Lecture 7: One-dimensional Dynamical Systems

MATH 303 ODE and Dynamical Systems

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Basic definitions

First-order differential equations

We consider first-order differential equations of the form

$$\frac{dx}{dt} = f(x),$$

that is, where the right hand side does not depend explicitly on the independent variable t. Such equations are separable and can in principle be solved as

$$\int \frac{dx}{f(x)} = t + c.$$

However, it turns out that we can understand many things about the properties of the solutions without explicitly computing $x(t)$. Actually, even if we can compute $x(t)$ the solution is often so algebraically complicated that it is difficult to understand its properties by looking at the algebraic expression.

Autonomous equations

Definition. A differential equation where the right hand side does not explicitly depend on t, i.e., of the form $x' = f(x)$, is called **autonomous**.

We will be assuming that f satisfies the conditions of the Existence and Uniqueness Theorem (that is, f is continuous and Lipschitz).

Lemma. Let $x_1(t)$ be the solution of the initial value problem $x' = f(x)$ with $x(0) = x_0$. Then the solution to the initial value problem $x' = f(x)$ with $x(t_0) = x_0$ is $x_2(t) = x_1(t - t_0)$.

Remark. The essence of this result is that to find all solutions of the equation $x' = f(x)$ it is sufficient to consider IVP with $t_0 = 0$. Solutions to all other IVP can be obtained by shifting time. This is not true for **non-autonomous** equations $x' = f(t, x)$.

Proof. We have $x_2(t_0) = x_1(t_0 - t_0) = x_1(0) = x_0$ so $x_2(t)$ indeed satisfies the given initial condition.

Moreover, $x_2(t)$ satisfies the given differential equation since

$$x_2'(t) = \frac{d}{dt}[x_1(t - t_0)] = x_1'(t - t_0)\frac{d(t - t_0)}{dt} = x_1'(t - t_0) = f(x_1(t - t_0)) = f(x_2(t))$$

Dynamical systems

One-dimensional dynamical systems

Definition. A (smooth) **dynamical system** is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that satisfies:

1. $\phi(0, x_0) = x_0$ for all $x_0 \in \mathbb{R}$;

2. $\phi(t + s, x_0) = \phi(t, \phi(s, x_0)) = \phi(s, \phi(t, x_0))$ for all $t, s, x_0 \in \mathbb{R}$.

Remark. The idea here is that we have a system which at $t = 0$ is at state x_0 and the function $\phi(t, x_0)$ tells us what is the state of the system at time t. The function ϕ is often called the **flow** of the system.

Example

Consider the IVP $x' = kx$ with $x(0) = x_0$. We have seen that the solution is $x = x_0 e^{kt}$.

Define $\phi(t, x_0) = x_0 e^{kt}$. Then we can check that ϕ satisfies the two properties of a flow.

1. $\phi(0, x_0) = x_0 e^0 = x_0$;

2. $\phi(t, \phi(s, x_0)) = \phi(t, x_0 e^{ks}) = x_0 e^{ks} e^{kt} = x_0 e^{k(t+s)} = \phi(t + s, x_0)$.

Therefore, the equation $x' = kx$ gives rise to a dynamical system.

From differential equations to dynamical systems

Lemma. Consider the differential equation $x' = f(x)$ and define a function ϕ by

$$\phi(t, x_0) = x(t),$$

where $x(t)$ is the solution to the IVP $x' = f(x)$ with $x(0) = x_0$. Then ϕ is the flow of a dynamical system.

Proof. For property (1) we check $\phi(0,x_0) = x(0) = x_0$.

For property (2), we want to show that

$$\phi(s + t, x_0) = \phi(s, \phi(t, x_0)).$$

Let $x_a(t)$ be the solution to the IVP with $x_a(0) = x_0$. Then we want to show that

$$x_a(s + t) = \phi(s, x_a(t)).$$

Fix t and let $x_1 = x_a(t)$. Then notice that $x_a(t)$ also satisfies the IVP $x_a(t) = x_1$. The relation $x_a(s + t) = \phi(s, x_a(t))$ that we want to prove can be written as $x_a(s + t) = x_b(s)$ where $x_b(t)$ is the solution to the IVP with $x_b(0) = x_1$.

Then we have $x_a(\tau) = x_b(\tau - t)$ (from the Lemma on the "Autonomous Equations" slide) so for $\tau = t + s$ we get

$$x_a(t + s) = x_b(t + s - t) = x_b(s).$$

From dynamical systems to differential equations

Lemma. Given a smooth dynamical system ϕ define

$$f(x) = \frac{\partial \phi}{\partial t}(0,x).$$

Then $\phi(t, x_0)$ is a solution to the IVP $x' = f(x)$ with $x(0) = x_0$.

Proof. We have $\phi(0,x_0) = x_0$, therefore the initial condition is satisfied. We also have

$$\frac{d}{dt}[\phi(t, x_0)] = \frac{\partial \phi}{\partial t}(t, x_0) = \lim_{h \to 0} \frac{\phi(t+h, x_0) - \phi(t, x_0)}{h}$$

$$= \lim_{h \to 0} \frac{\phi(h, \phi(t, x_0)) - \phi(0,\phi(t, x_0))}{h} = \frac{\partial \phi}{\partial t}(0,\phi(t, x_0)) = f(\phi(t, x_0)),$$

so we conclude that the differential equation is also satisfied.

