Lecture 10: Second-Order Non-Homogeneous Linear Equations

MATH 303 ODE and Dynamical Systems

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Non-homogeneous equations

We consider now equations of the form

$$ay'' + by' + cy = f(x),$$

where $f(x)$ is not identically zero. The method for solving such equations is based on the following considerations.

Suppose that we find **any** solution of the equation above. Such a solution will be called a **particular solution** and will be denoted by $y_p(x)$.

The solution method for non-homogeneous equations is based on the following fact.

Lemma. Suppose that $y_p(x)$ is a (known) particular solution of the non-homogeneous equation $ay'' + by' + cy = f(x)$ and that $y(x)$ is any other (unknown) solution of the same equation. Let

$$y_h(x) = y(x) - y_p(x).$$

Then $y_h(x)$ satisfies the homogeneous equation $ay'' + by' + cy = 0$.

Proof. We have

$$ay_h'' + by_h' + cy_h = ay'' + by' + cy - (ay_p'' + by_p' + cy_p) = f(x) - f(x) = 0.$$

Remark. This lemma means that any solution $y(x)$ of the non-homogeneous equation can be written as $y(x) = y_h(x) + y_p(x)$ where $y_h(x)$ is some solution of the homogeneous equation and $y_p(x)$ is the known particular solution.

Solution method for non-homogeneous equations

Step 1. Find the general solution of the homogeneous equation $ay'' + by' + cy = 0$ which has the form

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Step 2. Find a single particular solution $y_p(x)$ of the non-homogeneous equation $ay'' + by' + cy = f(x)$.

Step 3. Write the general solution of the non-homogeneous equation as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Finding a particular solution: the method of undetermined coefficients

Linear differential operators

Given a function $g(x)$ we denote by $L[g](x)$ the function

$$L[g](x) = ag''(x) + bg'(x) + cg(x).$$

Just as the operator T that we introduced when we discussed the existence and uniqueness theorem, L is also an operator: it takes as input a function g and produces a new function $L[g]$.

Using the operator L, the homogeneous equation we have been considering can be written as $L[y] = 0$ while the non-homogeneous equation is $L[y] = f$.

Definition. An operator L is **linear** if $L[g_1 + g_2] = L[g_1] + L[g_2]$ and $L[\lambda g] = \lambda L[g]$ for all functions g, g_1, g_2 and all numbers $\lambda \in \mathbb{R}$.

Lemma. The operator L defined by $L[g] = ag'' + bg' + cg$ is linear.

Proof. We have

$$L[g_1 + g_2] = ag_1'' + bg_1' + cg_1 + ag_2'' + bg_2' + cg_2 = L[g_1] + L[g_2],$$

and

$$L[\lambda g] = a(\lambda g)'' + b(\lambda g)' + c(\lambda g) = \lambda(ag'' + bg' + cg) = \lambda L[g].$$

Consider now the equation $L[y] = f$ for which we want to find a particular solution.

The method of undetermined coefficients is that if f has a specific type (e.g., polynomial, trigonometric, exponential, or a combination of these) then the particular solution y_p will belong in a space of functions W such that $f \in L[W]$.

We will see how this works in specific cases.

Polynomial f

Suppose that $f(x) = P(x)$ where $P(x)$ is a polynomial of degree $\deg P = n$, that is, the highest power of x in $P(x)$ is n.

Then we notice that if $Q(x)$ is any polynomial of degree n then $L[Q](x)$ will also be a polynomial of degree $\deg Q \leq n$ since the derivatives of $Q(x)$ will also be polynomials with degree $\leq n$.

To find $y_p = Q$ such that $L[Q] = P$ we can therefore write down the most general polynomial of degree n,

$$Q = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

compute $L[Q]$ and find the coefficients $a_0, a_1, \ldots, a_n$ such that $L[Q] = P$.

Example

Find a particular solution of the equation $y'' - y = 2 - x^2$.

