# Lecture 8: One-dimensional Maps

## MATH 303 ODE and Dynamical Systems

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# Discrete dynamical systems

# Discrete dynamics

A function $p : \mathbb{R} \to \mathbb{R}$ defines a discrete dynamical system through repeated applications of $p$. That is, if $x_0 \in \mathbb{R}$ is an initial condition (or initial point), then for $k = 0,1,2,3,\ldots$ we define

$$x_k = p(x_{k-1}) = p^k(x_0),$$

where $p^k = p \circ p \circ \cdots \circ p$ ($k$ times). If $p$ is invertible then we define $x_k$ for negative integers $k$ by

$$x_k = p^{-1}(x_{k+1}) = p^k(x_0),$$

where for $k < 0$ we have $p^k = p^{-1} \circ p^{-1} \circ \cdots \circ p^{-1}$ ($|k|$ times).

# Example

Let $p(x) = 2x$ and $x_0 = 1$.

Then $x_1 = 2x_0 = 2,\ x_2 = 2x_1 = 2^2,\ x_3 = 2x_2 = 2^3, \dots,\ x_k = 2^k$ for $k \geq 0$.

Moreover, $p^{-1}(x) = x/2$ so we find

$x_{-1} = x_0/2 = 1/2,\ x_{-2} = x_{-1}/2 = 1/2^2$, and in general
$x_k = 1/2^{|k|} = 2^{-|k|} = 2^k$ for $k < 0$.

We conclude that $x_k = 2^k$ for all $k \in \mathbb{Z}$.

# Fixed points

As we have seen before, the dynamics of dynamical systems $x' = f(x)$ with continuous time is organized around equilibria.

For discrete dynamical systems a similar role is played by **fixed points**.

**Definition.** A point $x_0 \in \mathbb{R}$ is called a **fixed point** of a map $p : \mathbb{R} \to \mathbb{R}$ if $p(x_0) = x_0$.

