Lecture 11: Higher-order Linear Differential Equations

MATH 303 ODE and Dynamical Systems

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Linear differential equations

Until now we have been considering second order linear differential equations with constant coefficients. Now we will consider a more general case.

An n-th order **linear differential equation** has the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

where the functions $a_0, a_1, \ldots, a_n, f$ are continuous in some interval $I \subseteq \mathbb{R}$.

Existence and uniqueness of solutions

Theorem. The initial value problem

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

with $y(x_0) = y_0$, $y'(x_0) = y_1$, ..., $y^{(n-1)}(x_0) = y_{n-1}$, where $y_0, \ldots, y_{n-1}$ are given real numbers has a unique solution $y(x)$ in the interval $I \subseteq \mathbb{R}$ provided that $a_0(x), \ldots, a_n(x), f(x)$ are continuous in I and $a_n(x) \neq 0$ for all $x \in I$.

Linear differential operators

Given a function $g(x)$ we denote by $L[g](x)$ the function

$$L[g](x) = a_n(x)g^{(n)}(x) + a_{n-1}g^{(n-1)}(x) + \cdots + a_1(x)g'(x) + a_0(x)g(x).$$

Using L the equation that we are considering can be written as $L[y] = f$.

Lemma. The operator L defined above is linear.

Sketch of the proof. Check that $L[g_1 + g_2] = L[g_1] + L[g_2]$ and $L[\lambda g] = \lambda L[g]$.

Solutions of homogeneous linear equations

Lemma. If $y_1, y_2, \ldots, y_m$ where $m \geq 1$ are solutions of the homogeneous linear equation $L[y] = 0$ then any linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$$

is also a solution of $L[y] = 0$.

Proof. We have

$$L[y] = L[c_1 y_1 + \cdots c_m y_m] = L[c_1 y_1] + \cdots + L[c_m y_m]$$

$$= c_1 L[y_1] + \cdots + c_m L[y_m] = 0 + \cdots + 0 = 0.$$

Question

Suppose that we are given the homogeneous initial value problem

$$L[y](x) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

with $y(x_0) = \gamma_0$, $y'(x_0) = \gamma_1$, ..., $y^{(n-1)}(x_0) = \gamma_{n-1}$ and we have found n solutions $y_1, \ldots, y_n$ of the equation $L[y] = 0$.

Can we find numbers $c_1, \ldots, c_n$ such that the solution to the given IVP is

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)?$$

This would mean that

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) + \cdots + c_n y_n(x_0) = \gamma_0$$

$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) + \cdots + c_n y_n'(x_0) = \gamma_1$$

$$\vdots$$

$$y^{(n)}(x_0) = c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = \gamma_{n-1}$$

We can view these equations as a linear system of equations with unknowns $c_1, c_2, \ldots, c_n$.

The same system in matrix form can be written as

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix}$$

We denote the $n \times n$ matrix appearing in the expression above by

$$M[y_1, \ldots, y_n](x_0)$$

and we call its determinant

$$W[y_1, \ldots, y_n](x_0) = \det M[y_1, \ldots, y_n](x_0)$$

the **Wronskian** of the solutions $y_1, \ldots, y_n$ at x_0.

From Linear Algebra we then know that the system on the previous slide has a unique solution $(c_1, \ldots, c_n)$ if and only if $W[y_1, \ldots, y_n](x_0) \neq 0$.

We can summarize this discussion in the following result.

Theorem. Suppose that $y_1, \ldots, y_n$ are solutions of the homogeneous equation

$$L[y](x) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

in an interval I and there is $x_0 \in I$ such that $W[y_1, \ldots, y_n](x_0) \neq 0$. Then every solution of $L[y] = 0$ in I can be written as

$$y(x) = c_1y_1(x) + \cdots + c_ny_n(x)$$

for a unique choice of $(c_1, \ldots, c_n)$.

Linearly independent functions

Definition. The functions $y_1, \ldots, y_m$ are **linearly dependent** in an interval I if there are $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, not all of them zero, such that

$$\lambda_1 y_1(x) + \cdots + \lambda_m y_m(x) = 0 \text{ for all } x \in I.$$

If the functions $y_1, \ldots, y_m$ do not satisfy the previous condition they are called **linearly independent**.

Linear in-/dependence and Wronskian

Theorem. Suppose that $y_1, \ldots, y_n$ are solutions of the homogeneous equation

$$L[y](x) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

in an interval I. Then the following statements are equivalent:

(a) The functions $y_1, \ldots, y_n$ are linearly dependent in I.

(b) $W[y_1, \ldots, y_n](x) = 0$ for all $x \in I$.

(c) There is $x_0 \in I$ such that $W[y_1, \ldots, y_n](x_0) = 0$.

Proof. We will first show that **(a) implies (b)**. That is, if the solutions $y_1, \ldots, y_n$ are linearly dependent in I then $W[y_1, \ldots, y_n](x) = 0$ for all $x \in I$.

