Lecture 12: Introduction to the Phase Plane

MATH 303 ODE and Dynamical Systems

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Planar systems of differential equations

We now consider a system of differential equations of the form

$$\begin{aligned} x' &= f(x, y), \\ y' &= g(x, y)\,. \end{aligned}$$

Here, x, y are unknown functions of the independent variable t and $'$ means d/dt.

A solution of this system is a pair of functions $x(t), y(t)$ such that

$$\begin{aligned} x'(t) &= f(x(t), y(t)), \\ y'(t) &= g(x(t), y(t))\,. \end{aligned}$$

Initial value problems

An initial value problem has the form

$$x(t_0) = x_0,$$
$$y(t_0) = y_0\,.$$

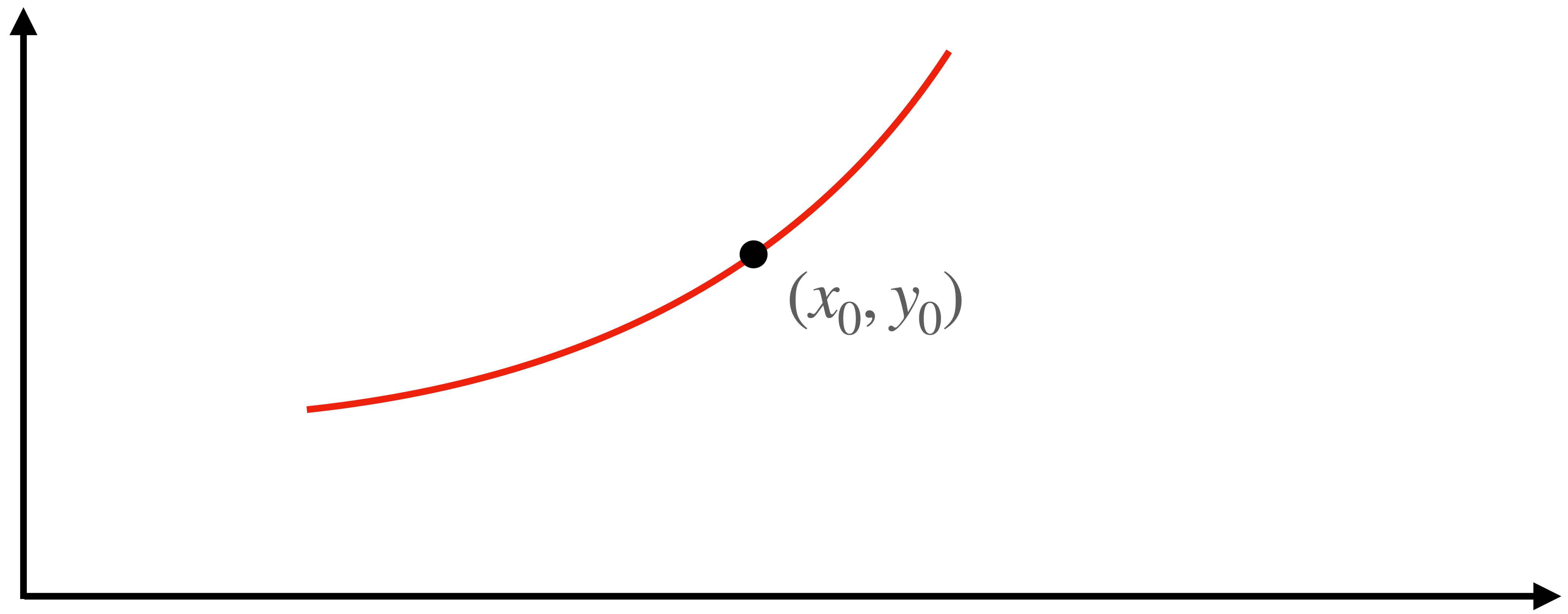
Because this system of equations is **autonomous** (that is, f, g do not explicitly depend on t) it is sufficient to consider initial conditions of the form

$$x(0) = x_0,$$
$$y(0) = y_0,$$

and this is what we will be doing from now on.