Example

Consider the flow $\phi(t, x_0) = x_0 e^{kt}$. Then

$$f(x) = \frac{\partial \phi}{\partial t}(0,x) = kxe^{kt}|_{t=0} = kx.$$

This shows that the differential equation corresponding to this dynamical system is $x' = f(x)$.

For another example, consider $\phi(t, x_0) = \dfrac{x_0}{x_0 - (x_0 - 1)e^{-kt}}$. Then

$$f(x) = \frac{\partial \phi}{\partial t}(0,x) = -\frac{x\left[k(x-1)e^{-kt}\right]}{(x-(x-1)e^{-kt})^2}|_{t=0} = kx(1-x).$$

Phase line

Equilibria

Definition. A point $x_e \in \mathbb{R}$ is called an **equilibrium** of $x' = f(x)$ if $f(x_e) = 0$.

In this case, the constant function $x(t) = x_e$ is the solution to the IVP $x' = f(x)$ with $x(0) = x_e$.

Outside equilibria we have either $f(x) > 0$ or $f(x) < 0$. In the first case, a solution that passes through the point x increases with time, while in the second case it decreases with time.

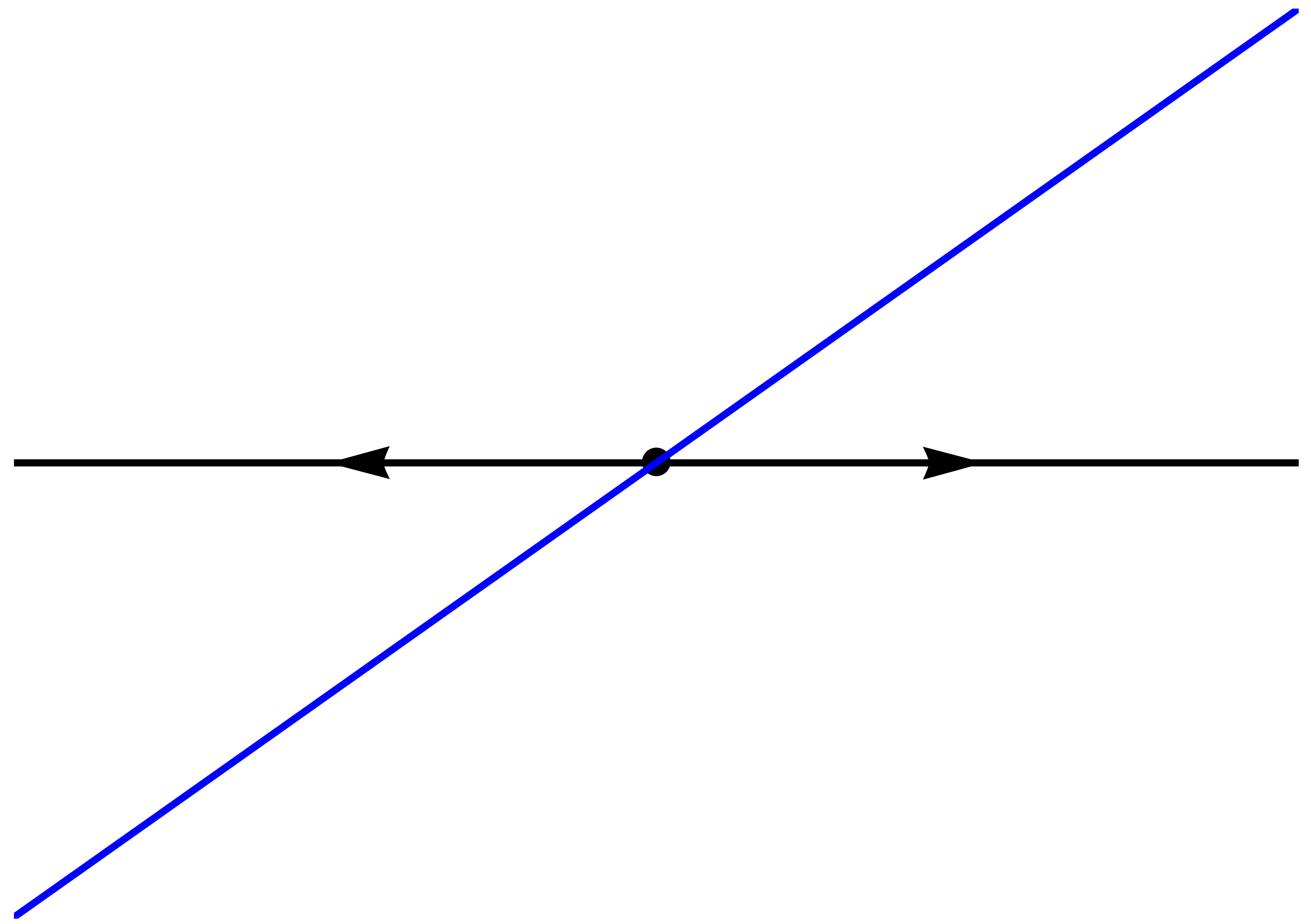
Phase line

For systems of the form $x' = f(x)$ we can represent their dynamics on the **phase line**.