Since $y_1, \ldots, y_n$ are assumed linearly dependent, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, not all of them zero, such that $\lambda_1 y_1(x) + \cdots + \lambda_n y_n(x) = 0$ for all $x \in I$. Taking derivatives of the last relation we have for every $x \in I$ that $\lambda_1 y_1^{(k)}(x) + \cdots + \lambda_n y_n^{(k)}(x) = 0$ for any $k \geq 0$. Therefore,

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the vector $[\lambda_1, \ldots, \lambda_n]^t$ is non-zero, we must have that the determinant of the $n \times n$ matrix, that is, the Wronskian $W[y_1, \ldots, y_n](x)$ equals zero for all $x \in I$.

It is obvious that **(b) implies (c)**.

We will finally show that **(c) implies (a)**. Suppose that there is $x_0 \in I$ such that $W[y_1, \ldots, y_n](x_0) = 0$ and consider the equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with unknowns $\lambda_1, \ldots, \lambda_n$.

Since the determinant of the $n \times n$ matrix, that is, $W[y_1, \ldots, y_n](x_0)$ equals zero, the equation must have a non-zero solution vector $[\lambda_1, \ldots, \lambda_n]^t$. Let

$$z(x) = \lambda_1 y_1(x) + \cdots + \lambda_n y_n(x).$$

Then

$$z^{(k)}(x_0) = \lambda_1 y_1^{(k)}(x_0) + \cdots + \lambda_n y_n^{(k)}(x_0) = 0 \text{ for all } k = 0,1,2,\ldots, n-1.$$

Therefore, z is a solution to the differential equation $L[y] = 0$ (as a linear combination of solutions) and $z(x_0) = z'(x_0) = \cdots = z^{(n-1)}(x_0) = 0$. From the existence and uniqueness theorem, the solution to this IVP has a unique solution in I. However, notice that the constant solution 0 also satisfies the same IVP in I. We conclude that $z(x) = 0$ for all $x \in I$, that is,

$$\lambda_1 y_1(x) + \cdots + \lambda_n y_n(x) = 0 \text{ for all } x \in I.$$

Linear in-/dependence and Wronskian

(Equivalent) Theorem. Suppose that $y_1, \ldots, y_n$ are solutions of the homogeneous equation

$$L[y](x) = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

in an interval I. Then the following statements are equivalent:

(a) The functions $y_1, \ldots, y_n$ are linearly independent in I.

(b) There is $x_0 \in I$ such that $W[y_1, \ldots, y_n](x_0) \neq 0$.

(c) $W[y_1, \ldots, y_n](x) \neq 0$ for all $x \in I$.

Example

For the homogeneous second order linear equation with constant coefficients we found the following three cases.

(a) Two distinct real roots $r_1 \neq r_2$. In this case two solutions are $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$. To check their linear independence through the Wronskian we compute

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = (r_1 - r_2)e^{(r_1+r_2)x} \neq 0.$$

(b) In the case of a double root r_0 the two solutions are $y_1 = e^{r_0 x}$, $y_2 = xe^{r_0 x}$. Then

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = e^{2r_0 x} \neq 0.$$

(c) In the case of complex conjugate roots $\alpha \pm i\beta$ we have $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$. Then

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \beta e^{2\alpha x} \neq 0.$$

Homogeneous linear equations with constant coefficients

Consider the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are constant and $a_n \neq 0$. The corresponding auxiliary equation, obtained by letting $y = e^{rx}$ is given by

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

This is a polynomial equation of degree n and therefore it has exactly n roots on the complex plane (counting multiplicity).

The structure of the roots will determine the form of the general solution. The general solution is a linear combination of solutions of the following form.

(a) For each real root $r \in \mathbb{R}$ that appears once we consider the solution e^{rx}.

(b) For each real root $r \in \mathbb{R}$ that appears with multiplicity k we consider the k solutions $e^{rx}, \ xe^{rx}, \ \ldots, \ x^{k-1}e^{rx}$.

(c) For a complex conjugate pair $\alpha \pm i\beta$ that appears once we consider the 2 solutions $e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$.

(d) For a complex conjugate pair $\alpha \pm i\beta$ that appears with multiplicity k we consider the $2k$ solutions $e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$, $xe^{\alpha x}\cos(\beta x)$, $xe^{\alpha x}\sin(\beta x)$, ..., $x^{k-1}e^{\alpha x}\cos(\beta x)$, $x^{k-1}e^{\alpha x}\sin(\beta x)$.

Example

Suppose that the auxiliary equation has degree 11 and roots

$$1,\ 1,\ 1,\ 2,\ -3,\ 1\pm i,\ 1\pm i,\ 2\pm i.$$

Then the general solution is

$$\begin{aligned} y = {} & c_1e^x + c_2xe^x + c_3x^2e^x + c_4e^{2x} + c_5e^{-3x} \\ & + c_6e^x\cos x + c_7e^x\sin x + c_8xe^x\cos x + c_9xe^x\sin x \\ & + c_{10}e^{2x}\cos x + c_{11}e^{2x}\sin x. \end{aligned}$$