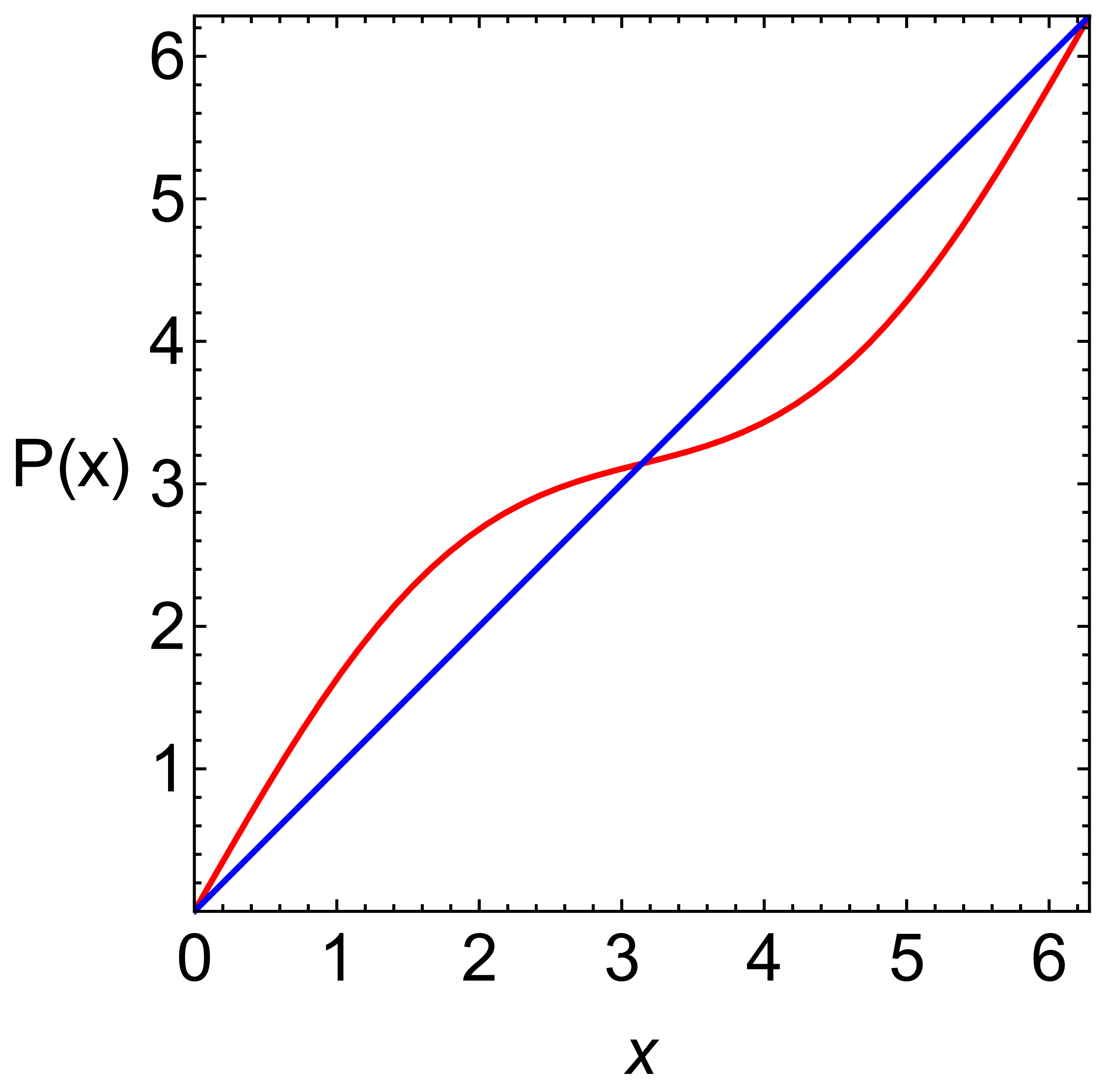
# Example

Consider the map $p(x) = x + \frac{3}{4}\sin x$. The fixed points of $p$ are given by solving $p(x) = x$.

This gives $\sin x = 0$, so we have $x = k\pi,\ k \in \mathbb{Z}$.

You can check that $p(k\pi) = k\pi$.

The fixed points can also be found graphically by considering the intersections of the graph of $p(x)$ with the diagonal $y = x$ as shown at the right.



# Linearization

To understand the dynamics near a fixed point $x_0$ we can consider the linearization of the map $p$ at $x_0$. We have

$$p(x) = p(x_0) + p'(x_0)(x - x_0) + O((x - x_0)^2) = x_0 + p'(x_0)(x - x_0) + O((x - x_0)^2)$$

Let $y = x - x_0$ represent the relative position with respect to the fixed point. Then the relative position of $p(x)$ with respect to the equilibrium is $p(x) - x_0$. This allows to define a function $q : \mathbb{R} \to \mathbb{R}$ given by $q(y) = p(x_0 + y) - x_0$.

This means $q(y) = p'(x_0)y + O(y^2)$. Assuming that $y$ is small we can neglect terms $O(y^2)$ to get the function $q(y) = \lambda y$ where $\lambda = p'(x_0)$.

# Dynamics of linear maps

The dynamics of the map $q(y) = \lambda y$ is $y_n = \lambda^n y_0$.

If $|\lambda| > 1$ then $|\lambda|^n \to \infty$ as $n \to \infty$. This implies that the fixed point is unstable.

If $|\lambda| < 1$ then $\lambda^n \to 0$ as $n \to \infty$. This implies that the fixed point is asymptotically stable.

If $|\lambda| = 1$ then there are examples where a fixed point is unstable (e.g., the fixed point $x = 0$ of $p(x) = x + x^3$) and examples where it is asymptotically stable (e.g., the fixed point $x = 0$ of $p(x) = x - x^3$).

# Poll

Consider the map $p(x) = x + \frac{3}{4}\sin x$. What is the linearized equation of the fixed point $x = 0$? Choose the correct answer at pollev.com/ke1.

A. $q(y) = \frac{7}{4}y$

B. $q(y) = \frac{3}{4}y$

C. $q(y) = y$

D. $q(y) = -\frac{3}{4}y$

# Poincaré maps

# Time-periodic non-autonomous systems

We consider differential equations of the form

$$x' = f(t, x),$$

where $f$ is a $2\pi$-periodic function of $t$, that is, $f(t + 2\pi, x) = f(t, x)$.