The phase line is the real axis where we have marked the positions of the equilibria and whether a solution increases or decreases with time by drawing arrows indicating the direction of the solutions as time increases.

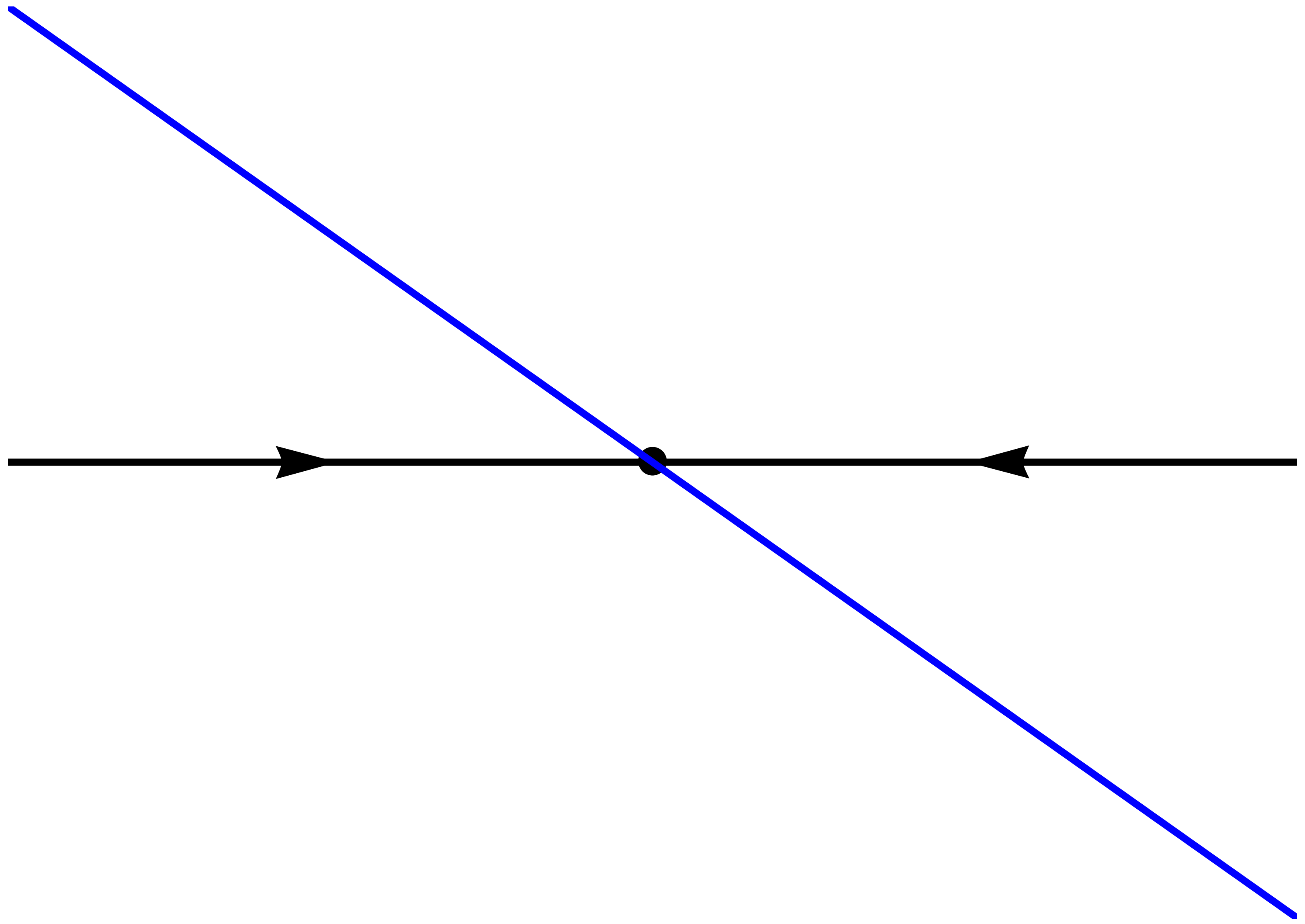
Example

Consider the equation $x' = x$. Then there is a single equilibrium $x = 0$ and the phase line (with the graph of $f(x) = x$ superimposed in blue) is shown below. The equilibrium in this case is **unstable** (there are nearby initial conditions that move away from the equilibrium).



