Lecture 6: Existence and Uniqueness Theorem

MATH 303 ODE and Dynamical Systems

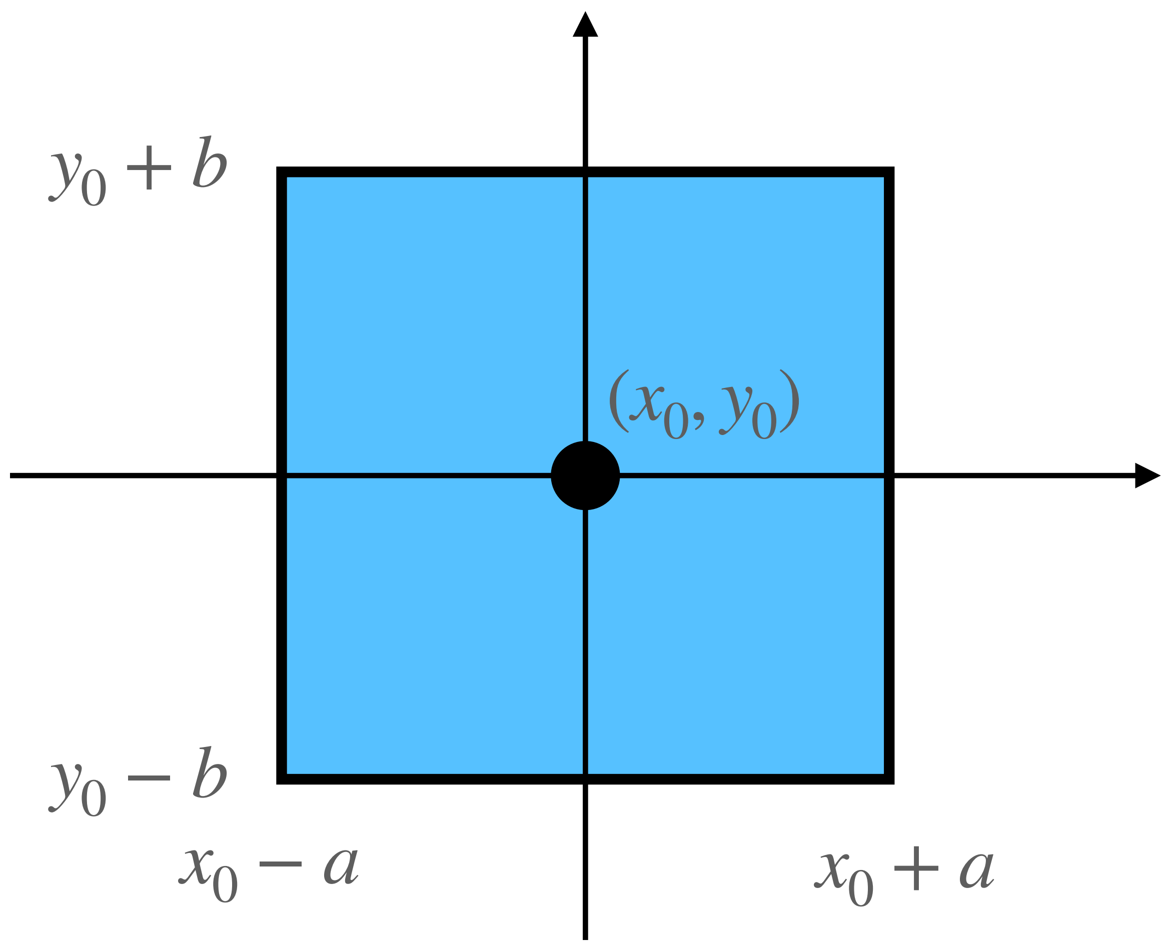
Konstantinos Efstathiou

Existence and Uniqueness Theorem

Theorem. Suppose that $f(x, y)$ is continuous on a rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subseteq \mathbb{R}^2$ and satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in R$ and some $K > 0$.



Then there exists $\delta > 0$ such that the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$ has a unique solution $y(x)$ defined for x in the interval $[x_0 - \delta, x_0 + \delta]$.

Remarks

The statement in the Textbook differs from the statement here in two ways.

First, it considers a rectangle R where (x_0, y_0) is not necessarily centered in R. However, the first step in the proof is to find a smaller rectangle R' such that $(x_0, y_0) \in R'$.

Second, instead of using a Lipschitz condition, it assumes continuity of $\partial f / \partial y$ in R. However, if $\partial f / \partial y$ is continuous on the closed rectangle R, then it must be bounded, i.e., $|\partial f / \partial y| \leq L$ for some $L > 0$. But then one can show that f must satisfy the Lipschitz condition of the Theorem stated here.