**Remark.** We only consider the case where the period is $2\pi$ but we could have chosen any other $T > 0$ as the period.

# A fundamental property

Recall that for autonomous systems $x' = f(x)$ we have that if $x_1(t)$ solves the IVP $x(0) = x_0$ then $x_2(t) = x_1(t - t_0)$ solves the IVP $x(t_0) = x_0$. This is not true for non-autonomous systems. However, the periodicity of $f$ leads to a similar result when $t_0 = 2k\pi,\ k \in \mathbb{Z}$.

**Lemma.** Consider the differential equation $x' = f(t, x)$ where $f(t + 2\pi, x) = f(t, x)$ and let $x_1(t)$ be the solution to the initial value problem $x(0) = x_0$. Then the solution to the initial value problem $x(2k\pi) = x_0,\ k \in \mathbb{Z}$ is given by $x_2(t) = x_1(t - 2k\pi)$.

**Proof.** The proof is almost a copy of the corresponding proof for the autonomous case.

We have $x_2(2k\pi) = x_1(2k\pi - 2k\pi) = x_1(0) = x_0$ so $x_2(t)$ indeed satisfies the given initial condition.

Moreover, $x_2(t)$ satisfies the given differential equation since

$$x_2'(t) = \frac{d}{dt}[x_1(t-2k\pi)] = x_1'(t-2k\pi)\frac{d(t-2k\pi)}{dt} = x_1'(t-2k\pi)$$

$$= f(t-2k\pi, x_1(t-2k\pi)) = f(t-2k\pi, x_2(t)) = f(t, x_2(t)).$$

# Poincaré map

Consider the differential equation $x' = f(t, x)$ where $f(t + 2\pi, x) = f(t, x)$. Then we define a function $P : \mathbb{R} \to \mathbb{R}$ in the following way.

For any $x_0 \in \mathbb{R}$, let $x(t)$ be the solution to the initial value problem $x' = f(t, x)$ with $x(0) = x_0$, and define $P(x_0) = x(2\pi)$.

The function $P$ defined in this way is called the **Poincaré map** for the system.

# Poincaré map and solutions of the differential equation

**Lemma.** Consider the differential equation $x' = f(t, x)$ where $f(t + 2\pi, x) = f(t, x)$ and the corresponding Poincaré map $P$. Let $x(t)$ be the solution to the IVP $x' = f(t, x)$ with $x(0) = x_0$. Then for all $k \in \mathbb{Z}$ we have

$$x(2k\pi) = P^k(x_0).$$

**Proof.** For $k \geq 0$ we use induction. By the definition of the Poincaré map, the required relation is true for $k = 1$. Assume that it is true up to some $k \geq 1$. Then for $k + 1$ we have $P^{k+1}(x_0) = P(P^k(x_0)) = P(x(2k\pi)) = P(x_1)$.

Let $x_a(t)$ be the solution to the initial value problem $x_a(0) = x_1$ so that $P(x_1) = x_a(2\pi)$.