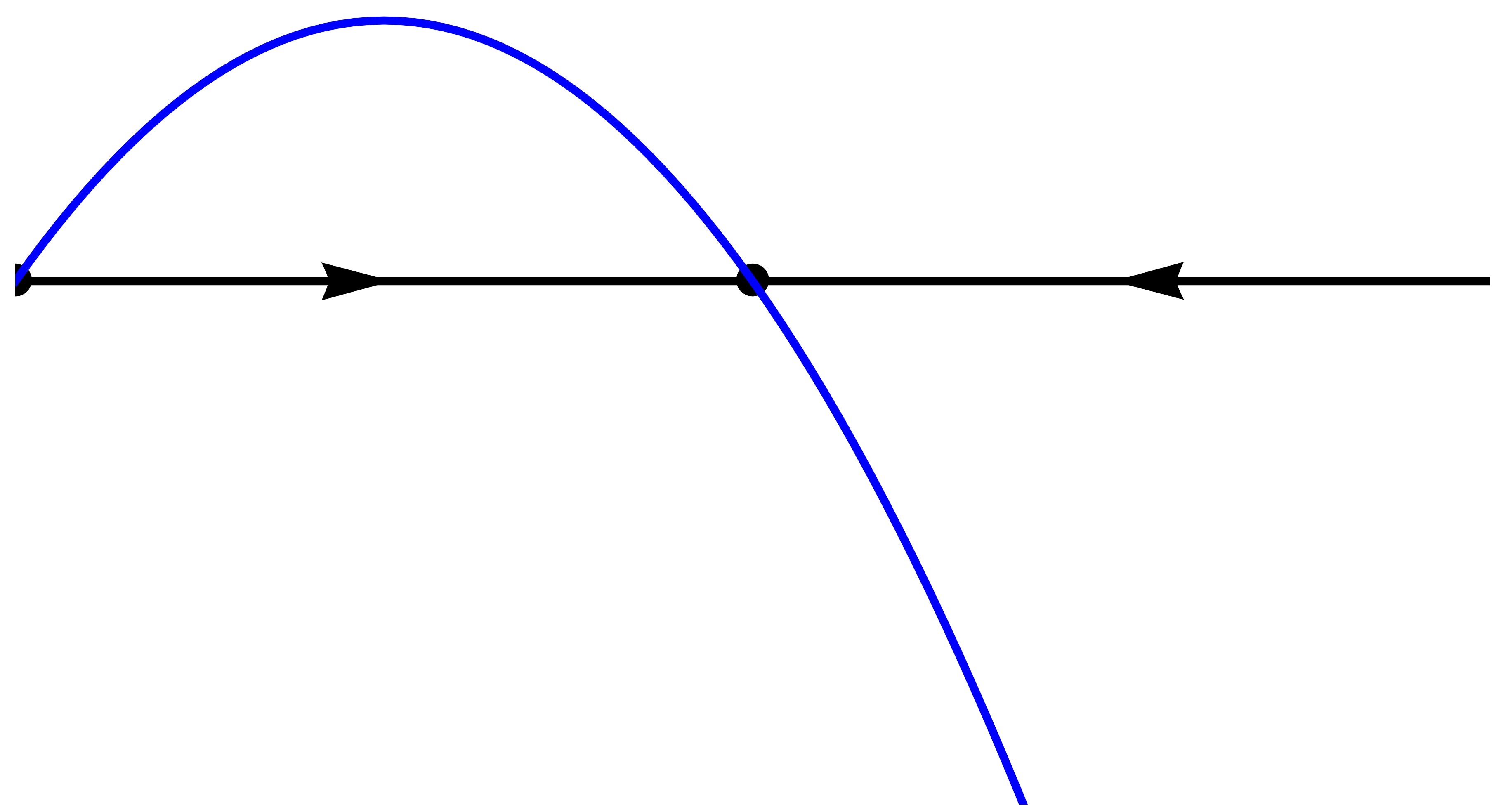
Example

Consider the equation $x' = -x$. Then there is a single equilibrium $x = 0$ and the phase line (with the graph of $f(x) = -x$ superimposed in blue) is shown below. The equilibrium in this case is **asymptotically stable** (there is a neighborhood of the equilibrium such that all initial conditions in this neighborhood move toward the equilibrium).