Poll

Consider the initial value problem $y' = |y|^{1/2}$ with $y(1) = 0$. Are the conditions of the Existence and Uniqueness Theorem satisfied in $R = [0,2] \times [-1,1]$?

A. Yes

B. No, the continuity condition fails

C. No, the Lipschitz condition fails

D. No, both the continuity and the Lipschitz conditions fail

Answer

No. The condition that fails here is the Lipschitz condition.

To see this let $y_1 = y > 0$ and $y_2 = 0$. Then the Lipschitz condition implies that $|y^{1/2} - 0| \leq K|y - 0|$ for some $K > 0$ which is the same for all $y \in (0,1]$.

The last inequality becomes $y \geq \dfrac{1}{K^2}$. However, this cannot be satisfied for all $y \in (0,1]$ since for any choice of K we can find $y \in (0,1]$ with $y < \dfrac{1}{K^2}$.

Poll

Consider the initial value problem $y' = y^2$ with $y(0) = 1$. Are the conditions of the Existence and Uniqueness Theorem satisfied in $R = [-1,1] \times [0,2]$?

A. Yes

B. No, the continuity condition fails

C. No, the Lipschitz condition fails

D. No, both the continuity and the Lipschitz conditions fail

Initial value problems as fixed point problems

Integral form of initial value problems

We consider the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$. If $y(x)$ is a solution then we have

$$y'(x) = f(x, y(x)).$$

Integrating both sides from x_0 to some arbitrary x we find

$$\int_{x_0}^{x} y'(s)\, ds = \int_{x_0}^{x} f(s, y(s))\, ds.$$

The integral at the left hand side gives

$$y(x) - y(x_0) = y(x) - y_0 = \int_{x_0}^{x} f(s, y(s))\, ds.$$

Therefore, we get

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

The equation that we arrived at is equivalent to the initial value problem but is written in an integral form.

Another way to write this integral equation is by introducing an operator T, that is, an object which takes as input a function and returns a new function. In our case, the operator T is defined by the following relation:

$$T[g](x) = y_0 + \int_{x_0}^{x} f(s, g(s))\, ds.$$

The way to read this relation is that given the function g we define a new function that we denote by $T[g]$ and its value at x is given by the relation above.

Fixed point problem

Comparing the integral equation $y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds$ and the definition $T[g](x) = y_0 + \int_{x_0}^{x} f(s, g(s))\, ds$ we see that the right hand side of the integral equation is $T[y](x)$. Therefore, the integral equation can be written

$$y(x) = T[y](x)$$

which implies that the function $T[y]$ equals the function y, i.e., $T[y] = y$.

Such a function y is called a **fixed point** of the operator T.

It may appear that until now we have made no actual progress. We have just restated the original initial value problem as the problem of finding a fixed point of the operator T.

However, we have actually made huge progress.

The reason is that fixed point problems are some of the best studied problems in Mathematics and there is a huge amount of theoretical results related to such problems.

In general, translating a problem to the form of a fixed point problem opens the door to using the whole theory associated to fixed point problems.

Metric spaces and contractions

Contractions

A very powerful result for fixed point problems is the **Banach fixed point theorem**. The theorem applies to fixed points of functions (or operators) called contractions. We give the basic definitions.

Assume that we have a space X where the distance between two points g, h in X is denoted by $d(g, h)$. The distance should satisfy the following properties for all $g, h, k \in X$:

1. $d(g, h) \geq 0$; $d(g, h) = 0$ if and only if $g = h$.

2. $d(g, h) = d(h, g)$.

3. $d(g, h) \leq d(g, k) + d(k, h)$ (triangle inequality).

Such a space (X, d) is called a **metric space**.

Definition. A function (or operator) $T : X \to X$ is called a **contraction** if there is a constant $K < 1$ such that for all $g, h \in X$ we have

$$d(T[g], T[h]) \leq K\, d(g, h)\ .$$

In our case the space X will be the set of continuous functions defined on an interval $[x_0 - \delta, x_0 + \delta]$ where $\delta > 0$ will be determined later.

The "points" in this space are **continuous functions**.

Given two continuous functions $g, h \in X$ a **distance** between them can be defined as

$$d(g, h) = \|g - h\| = \max_{x \in [x_0 - \delta, x_0 + \delta]} |\, g(x) - h(x)\,|\ .$$

You can check that d satisfies the three properties in the previous slide that any distance function must satisfy.

Limits

Definition. A sequence $\{g_n\}$ of points in X **converges** to a limit y in X if $\lim_{n\to\infty} d(g_n, y) = 0$, i.e., if $\lim_{n\to\infty} ||g_n - y|| = 0$.

Lemma. If a sequence in X converges to a limit then the limit is unique.