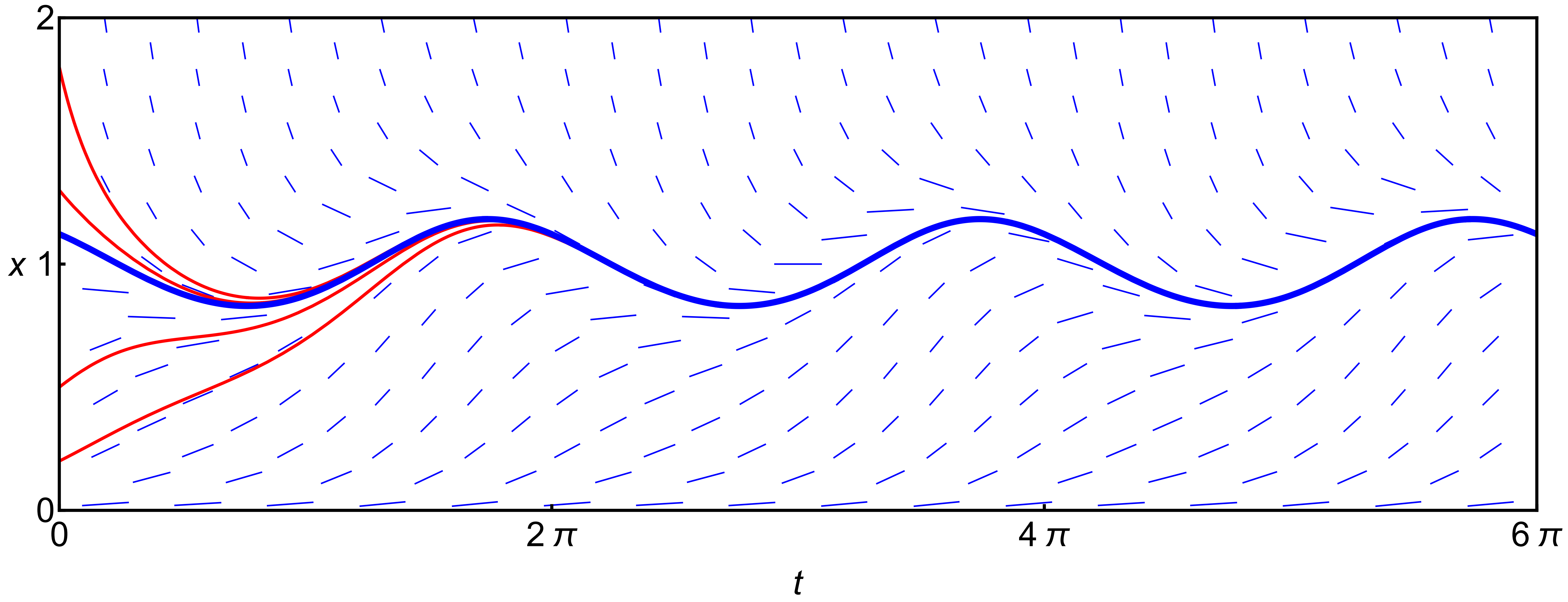
Then note that $x(t)$ is also the solution to the initial value problem $x(2k\pi) = x_1$ . But then we know that $x(\tau) = x_a(\tau - 2k\pi)$ so for $\tau = 2(k+1)\pi$ we find $x_a(2\pi) = x(2(k+1)\pi)$.

# Example

Consider the logistic equation with periodic harvesting

$$x' = x\,(1 - x) - h\,x\sin t.$$

Below we see the direction field for the given equation and a few solutions. We notice that $x = 0$ is still an equilibrium solution but we no longer have the equilibrium at $x = 1$.



Instead of the asymptotically stable equilibrium $x = 1$ we see that the solutions as $t$ increases come closer to a solution that appears to be periodic with period $2\pi$.