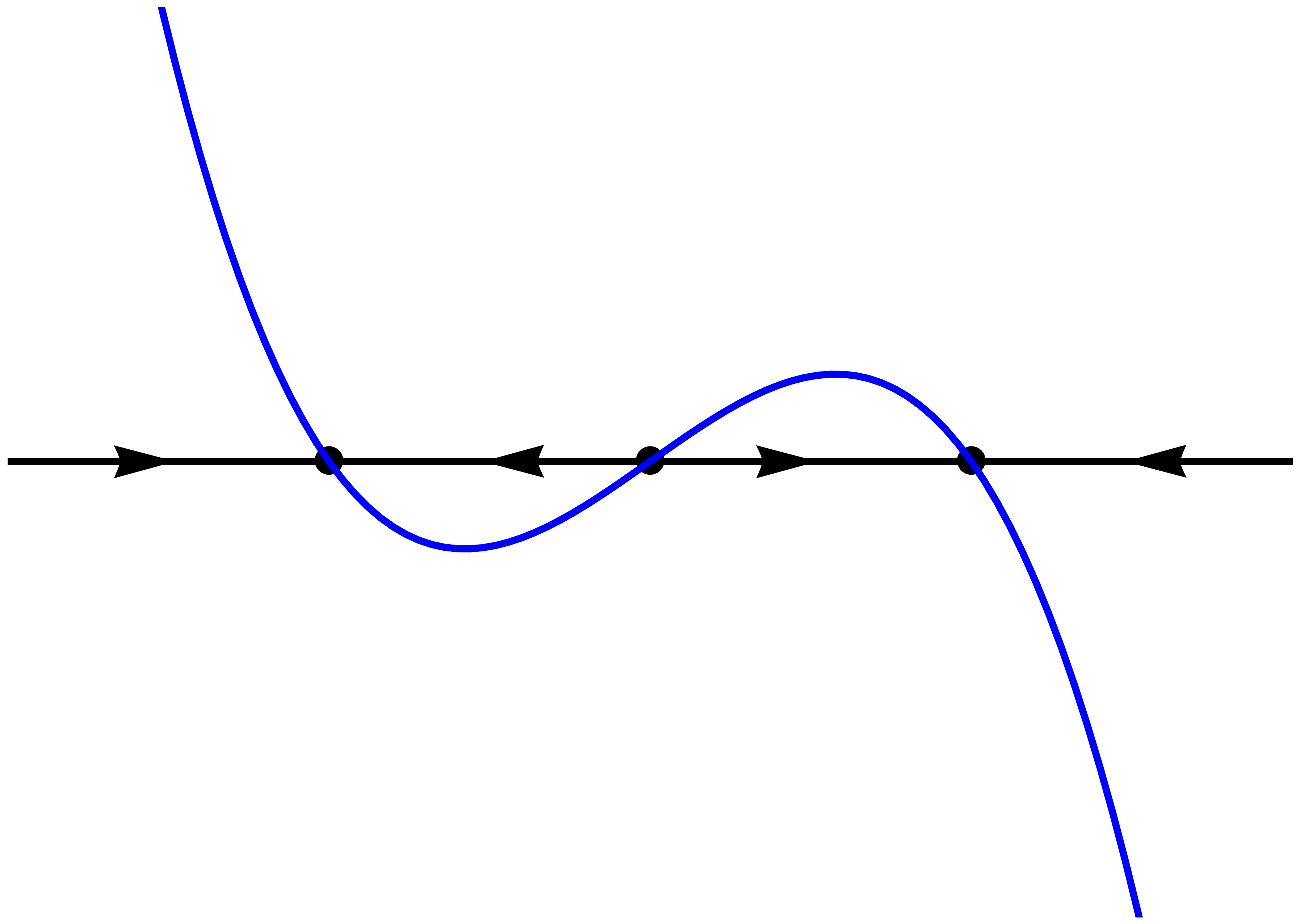
Example

Consider the logistic equation $x' = x(1-x)$ where $x \geq 0$. Then there are two equilibria $x = 0$ and $x = 1$ and the phase line (with the graph of $f(x) = x(1-x)$ for $x \geq 0$ superimposed in blue) is shown below. Here $x = 0$ is unstable and $x = 1$ is asymptotically stable.



Example

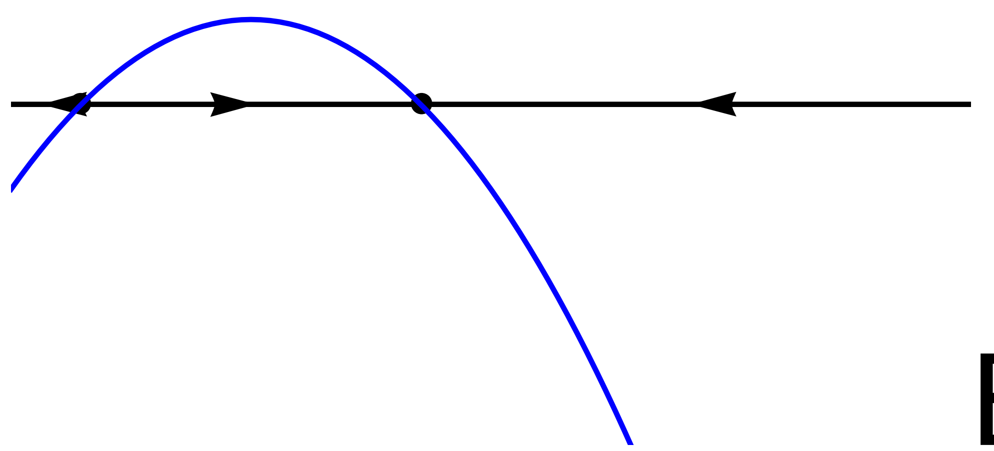
Consider the equation $x' = x(1 - x^2)$. Then there are three equilibria $x = -1$, $x = 0$, and $x = 1$ and the phase line (with the graph of $f(x) = x(1 - x^2)$ superimposed in blue) is shown below. Here $x = 0$ is unstable and $x = -1$, $x = 1$ are asymptotically stable.



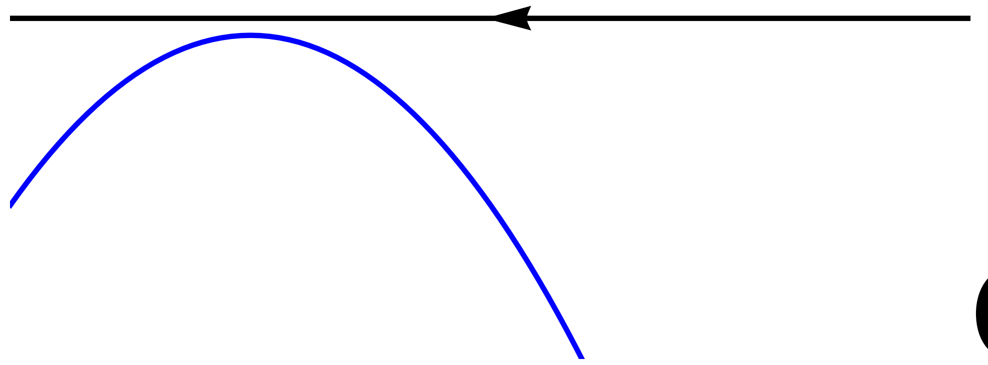
Poll

Consider the logistic equation with constant harvesting $x' = x(1-x) - \frac{1}{8}$.
Which of the following is the corresponding phase line?