Definition. A sequence $\{g_n\}$ of points in X converges is a **Cauchy sequence** if for every $\varepsilon > 0$ there is N such that $d(g_n, g_m) = ||g_n - g_m|| < \varepsilon$ for all $n, m \geq N$.

Definition. A metric space (X, d) is **complete** if every Cauchy sequence converges to some point in X.

Banach fixed point theorem

Theorem (Banach fixed point theorem). If X is a **complete** metric space and $T : X \to X$ is a **contraction** then T has a **unique fixed point** in X.

Lemma. The space X of continuous functions defined on an interval $[x_0 - \delta, x_0 + \delta]$ with $||g(x) - y_0|| \leq b$ for all $g \in X$ is a complete metric space.

Sketch of the proof of the Banach fixed point theorem

Start with an arbitrary point $g_0 \in X$ and define recursively the sequence $\{g_n\}$ by $g_{n+1} = T[g_n]$.

Then one can show that for $m > n$ we have

$$\|g_n - g_m\| < \frac{K^n}{1-K}\|g_1 - g_0\|.$$

Since $0 \leq K < 1$ we have that $K^n \to 0$ as $n \to \infty$. This means that $\|g_n - g_m\|$ can be made arbitrarily small by choosing n large enough. Therefore, the sequence $\{g_n\}$ is Cauchy and, since X is complete, it has a limit $y \in X$.

Moreover, it turns out that T is a continuous operator on X, therefore $T[g_n] \to T[y]$. However, we must have $T[g_n] \to y$ and since limits are unique we conclude that $T[y] = y$. This shows that $y \in X$ is a fixed point, i.e., the equation $T[y] = y$ has a solution.

For the uniqueness, suppose that there are two fixed points y_1, y_2. Then

$$\|y_1 - y_2\| = \|T[y1] - T[y_2]\| \leq K\|y_1 - y_2\|.$$

Since $K < 1$ this is a contradiction unless $\|y_1 - y_2\| = 0$ which implies $y_1 = y_2$.