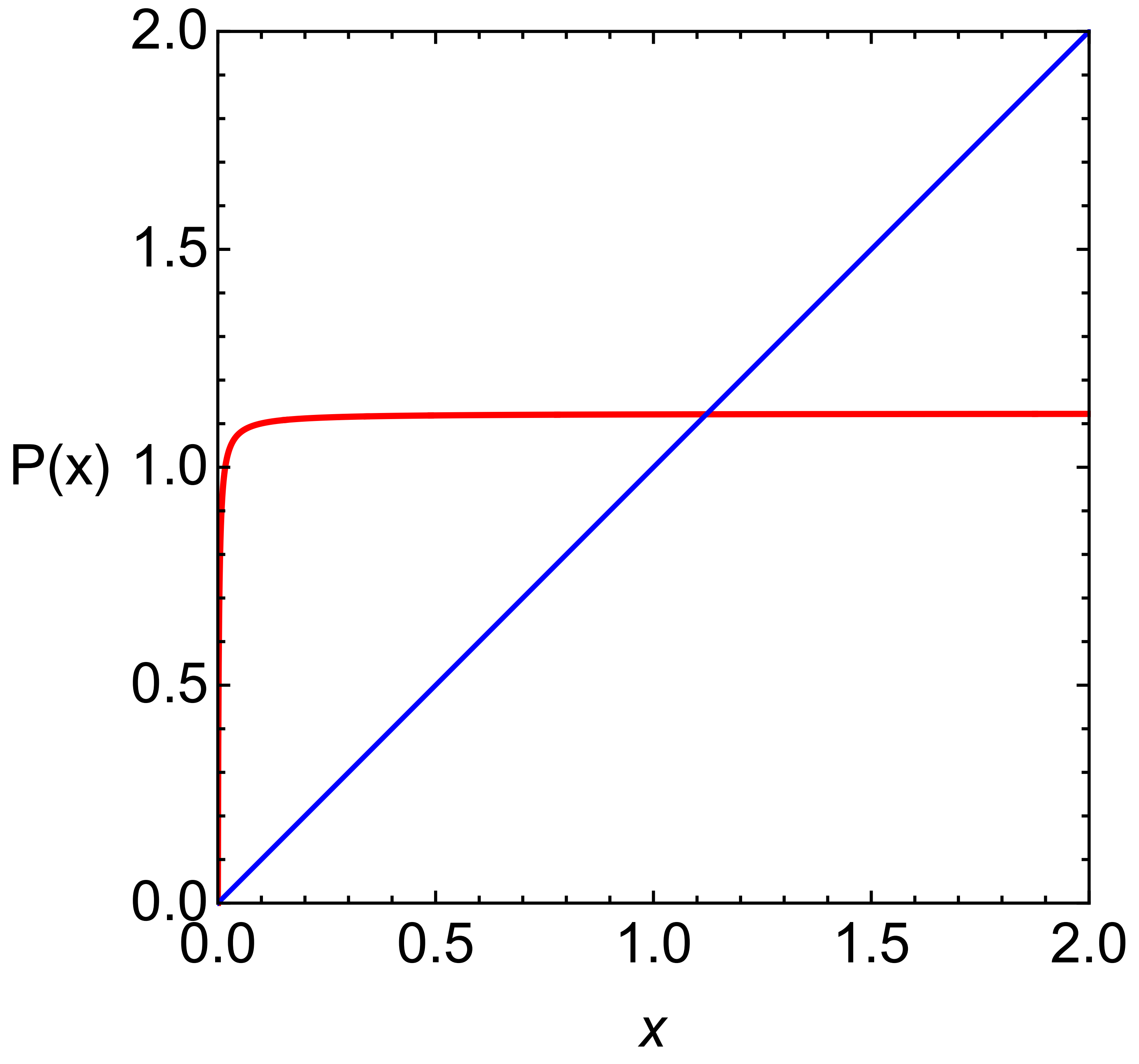
**The Poincaré map is very useful for analyzing such periodic solutions.** The reason is that if a solution is periodic with period $2\pi$ and $x(0) = x_0$ then $x(2\pi) = x_0$.

Recall that $x(2\pi) = P(x_0)$ and we conclude that **periodic solutions of period $2\pi$ correspond to points $x_0$ with $P(x_0) = x_0$**, that is, fixed points of the Poincaré map.