Representation of solutions

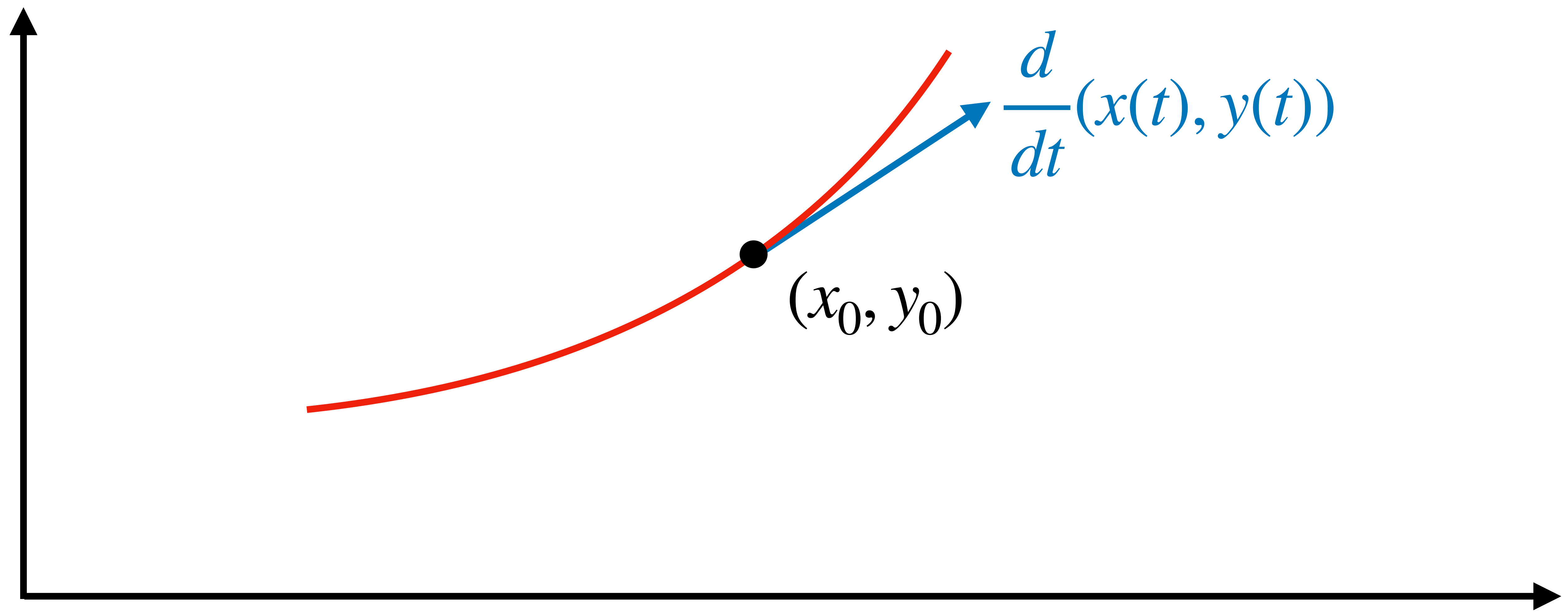
A solution to an IVP $x(0) = x_0,\ y(0) = y_0$ is a pair of functions $x(t),\ y(t)$. If we consider the xy-plane then these two functions describe a curve parameterized by t. As t changes the point $x(t),\ y(t)$ moves on the xy-plane and traces a curve, called the **integral curve**.



Geometric meaning of solutions

The **velocity vector** associated to the integral curve $(x(t), y(t))$ is given by

$$\frac{d}{dt}(x(t), y(t)) = (x'(t), y'(t)) = \big(f(x(t), y(t)), g(x(t), y(t))\big)$$



We conclude that the vector $(f(x, y), g(x, y))$ at a point (x, y) is the velocity vector for the integral curve passing through this point which geometrically implies that it is tangent to the interval curve at this point.

Therefore, solving the planar system means to find curves such that at each point (x, y) along the curves the tangent direction is given by the vector $(f(x, y), g(x, y))$.