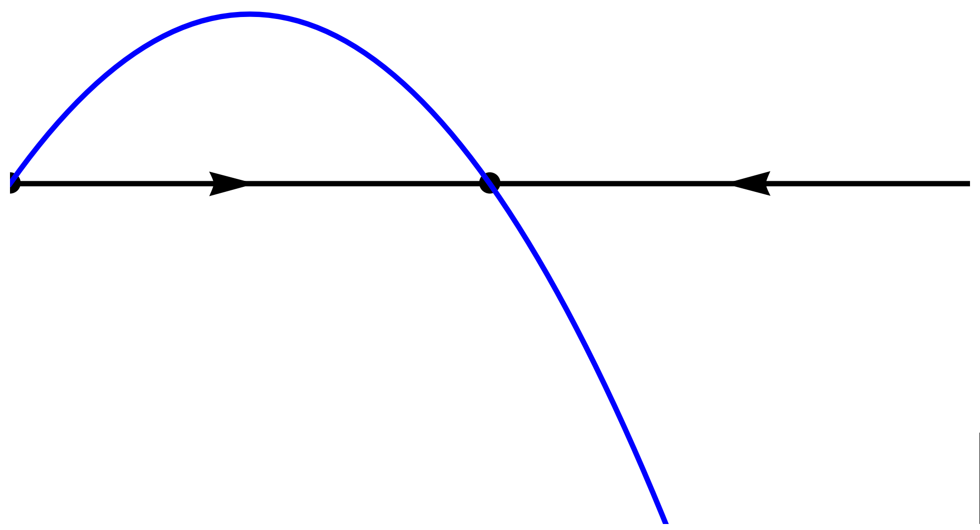
Give your answer at pollev.com/ke1.



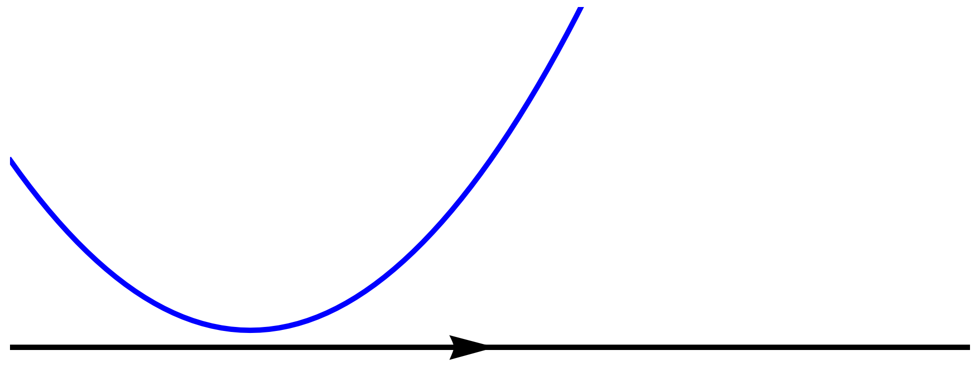
A.



B.



C.



D.

Stability & linearization

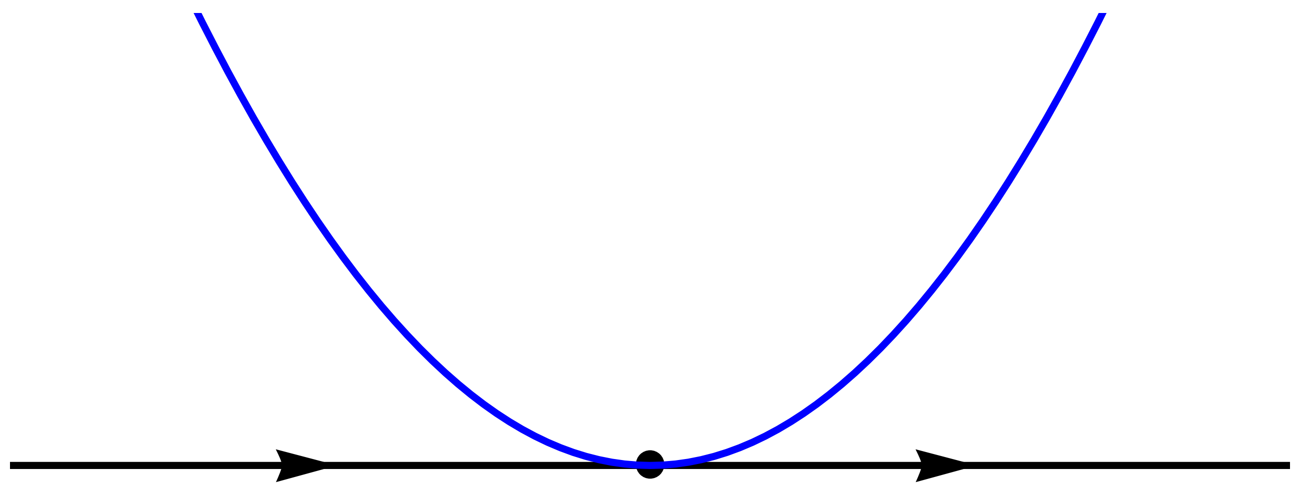
More about stability

From the previous examples it is evident that if as x increases $f(x)$ changes sign from negative to positive at the equilibrium x_e then the equilibrium is unstable. This is true if $f'(x_e) > 0$.

