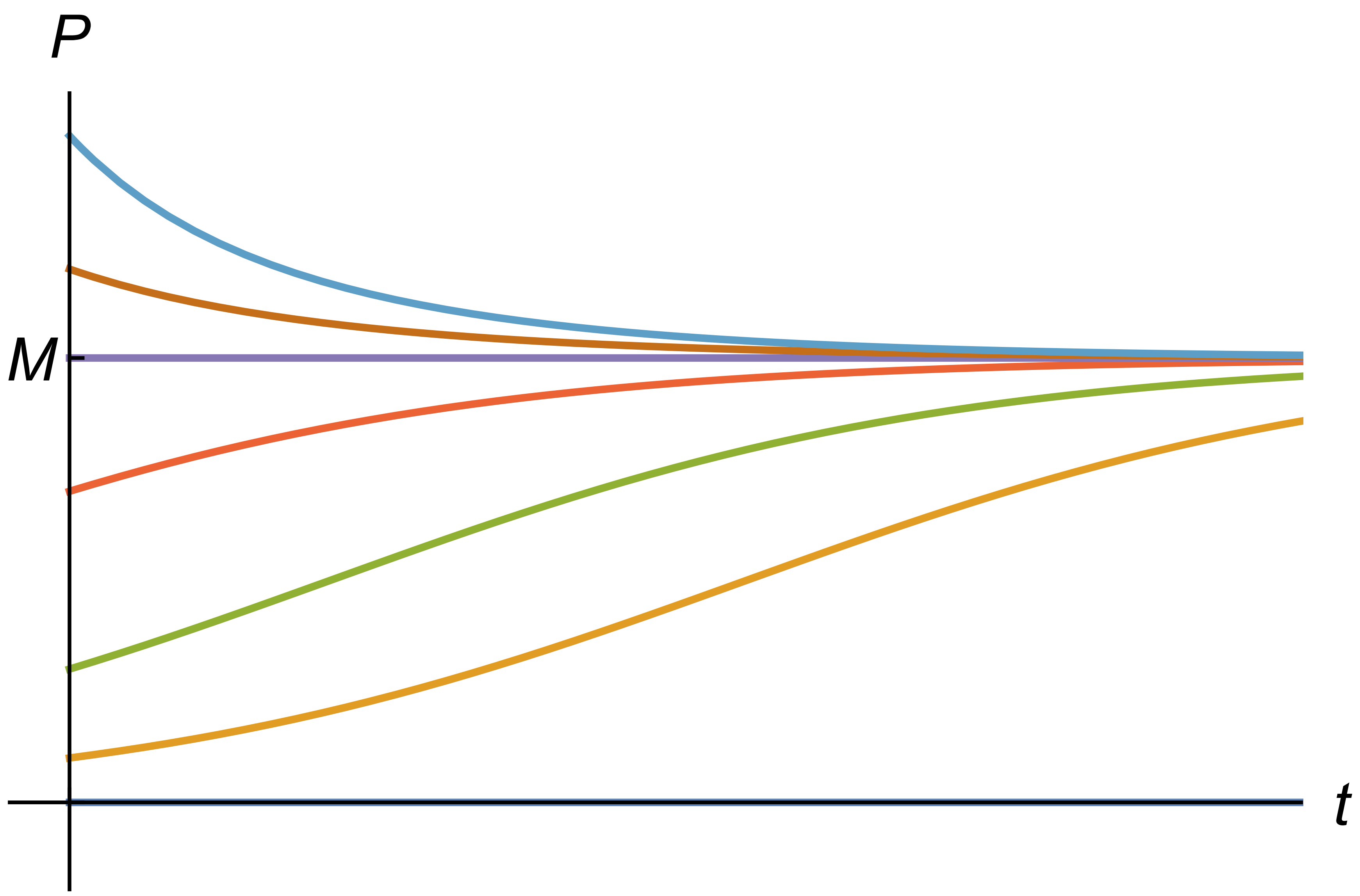
Note that as $t \to \infty$ we have $P \to M$.

Moreover, for $P_0 = 0$ we find $P = 0$, and for $P_0 = M$ we find $P = M$.