Phase plane and phase portraits

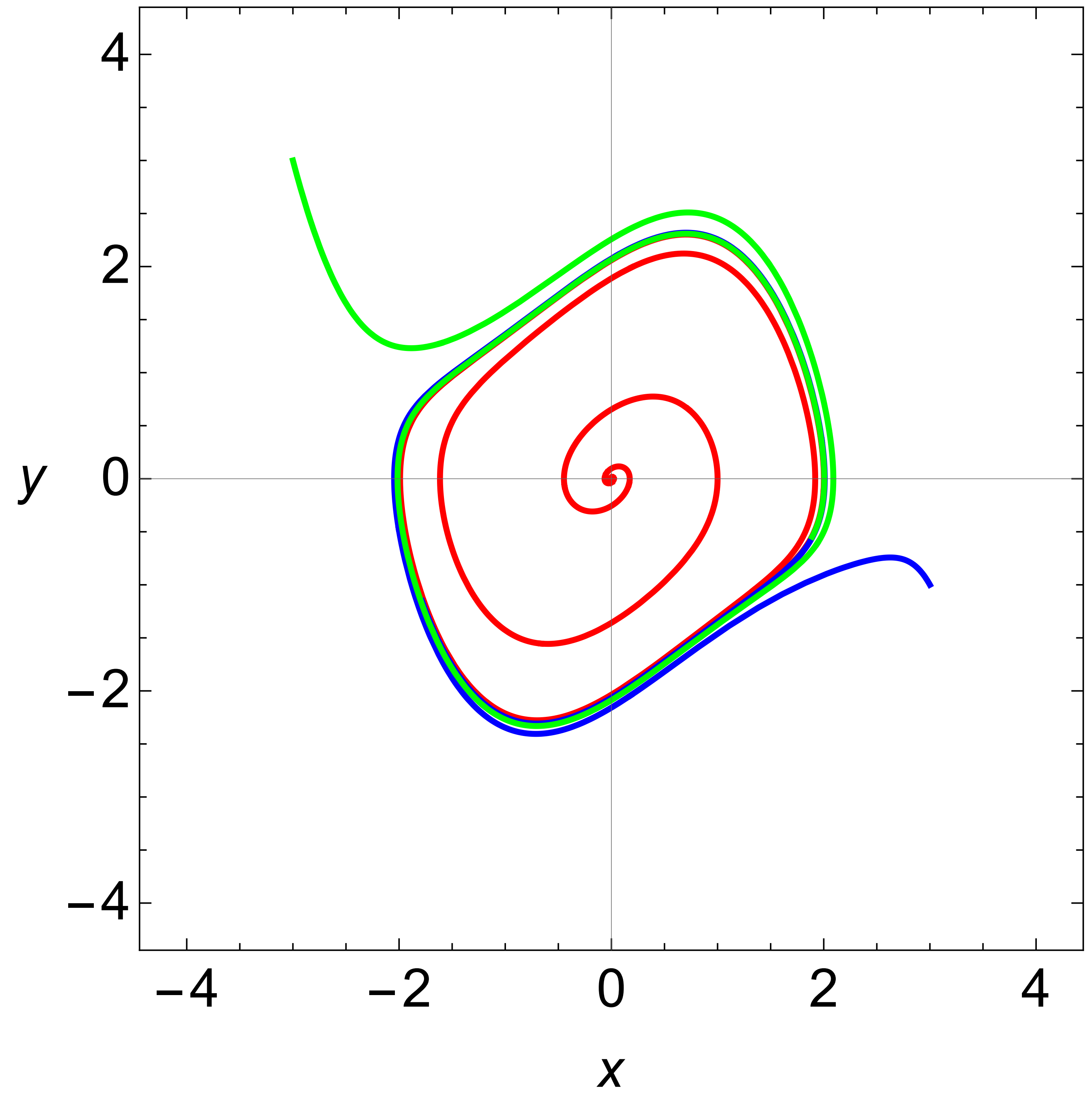
The xy-plane is called the **phase plane**.

Recall that for equations of the form $x' = f(x),\ x \in \mathbb{R}$ we talked about the phase line. If the space of unknown functions of t is not a line or a plane then we talk about the **phase space**, a term that also encompasses the notions of phase line and phase plane.

The **phase portrait** of the system $x' = f(x, y),\ y' = g(x, y)$ is the phase plane together with several solutions of the given system.

