# Lecture 9: Second-Order Homogeneous Linear Equations

## MATH 303 ODE and Dynamical Systems

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# Second-order linear equations

A second order **linear** differential equation has the form

$$a(x)\, y'' + b(x)y' + c(x)y = f(x).$$

We will consider the general theory of linear differential equation (of any order) in another lecture.

In this lecture we will focus on solving a very special case of this equation, the case where $a(x), b(x), c(x)$ are constant (we say that the equation has constant coefficients) and $f(x) \equiv 0$ (we say that the equation is homogeneous). That is, we consider the equation

$$ay'' + by' + cy = 0,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. This is a **second order homogeneous linear equation with constant coefficients**.

# Solution method

Finding the general solution such equations follows a very simple algorithm. The theory of homogeneous linear equations shows that the general solution is a linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

of two functions $y_1(x), y_2(x)$ that **(i)** solve the equation and **(ii)** are linearly independent in the sense that $y_1(x)/y_2(x)$ is not constant.

To find two such functions we try the solution $y = e^{rx}$ where $r$ is a number that must be determined.

Substituting $y = e^{rx}$ into $ay'' + by' + cy = 0$ we find

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0,$$

which is satisfied if and only if

$$ar^2 + br + c = 0.$$

The last equation is called the **auxiliary equation** corresponding to the differential equation $ay'' + by' + cy = 0$.

For the solution of the auxiliary equation we distinguish 3 cases.

**A. Two real distinct roots** $r_1, r_2$ when the discriminant $\Delta = b^2 - 4ac > 0$.

**B. A double real root** $r_0$ when $\Delta = 0$.

**C. Two complex conjugate roots** $\alpha \pm i\beta$ when $\Delta < 0$.

We will summarize now the type of solutions we get in each case and then we will explain them in more detail.

**A.** When we have **two distinct real roots** $r_1, r_2$ the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x},$$

where $c_1, c_2$ are arbitrary constants.

**B.** When there is a **double real root** $r_0$ the general solution is

$$y = c_1 e^{r_0 x} + c_2 x e^{r_0 x} = (c_1 + c_2 x) e^{r_0 x},$$

where $c_1, c_2$ are arbitrary constants.

**C.** When we have **two complex conjugate roots** $\alpha \pm i\beta$ the general solution is

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)),$$

where $c_1, c_2$ are arbitrary constants

# Two distinct real roots $r_1, r_2$

In this case the functions $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}$ are solutions and are linearly independent since

$$\frac{y_1}{y_2} = e^{(r_1 - r_2)x} \neq \text{constant.}$$

Therefore, we can directly use the theory to claim that the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

# Poll

Solve the initial value problem $y'' + 2y' - 3y = 0$ with $y(0) = 0$ and $y'(0) = 4$.

Choose the correct answer at pollev.com/ke1.

A. $y = e^{3x} - e^{-x}$

B. $y = e^{x} - e^{-3x}$

C. $y = -e^{x} + e^{-3x}$

D. $y = -e^{3x} + e^{-x}$

# Double real root $r_0$

This case appears when $\Delta = b^2 - 4ac = 0$. The trial solution $y = e^{rx}$ gives a single solution $y_1 = e^{r_0 x}$. To write the general solution we need a second solution $y_2$ that is linearly independent from $y_1$.

It turns out that $y_2 = xe^{r_0 x}$ has these properties. To check that it is indeed a solution we compute:

$$y_2' = e^{r_0 x} + r_0 x e^{r_0 x} = e^{r_0 x}(1 + r_0 x),$$

$$y_2'' = r_0 e^{r_0 x}(1 + r_0 x) + e^{r_0 x} r_0 = e^{r_0 x}(2r_0 + r_0^2 x).$$

Substituting into the left hand side of the differential equation $ay'' + by' + cy = 0$ we find

$$ay_2'' + by_2' + cy_2 = e^{r_0 x}(a(2r_0 + r_0^2 x) + b(1 + r_0 x) + cx)$$

from where we find the expression

$$e^{r_0 x}[(2ar_0 + b) + x(ar_0^2 + br_0 + c)].$$

We now notice that since $r_0$ is a root, it satisfies $ar_0^2 + br_0 + c = 0$. Moreover, since $\Delta = 0$ we have that $r_0 = -\dfrac{b}{2a}$. Therefore, $2ar_0 + b = 0$. We conclude that $y_2 = xe^{r_0 x}$ is indeed a solution.