Here, $f(x) = 2 - x^2$ is a polynomial of degree 2. Therefore, we try

$$y_p(x) = a_2x^2 + a_1x + a_0.$$

Then $y_p' = 2a_2x + a_1$, $y_p'' = 2a_2$, and substituting into the equation we find

$$2a_2 - (a_2x^2 + a_1x + a_0) = 2 - x^2,$$

or

$$-a_2x^2 - a_1x + (2a_2 - a_0) = 2 - x^2.$$

From the equation $-a_2x^2 - a_1x + (2a_2 - a_0) = 2 - x^2$ we get

$$-a_2 = -1, \; -a_1 = 0, \; 2a_2 - a_0 = 2,$$

with solution $a_2 = 1, \; a_1 = a_0 = 0$.

Therefore, $y_p(x) = x^2$.

In this case, the general solution is

$$y = c_1e^x + c_2e^{-x} + x^2,$$

where $c_1e^x + c_2e^{-x}$ is the general solution of the homogeneous equation $y'' - y = 0$.

Exponential f

Suppose that $f(x) = P(x)e^{\lambda x}$ where $P(x)$ is again a polynomial with $\deg P = n$. Then we notice that if $Q(x)$ is a polynomial with $\deg Q = n$ then $L[Qe^{\lambda x}]$ has the form $R(x)e^{\lambda x}$ where $\deg R \leq n$.

In this case, try

$$y_p = (a_n x^n + \cdots + a_1 x + a_0)e^{\lambda x}.$$

But there is a catch that we will see in a moment.

Example 1

Find a particular solution for $y'' - y = 2xe^{2x}$.

Here we try $y_p = (a_1x + a_0)e^{2x}$. Then $y_p' = (2a_1x + 2a_0 + a_1)e^{2x}$ and $y_p'' = (4a_1x + 4a_1 + 4a_0)e^{2x}$.

Substituting into the equation we find $(3a_1x + 4a_1 + 3a_0)e^{2x} = 2xe^{2x}$. Equating coefficients for same powers of x we get the equations $3a_1 = 2$, $4a_1 + 3a_0 = 0$ with solution $a_1 = 2/3$, $a_0 = -8/9$. Therefore, a particular solution is

$$y_p(x) = \left(\frac{2}{3}x - \frac{8}{9}\right)e^{2x}.$$

Example 2

Find a particular solution for $y'' - y = 2xe^x$.

First, this is what NOT to do. Try $y_p = (a_1 x + a_0)e^x$. Then $y_p'' - y_p = 2a_1 e^x$ which clearly can't give $2xe^x$. The thing that went wrong here is that e^x is a solution to the homogeneous equation $y'' - y = 0$ and because of that the degree of the polynomial in front of e^x is reduced by one when we compute $y'' - y$.

Therefore, to match the polynomial $2x$ of degree one we need to start with a polynomial of degree 2. The easiest way to achieve this is to try

$$y_p = x(a_1 x + a_0)e^x.$$

Then we compute $y_p'' - y_p = (4a_1 x + 2a_1 + 2a_0)e^x = 2xe^x$ which gives $a_1 = 1/2,\ a_0 = -1/2$. Therefore,

$$y_p = \frac{1}{2}x(x-1)e^x.$$

Exponential f revisited

The rules for choosing y_p when $f(x) = P(x)e^{\lambda x}$ are as follows.

(a) If λ is not a root of the associated auxiliary equation then try $y_p = Q(x)e^{\lambda x}$ where $\deg Q = \deg P$.

(b) If the associated auxiliary equation has two distinct real roots and λ is one of them then try $y_p = x\,Q(x)e^{\lambda x}$ where $\deg Q = \deg P$.

(c) If the associated auxiliary equation has a double real root which equals λ then try $y_p = x^2\,Q(x)e^{\lambda x}$ where $\deg Q = \deg P$.

Trigonometric f

The rules for choosing y_p when $f(x) = P_1(x)e^{\lambda x}\cos(\mu x) + P_2(x)e^{\lambda x}\sin(\mu x)$ are as follows.

(a) If $\lambda + i\mu$ is not a root of the associated auxiliary equation then try

$$y_p = Q_1(x)e^{\lambda x}\cos(\mu x) + Q_2(x)e^{\lambda x}\sin(\mu x)$$

where $\deg Q_1 = \deg Q_2 = \max(\deg P_1, \deg P_2)$.