Sketch of the Proof of the Existence and Uniqueness Theorem

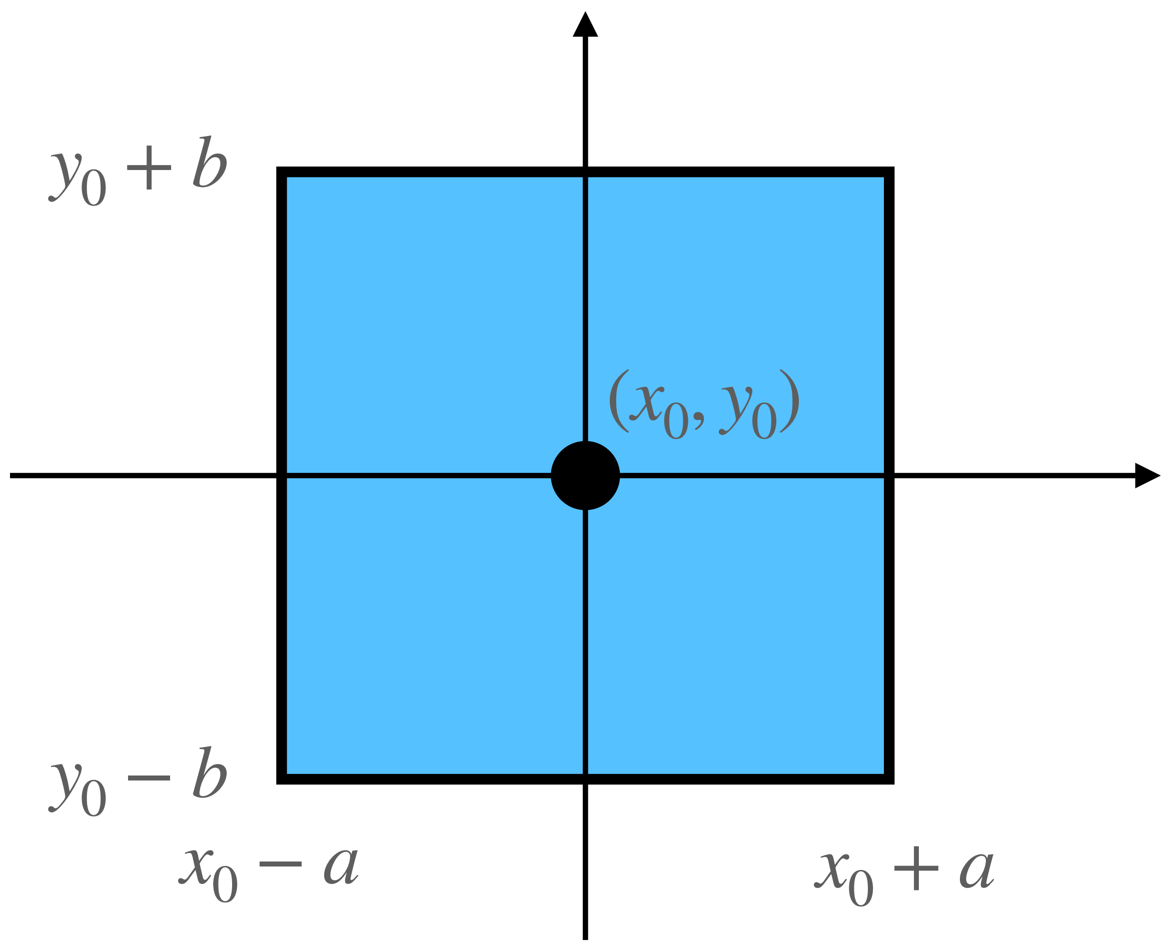
Existence and Uniqueness Theorem

Theorem. Suppose that $f(x, y)$ is continuous on a rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \subseteq \mathbb{R}^2$ and satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in R$ and some $K > 0$.

Then there exists $\delta > 0$ such that the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$ has a unique solution $y(x)$ defined for x in the interval $[x_0 - \delta, x_0 + \delta]$.



Complete metric space and contraction

We will show that the operator T **is a contraction** when restricted to the **complete metric space** X of continuous functions defined on an interval $[x_0 - \delta, x_0 + \delta]$ and $||g(x) - y_0|| \leq b$ for all $g \in X$. Here δ is chosen so that

$$0 < \delta < \min\left\{a, \frac{b}{M}, \frac{1}{K}\right\},$$

where K is the Lipschitz constant and M is the maximal value of $f(x, y)$ in the rectangle R.

We will show two things. First, that T maps X into X. Second, that T is a contraction on X.

T maps X into X

Suppose that $g \in X$, that is, g is continuous and $\|g - y_0\| \leq b$. Clearly,

$$T[g](x) = y_0 + \int_{x_0}^{x} f(s, g(s))\, ds$$

is continuous as the integral of a continuous function. Moreover,

$$\|T[g] - y_0\| = \max_{x \in [x_0 - \delta, x_0 + \delta]} \left| T[g](x) - y_0 \right|.$$

We have

$$|T[g](x) - y_0| = \left| \int_{x_0}^{x} f(s, g(s))\, ds \right| \leq \left| \int_{x_0}^{x} |f(s, g(s))|\, ds \right| \leq M|x - x_0| \leq M\delta \leq M\frac{b}{M} = b$$

.

Therefore, $\|T[g] - y_0\| \leq b$.

T is a contraction on X

We have that

$$|T[g](x) - T[h](x)| = \left| \int_{x_0}^{x} f(s, g(s))ds - \int_{x_0}^{x} f(s, h(s))ds \right| \leq \left| \int_{x_0}^{x} |f(s, g(s)) - f(s, h(s))| ds \right|$$

Using the Lipschitz property we get

$$|T[g](x) - T[h](x)| \leq \left| \int_{x_0}^{x} K|g(s) - h(s)|ds \right| \leq \left| \int_{x_0}^{x} K\|g - h\|ds \right| = K\|g - h\||x - x_0| \leq K\delta\|g - h\|$$

Therefore,

$$\|T[g] - T[h]\| = \max_{x \in [x_0 - \delta, x_0 + \delta]} |T[g](x) - T[h](x)| \leq K\delta\|g - h\|$$

where we note that $K\delta < 1$ so that T is a contraction.

Proof of the existence and uniqueness theorem

Recall that we work in the space X of continuous functions defined on an interval $[x_0 - \delta, x_0 + \delta]$ with $||g(x) - y_0|| \leq b$ for all $g \in X$ which is a complete metric space.

Moreover, we have shown that $T : X \to X$ is a contraction on X.

Therefore, applying the Banach fixed point theorem we conclude that there must be a unique fixed point $T[y] = y$ in X.

Picard iteration

Picard iteration

In the proof of the Banach fixed point theorem we saw that the fixed point is the limit of the sequence $\{g_n\}$ defined by $g_{n+1} = T[g_n]$.

The procedure of successively applying the operator T where

$$T[g](x) = y_0 + \int_{x_0}^{x} f(s, g(s))\, ds$$

to an initial function in X is called **Picard iteration**. We define $g_0(x) = y_0$ (constant). Then

$$g_1(x) = T[g_0](x) = y_0 + \int_{x_0}^{x} f(s, y_0) ds,$$

$$g_2(x) = T[g_1](x) = y_0 + \int_{x_0}^{x} f(s, g_1(s)) ds, \ldots$$

The limit of this iterative procedure is the solution $y(x)$ of the initial value problem.