Example of phase portrait for the van der Pol system

$$x' = y,\ y' = \mu(1 - x^2)y - x.$$



Where planar systems come from?

1. Consider the second order equation $x'' = f(x, x')$ and let $y = x'$. Then we have the planar system

$$\begin{aligned} x' &= y, \\ y' &= f(x, y), \end{aligned}$$

which is equivalent to the original equation $x'' = f(x, x')$.

2. Consider the non-autonomous first order equation $x' = f(t, x)$ and the planar system

$$\begin{aligned} x' &= f(y, x), \\ y' &= 1. \end{aligned}$$

The two systems are equivalent.

3. Suppose that we have two interacting populations (rabbits x and foxes y). Then the rate of change of the population of rabbits will depend on how many rabbits are there but also how many foxes are there. The same is true for the population of foxes. Therefore, the rates of change of x, y can be written as

$$
\begin{aligned}
x' &= f(x, y), \\
y' &= g(x, y)\,.
\end{aligned}
$$

This is a planar system in general form.

Equilibria

Definition. A point $(x_e, y_e) \in \mathbb{R}^2$ is called an **equilibrium** of the system

$$\begin{aligned} x' &= f(x, y), \\ y' &= g(x, y), \end{aligned}$$

if $f(x_e, y_e) = g(x_e, y_e) = 0$.

The pair of functions $(x(t), y(t)) = (x_e, y_e)$ is a solution to the IVP $(x(0), y(0)) = (x_e, y_e)$ and is called an **equilibrium solution**.

Poll

Find the equilibria of the system $x' = y$, $y' = (1 - x^2)(y - x)$.

Choose the correct answer at pollev.com/ke1.