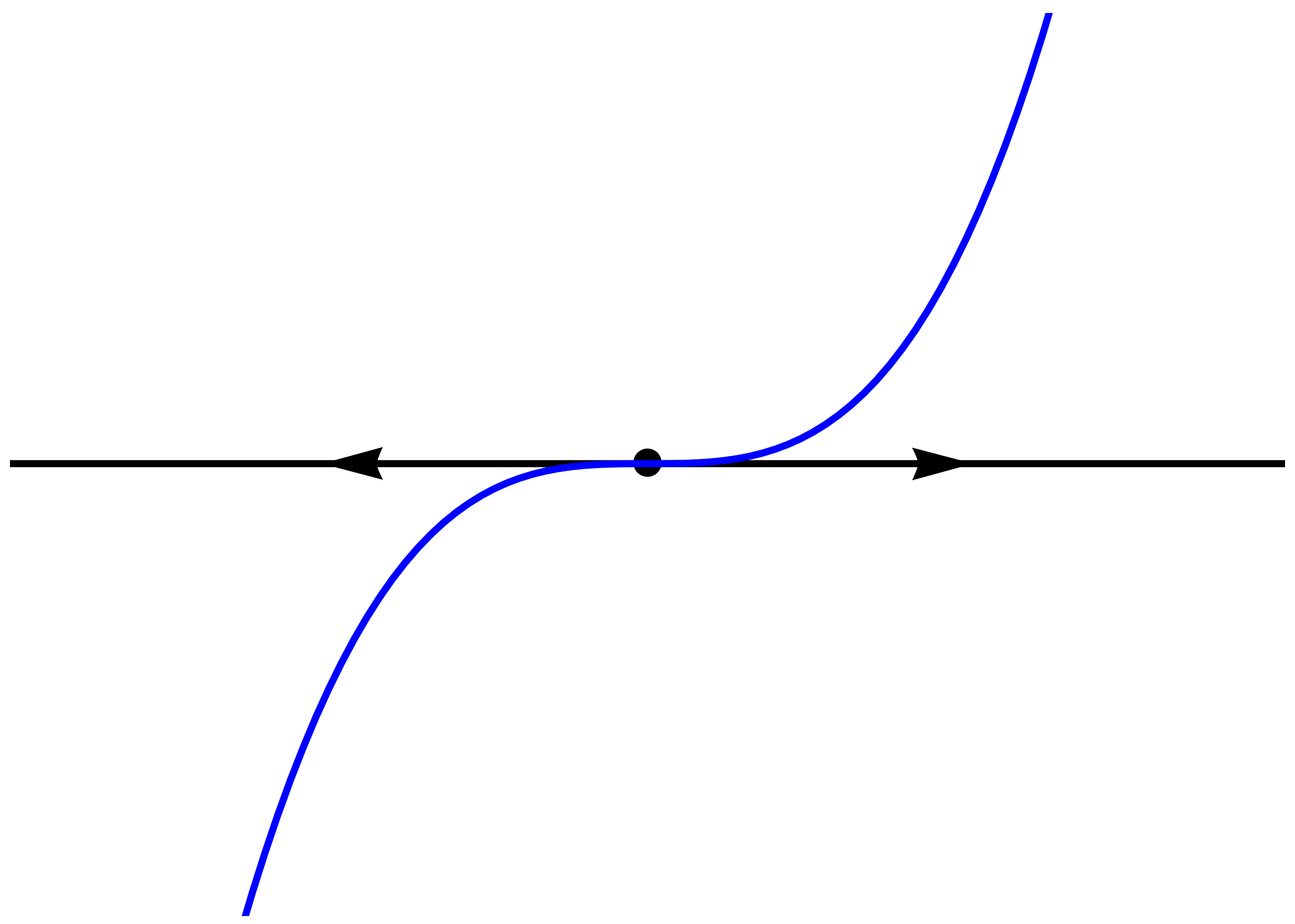
If $f(x)$ changes sign from positive to negative at the equilibrium x_e then the equilibrium is asymptotically stable. This is true if $f'(x_e) < 0$.

Note that in the case $f'(x_e) = 0$ we cannot directly conclude anything about stability and we have to check how the sign of $f(x)$ changes at x_e.