(b) If $\lambda + i\mu$ is a root of the associated auxiliary equation then try

$$y_p = x\left[Q_1(x)e^{\lambda x}\cos(\mu x) + Q_2(x)e^{\lambda x}\sin(\mu x)\right]$$

where $\deg Q_1 = \deg Q_2 = \max(\deg P_1, \deg P_2)$.

Example

Find a particular solution for $y'' + 2y' + 2y = 5e^{-x}\cos x$. Here $\lambda = -1$, $\mu = 1$, $P_1(x) = 5$, $P_2(x) = 0$. Therefore $\deg Q_1 = \deg Q_2 = 0$, that is, constant polynomials with $Q_1(x) = a_1$ and $Q_2(x) = a_2$.

So, the first guess would be

$$y_p(x) = a_1 e^{-x}\cos x + a_2 e^{-x}\sin x.$$

However, we notice that the auxiliary equation $r^2 + 2r + 2 = 0$ has roots $-1 \pm i = \lambda \pm i\mu$. This means that we should try instead the form

$$y_p(x) = x\left[a_1 e^{-x}\cos x + a_2 e^{-x}\sin x\right].$$

Some (heavy) computations give

$$y_p'' + 2y_p' + 2y_p = e^{-x}(2a_2 \cos x - 2a_1 \sin x) = 5e^{-x} \cos x.$$

We conclude that $a_2 = \frac{5}{2}$ and $a_1 = 0$.

Poll

Which of the following forms you should try for finding a particular solution to $y'' + 2y' + 2y = x\cos x$?

Choose the correct answer at pollev.com/ke1.

A. $y_p = (a_1x + a_0)\cos x + (b_1x + b_0)\sin x$

B. $y_p = x(a_1x + a_0)\cos x + x(b_1x + b_0)\sin x$

C. $y_p = (a_1x + a_0)\cos x + (a_1x + a_0)\sin x$

D. $y_p = (a_1x + a_0)\cos x$

Poll

Which of the following forms you should try for finding a particular solution to $y'' - 3y' + 2y = e^{2x} \cos x$?

Choose the correct answer at pollev.com/ke1.

A. $y_p = a_1 e^{2x} \cos x + b_1 e^{2x} \sin x$

B. $y_p = x(a_1 e^{2x} \cos x + b_1 e^{2x} \sin x)$

C. $y_p = a_1 e^{2x} \cos x$

D. $y_p = a_1 e^{2x} \cos x + b_1 e^{x} \sin x$

Poll

Which of the following forms you should try for finding a particular solution to $y'' - 3y' + 2y = x^2 e^{2x}$?

Choose the correct answer at pollev.com/ke1.

A. $y_p = (a_2 x^2 + a_1 x + a_0) e^{2x}$

B. $y_p = x(a_2 x^2 + a_1 x + a_0) e^{2x}$

C. $y_p = (a_1 x + a_0) e^{2x}$

D. $y_p = (a_3 x^3 + a_2 x^2 + a_1 x + a_0) e^{2x}$

Right hand side $f = f_1 + f_2$

Suppose that the right hand side is a sum of the form $f(x) = f_1(x) + f_2(x)$ and that a particular solution corresponding to $f_1(x)$ is $y_{p,1}(x)$ while a particular solution corresponding to $f_2(x)$ is $y_{p,2}(x)$.

Then a particular solution corresponding to $f(x)$ is $y_p(x) = y_{p,1}(x) + y_{p,2}(x)$.

Example

Consider the equation $y'' - y = 2 - x^2 + 2xe^x$.

We have seen that a particular solution of $y'' - y = 2 - x^2$ is $y_{p,1} = x^2$,

and a particular solution of $y'' - y = 2xe^x$ is $y_{p,2} = \frac{1}{2}x(x-1)e^x$.

Therefore, a particular solution for the equation $y'' - y = 2 - x^2 + 2xe^x$ is

$$y_p = x^2 + \frac{1}{2}x(x-1)e^x.$$