The graph of the solutions

$$P = \frac{MP_0}{P_0 - (P_0 - M)e^{-kt}}$$

for different P_0 is the following.



Constant harvesting

A variation of the previous model is to assume that the population is reduced at a constant rate (harvesting). Then the logistic model can be modified to

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - h.$$

Then by defining again $x = P/M$ we get

$$\frac{dx}{dt} = kx(1 - x) - a,$$

where $a = h/M$.

This problem can be solved using the same methods used for the logistic model without harvesting ($a = 0$) but one should be careful that for $a > k/4$ the expression $kx(1 - x) - a$ has no real roots. We will later revisit this model from the point of view of dynamical systems.

Cooling and Heating of Buildings

Newton's law of cooling

Temperature inside building: $T(t)$

Outside temperature: $M(t)$

Additional heating or cooling factors (e.g., central heating, air conditioning, etc.) induce a rate of change of the temperature $T(t)$ given by a function $F(t)$.

Rate of change of the temperature inside the building:

$$\frac{dT}{dt} = k(M - T) + F.$$

The equation

$$T' = k(M - T) + F.$$

can be written as

$$T' = - kT + (kM + F).$$

This is a linear equation with standard form

$$T' + kT = kM + F.$$

The integrating factor is $\mu = e^{kt}$ and multiplying both sides by it we get

$$(e^{kt}T)' = e^{kt}(kM + F).$$

Then the solution is

$$T = e^{-kt}\left(c + \int e^{kt}(kM + F)\,dt\right).$$

Initial Value Problem

Another way to write the solution

$$T(t) = e^{-kt}\left(c + \int e^{kt}(kM(t) + F(t))\,dt\right),$$

is in the form

$$T(t) = e^{-kt}\left(c + \int_{t_0}^{t} e^{ks}(kM(s) + F(s))\,ds\right).$$

Consider now the IVP $T(t_0) = T_0$. Then we have

$$T_0 = T(t_0) = e^{-kt_0}\left(c + \int_{t_0}^{t_0} e^{ks}(kM(s) + F(s))\,ds\right) = ce^{-kt_0},$$

giving $c = e^{kt_0}T_0$. Therefore, the solution to the given IVP is

$$T(t) = T_0 e^{-k(t-t_0)} + \int_{t_0}^{t} e^{k(s-t)}(kM(s) + F(s))\,ds.$$

Constant M, F

If we assume that M, F are constant then we find

$$T = ce^{-kt} + M + \frac{F}{k}.$$

In this case, we can also solve the equation $T' = k(M - T) + F$ without using an integrating factor. If we make the substitution

$$D = T - M - \frac{F}{k}$$

we get

$$D' = T' = -k\left(T - M - \frac{F}{k}\right) = -kD.$$

with solution $D = ce^{-kt}$ so that $T = ce^{-kt} + M + F/k$.

Substitutions

Poll

Which of the following equations is obtained by making the substitution $u = 1/y$ to the equation

$$y' - 5y = xy^2.$$

Select the correct answer at pollev.com/ke1.

A. $u' - 5u = x$

B. $u' + 5xu = x^2$

C. $u' + 5u = -x$

D. $u' + 5u/x = x$

Bernoulli Equation

Exploration

The Bernoulli equation has the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n,$$

where n is a real number and P, Q are continuous functions in an interval $I \subseteq \mathbb{R}$.

This is similar to a linear equation, except for the y^n term at the right hand side. The idea here is to make a substitution $u = y^a$ for an appropriately chosen a so that the transformed equation for u will be a linear equation.

Determine a and give the differential equation satisfied by u.

Solution

We have $u = y^a$ so

$$\frac{du}{dx} = ay^{a-1}\frac{dy}{dx} = ay^{a-1}\left(-Py + Qy^n\right) = -aPy^a + aQy^{n+a-1}.$$

Expressing y^a in terms of u we find

$$\frac{du}{dx} = -aPu + aQy^{n+a-1}.$$

This is again almost a linear equation except for the term y^{n+a-1}. However, now we can control a so let's choose $a = 1 - n$ so that $y^{n+a-1} = 1$.

Then we find that in terms of u we have the linear equation

$$\frac{du}{dx} + aPu = aQ.$$