Moreover, $y_2/y_1 = x$ is not constant, and thus the two functions are linearly independent. We conclude that the general solution is

$$y = c_1 e^{r_0 x} + c_2 x e^{r_0 x} = e^{r_0 x}(c_1 + c_2 x).$$

# Poll

Solve the initial value problem $y'' + 2y' + y = 0$ with $y(0) = 0$ and $y'(0) = 4$.
Choose the correct answer at pollev.com/ke1.

A. $y = 4e^{-x}x$

B. $y = 4e^{-x}(x + 1)$

C. $y = 4e^{x}x$

D. $y = 2e^{x} - 2e^{-x}$

# Complex conjugate roots $\alpha \pm i\beta$

Here we consider the solutions $z = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$ and

$\bar{z} = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$.

These are complex valued solutions but here we are looking for real valued solutions. These can be obtained by noticing that linear combinations of solutions are still solutions. In particular, the real and imaginary parts of a complex number can be obtained as linear combinations of the number with its complex conjugate. We have

$$y_1 = e^{\alpha x}\cos(\beta x) = \frac{1}{2}(z + \bar{z}) \text{ and } y_2 = e^{\alpha x}\sin(\beta x) = \frac{1}{2i}(z - \bar{z}).$$

The solutions $y_1 = e^{\alpha x} \cos(\beta x)$ and $y_2 = e^{\alpha x} \sin(\beta x)$ are linearly independent since

$$\frac{y_2}{y_1} = \tan(\beta x) \neq \text{constant.}$$

We conclude that the general solution in this case must be

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

# Poll

Solve the initial value problem $y'' + 2y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 4$.
Choose the correct answer at pollev.com/ke1.

A. $y = 4e^x \sin x$

B. $y = 4e^x \cos x$

C. $y = 4e^{-x} \sin x$

D. $y = 4e^{-x}(\sin x + \cos x)$

# Complex conjugate roots $\alpha \pm i\beta$: Alternative solution form (1)

In the case of roots $\alpha \pm i\beta$ we have seen that the general solution is

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

We will show that the term $c_1 \cos(\beta x) + c_2 \sin(\beta x)$ can always be written in the form $A\cos(\beta x - \phi)$. We have

$$A\cos(\beta x - \phi) = A\cos\phi\cos(\beta x) + A\sin\phi\sin(\beta x).$$

Comparing the two expressions we find $A\cos\phi = c_1$, $A\sin\phi = c_2$. One way to read these equations is that $(A, \phi)$ are the polar coordinates of the point $(c_1, c_2)$ in the plane. Therefore,

$$A = \sqrt{c_1^2 + c_2^2}, \quad \phi = \arg(c_1 + ic_2).$$

Using this form, the general solution

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

can be written as

$$y = Ae^{\alpha x} \cos(\beta x - \phi).$$

# Poll

Consider the function $y = \sin x - \cos x$. Find $A, \phi$ so that $y = A\cos(x - \phi)$.
Choose the correct answer at pollev.com/ke1.

A. $A = \sqrt{2},\ \phi = -\pi/4$

B. $A = 1,\ \phi = \pi/4$

C. $A = \sqrt{2},\ \phi = \pi/4$

D. $A = \sqrt{2},\ \phi = 3\pi/4$

# Complex conjugate roots $\alpha \pm i\beta$: Alternative solution form (2)

In the case of roots $\alpha \pm i\beta$ we have seen that the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 \frac{z + \bar{z}}{2} + c_2 \frac{z - \bar{z}}{2i},$$

where $z = e^{(\alpha + i\beta)x}$. Then we can write

$$y = \frac{c_1}{2}(z + \bar{z}) - \frac{ic_2}{2}(z - \bar{z}) = \frac{c_1 - ic_2}{2} z + \frac{c_1 + ic_2}{2} \bar{z} = \frac{cz + \bar{c}\bar{z}}{2} = \mathrm{Re}(cz),$$

where $c = c_1 - ic_2$. Therefore the general **real-valued** solution can be written in the form

$$y = ae^{(\alpha + i\beta)x} + \bar{a}e^{(\alpha - i\beta)x},$$

where $a$ is an arbitrary **complex** constant.

**Remark.** If we were trying to find complex-valued solutions $y$ then the general solution would be written as

$$y = a_1 e^{(\alpha + i\beta)x} + a_2 e^{(\alpha - i\beta)x},$$

where $a_1, a_2$ are arbitrary **complex** constants.