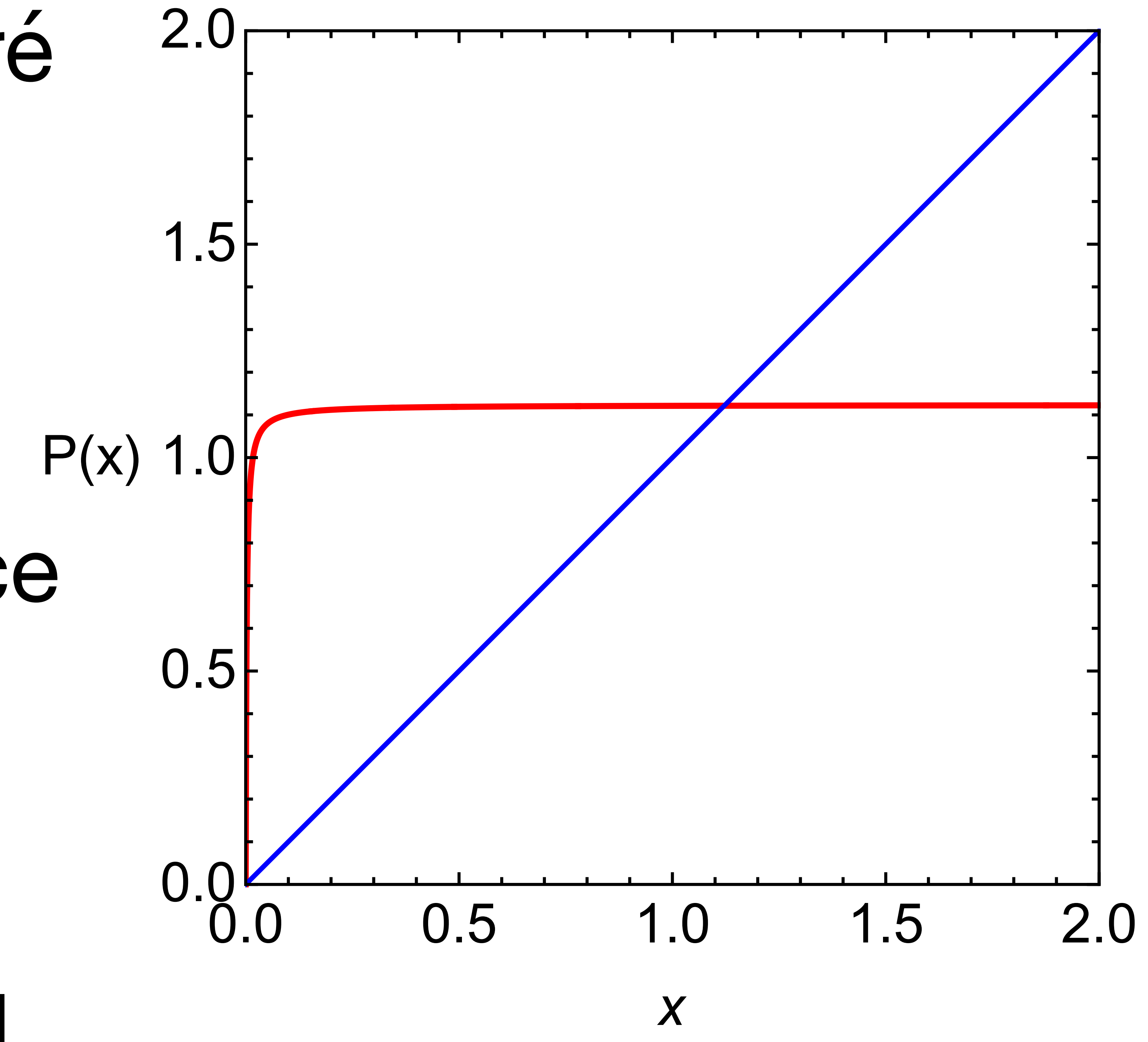
We have also numerically computed the Poincaré map using Mathematica. This is done using the following function definition which numerically integrates the differential equation from the initial condition $x(0) = x_0$ for time $t = 2\pi$ and returns $x(2\pi)$.

```
P[h_,x0_] := NDSolveValue[
  x'[t] == x[t](1-x[t]) - h x[t] Sin[t] &&
  x[0] == x0,
  x[2 Pi], {t, 0, 2 Pi}]
```

The graph of this function for $h = 0.25$ is shown at the right (red curve). In the same plot we show the graph of the identity function $\mathrm{id}(x) = x$ (blue line). The intersection point of these graphs around $x = 1$ is a fixed point of the Poincaré map and thus corresponds to the periodic orbit.



From the numerically computed graph of the Poincaré map we observe that there are two fixed points. One at $x = 0$ and the other one around $x \approxeq 1.12$.




The second fixed point corresponds to a periodic orbit of period $2\pi$ and it is asymptotically stable, since the derivative $\lambda$ of the Poincaré map at $x \approxeq 1.12$ can be seen in the plot to be close to $0$ (actually, $\lambda \approxeq 0.0018$), that is, $|\lambda| < 1$. This means that if we start at an initial point close to the fixed point, we will approach the fixed point.

This also means that the corresponding solution for the differential equation will approach the periodic solution and we conclude that the periodic solution is asymptotically stable.