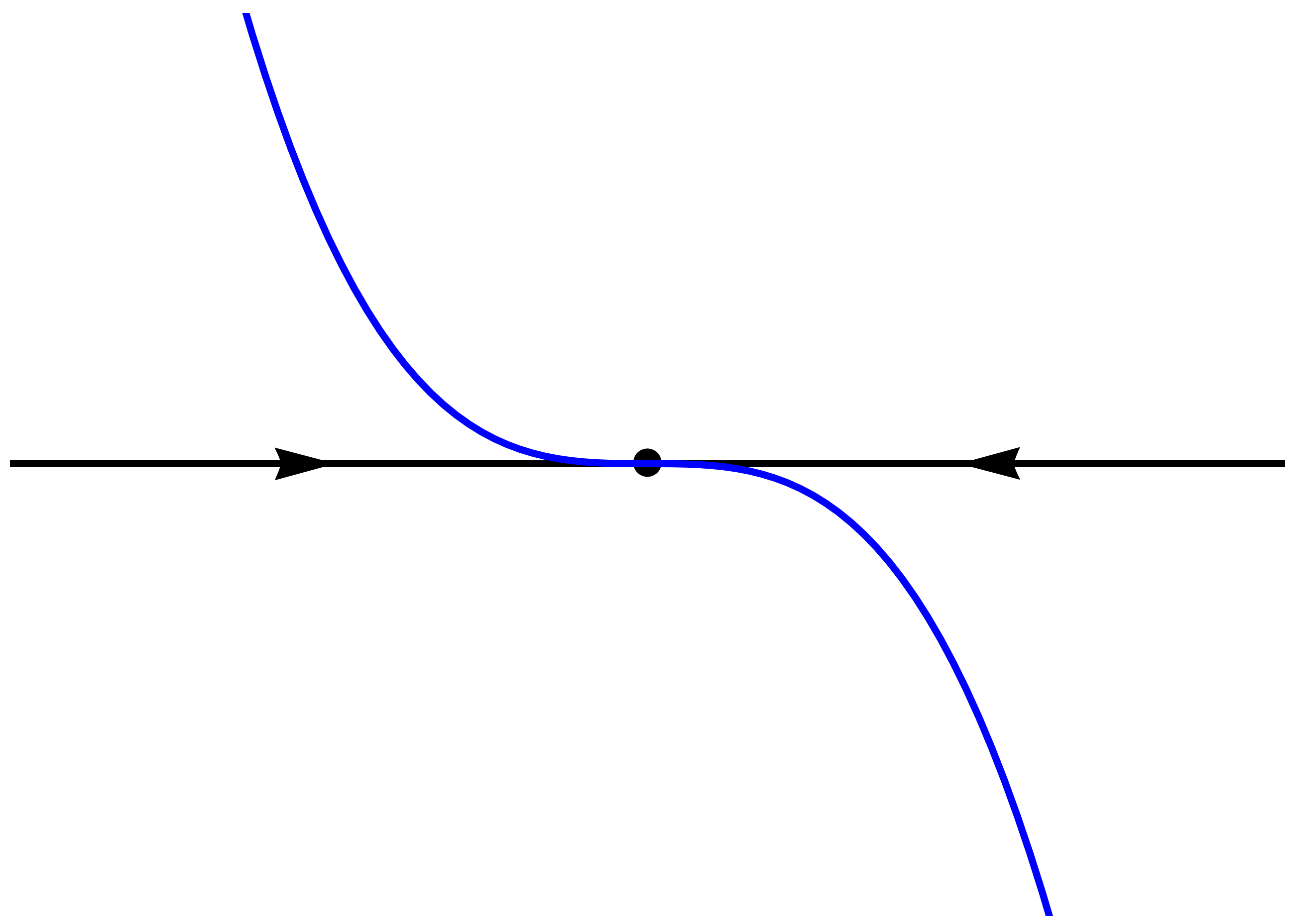
Examples



$$x' = x^2$$
$$x_e = 0$$
$$f'(x_e) = 0$$
Unstable



$$x' = x^3$$
$$x_e = 0$$
$$f'(x_e) = 0$$
Unstable



$$x' = -x^3$$
$$x_e = 0$$
$$f'(x_e) = 0$$
Asymptotically stable

Linearization

One method to understand the dynamics near an equilibrium is through linearization. The idea here is that the dynamics near the equilibrium x_e is determined by the first terms in the Taylor expansion of f at x_e. We have

$$x' = f(x_e) + f'(x_e)\,(x - x_e) + O((x - x_e)^2) = f'(x_e)\,(x - x_e) + O((x - x_e)^2).$$

Let $y = x - x_e$ represent the position relative to the equilibrium. Then

$$y' = x' = f'(x_e)\,y + O(y^2).$$

The linearization at x_e involves ignoring the terms $O(y^2)$. Therefore we get the equation

$$y' = f'(x_e)\,y.$$

Solutions of the linearized equation

Let $\lambda = f'(x_e)$. Then the linearized equation is $y' = \lambda y$ with solution $y = y_0 e^{\lambda t}$.

When $\lambda > 0$ we have exponentially growing solutions. This corresponds to an unstable equilibrium.

When $\lambda < 0$ we have $y_0 e^{\lambda t} \to 0$ as $t \to \infty$. This corresponds to asymptotically stable solutions.

Finally, when $\lambda = 0$ the linearization does not make sense since the higher order terms $O(y^2)$ in the Taylor expansion cannot be ignored.

Poll

What is the linearization of $x' = x - x^3$ at $x_e = 1$?

Give your answer at pollev.com/ke1.

A. $y' = y$

B. $y' = 2y$

C. $y' = -y$

D. $y' = -2y$