A. $1, -1, y = x$

B. $(0,0)$

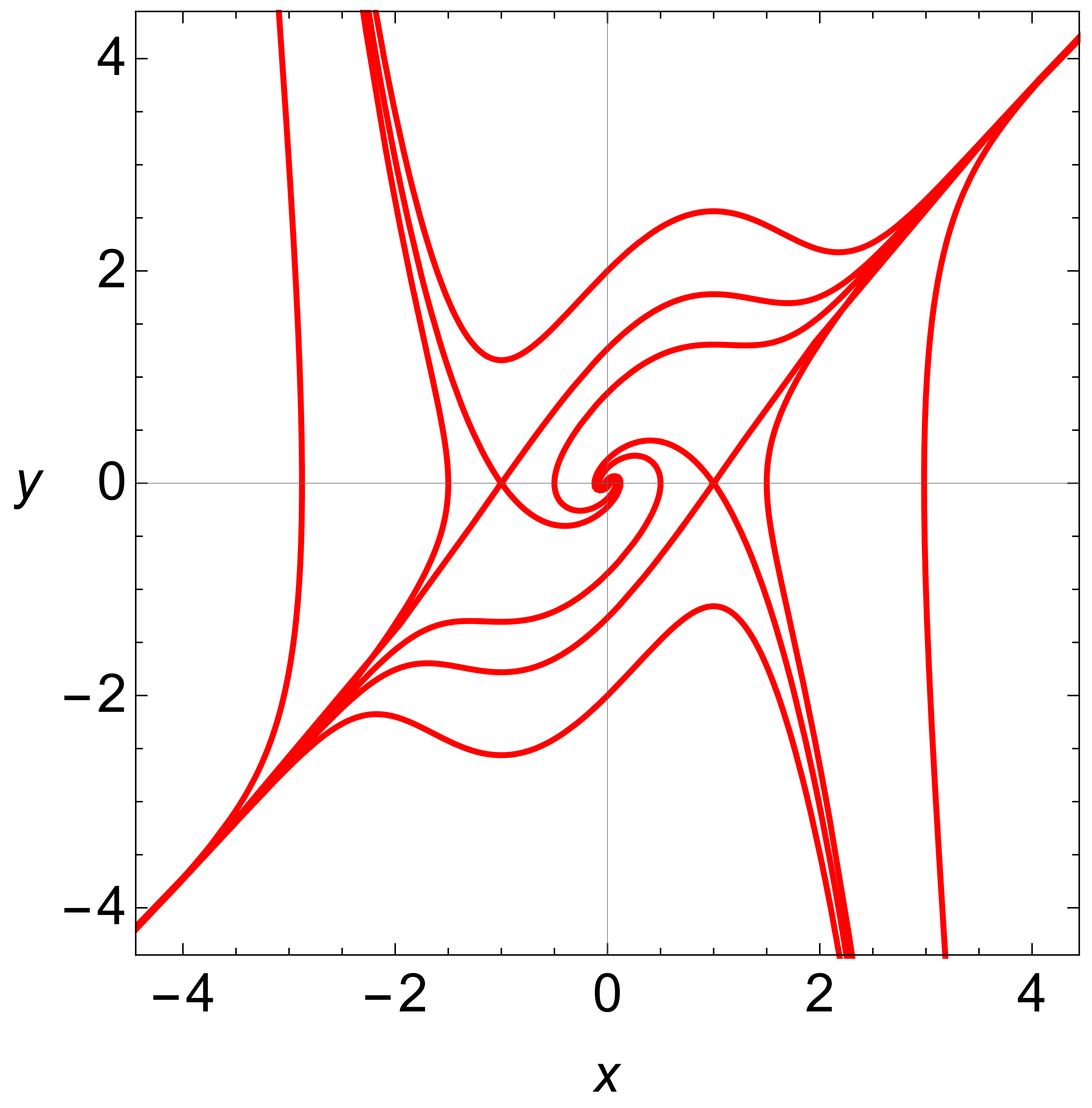
C. $(0, -1), (0,0), (0,1)$

D. $(0,1), (0, -1)$

Answer

There are three equilibria: $(-1,0)$, $(0,0)$, $(1,0)$.

The phase portrait for $x' = y$, $y' = (1 - x^2)(y - x)$ is shown below.



Example

Consider the planar system $x' = y,\ y' = -x$. We will see how to obtain the solutions to this system.

One way to approach this problem is to convert this system to a second order equation. We have

$$x'' = y' = -x.$$

Therefore we obtain the second order equation $x'' = -x$ which has general solution $x = c_1 \cos t + c_2 \sin t$.

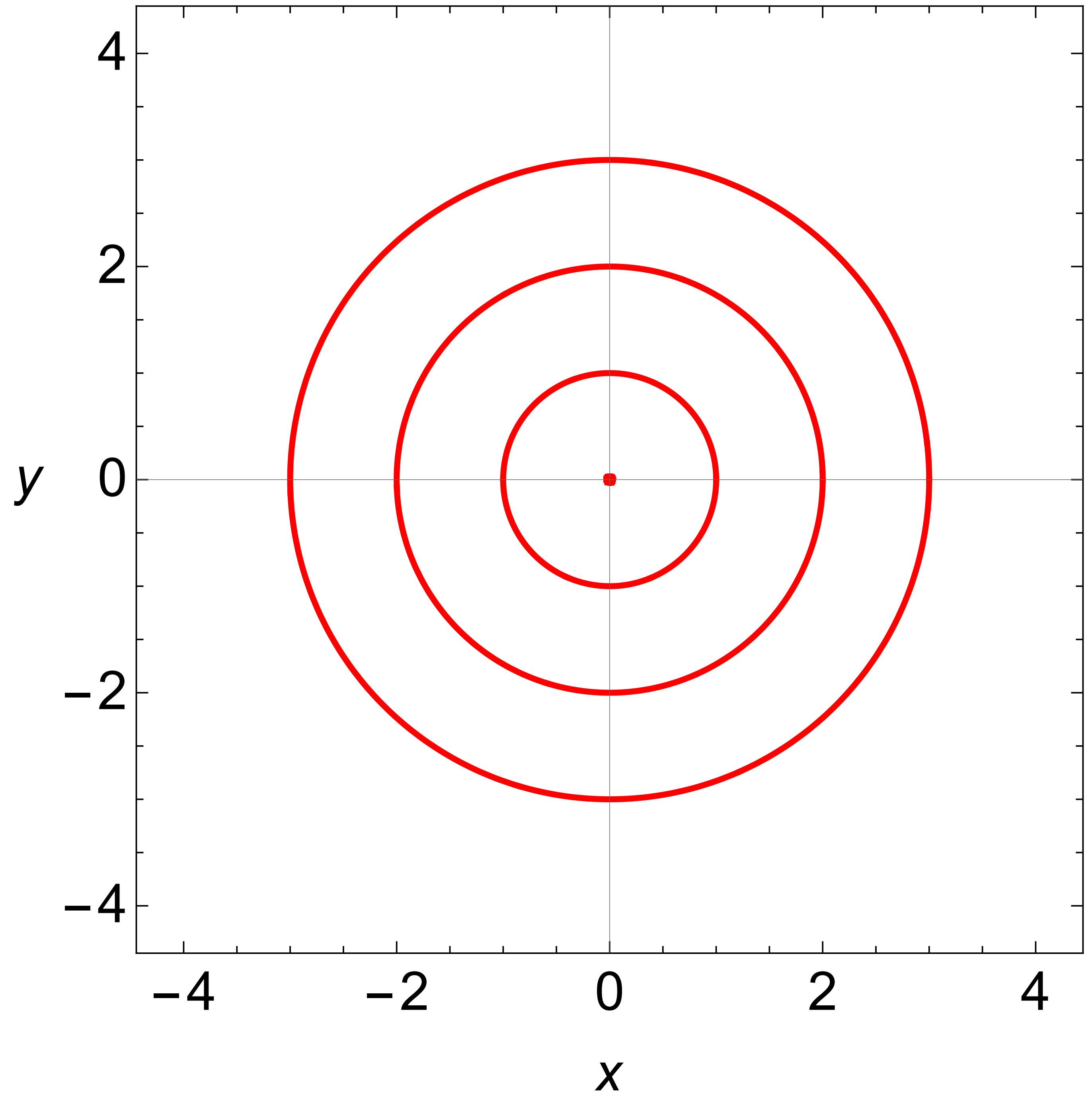
Then we also have $y = x' = -c_1 \sin t + c_2 \cos t$.

From the IVP $x(0) = x_0,\ y(0) = y_0$ we get $c_1 = x_0,\ c_2 = y_0$.

In matrix form the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

The 2×2 matrix in the expression above represents clockwise rotation on the xy-plane. Note that this is a complete solution to the IVP.



Another way to approach this problem is the following. We have the equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x.$$

Here x and y are functions of t. However, suppose that we can invert the function $x(t)$ and find $t(x)$, that is, express the time as a function of x. Then we can also define a function $y(x)$ by replacing t in $y(t)$ by $t(x)$.

The differential equation satisfied by $y(x)$ will then be

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}.$$

The last relation always holds. In our example we have $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

The equation $\frac{dy}{dx} = -\frac{x}{y}$ is a separable equation from where we find

$y\,dy = -x\,dx$. Integrating we get $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$. We can now solve for y and we find

$$y = \pm\sqrt{2C - x^2}.$$

In instead of solving for y we solve for the constant C we get

$$C = \frac{1}{2}(y^2 + x^2).$$

This means that the expression $x^2 + y^2$ is constant ($= 2C$) along any integral curve. Therefore, the motion takes place on a circle of radius $\sqrt{2C}$ on the phase plane. Quantities that are constant along integral curves are called **conserved quantities** or **integrals of motion**.

It is also possible to directly check that a function such as $x^2 + y^2$ is a conserved quantity.

We compute

$$\frac{d}{dt}(x^2 + y^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2x(y) + 2y(-x) = 0.$$

This again shows that $x^2 + y^2$ remains constant in time.

Poll

Find the conserved quantity for the system $x' = y,\ y' = x$.

Choose the correct answer at pollev.com/ke1.

A. $x^2 + y^2$

B. xy

C. $x^2 - y^2$

D. $x - y$

Remark

As we will see later, all systems of the form

$$x' = y,$$
$$y' = f(x),$$

can be treated in a similar way and we find that there is a function $E(x, y)$ that remains constant along integral curves.

Systems of this form typically come from problems in Physics, where x represents position, $y = x'$ represents velocity, $f(x)$ represents the exerted forces, and $E(x, y)$ represents the total mechanical energy (kinetic + potential).

We will see later in detail how to use the fact that $E(x, y)$ remains constant to draw the phase portrait when we discuss the "Energy method".