Lecture 14: Examples and Applications

MATH 303 ODE and Dynamical Systems

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Recall

Reduction of a planar system to a non-autonomous first order equation

As we saw in the previous lecture, given a system

$$x' = f(x, y), \; y' = g(x, y)$$

we can write

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'} = \frac{f(x, y)}{g(x, y)}.$$

The equation

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

is a first order equation and it may be solvable whereas the planar system is not directly solvable. Note that by solving the first order equation we can find the shape of the integral curves but not how fast we move along these curves. This is because we have eliminated t from the original system.

Biomathematics

Lotka-Volterra model

The Lotka-Volterra population model is given by

$$x' = Ax - Bxy, \quad y' = -Cy + Dxy,$$

where $A, B, C, D > 0$.

The equilibria are $(0{,}0)$ and $\left(\dfrac{C}{D}, \dfrac{B}{A}\right)$.

Then we have the reduced equation

$$\frac{dy}{dx} = \frac{-Cy + Dxy}{Ax - Bxy} = \frac{y(Dx - C)}{x(A - By)} = \frac{Dx - C}{x} \frac{y}{A - By}.$$

This is a separable equation with

$$\left(\frac{A}{y} - B\right) dy = \left(D - \frac{C}{x}\right) dx.$$

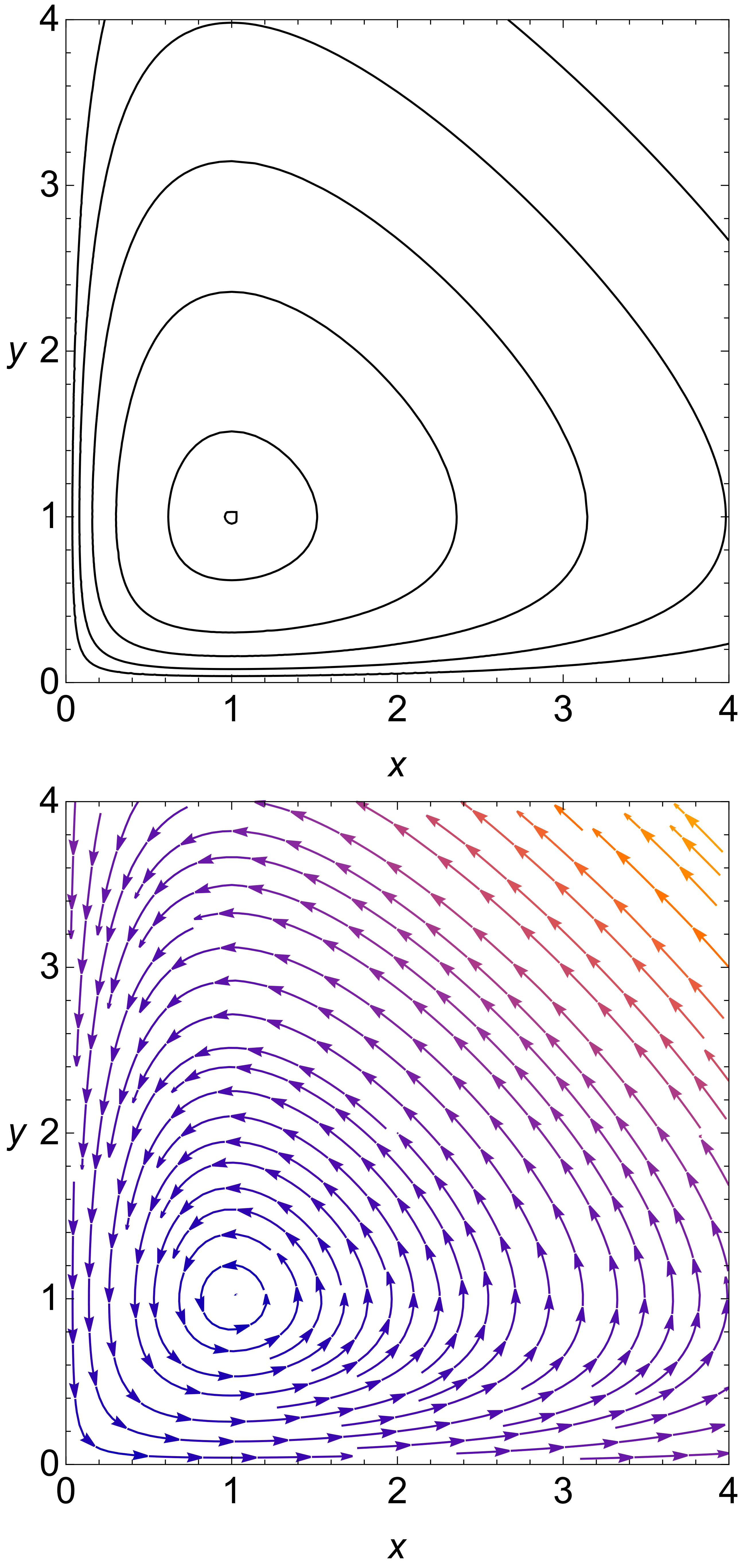
Integrating gives

$$A \ln y - By = Dx - C \ln x + K.$$

Solving for K gives the function

$$K = A \ln y + C \ln x - By - Dx,$$

which is constant along integral curves.



SIR model

Epidemic spread model.

$s(t)$: fraction of susceptible individuals;

$i(t)$: fraction of currently infected individuals;

$r(t)$: fraction of recovered individuals.

System

$$\frac{ds}{dt} = -a\,si, \quad \frac{di}{dt} = a\,si - k\,i, \quad \frac{dr}{dt} = k\,i.$$

We have $(s+i+r)' = 0$ which implies $s+i+r = 1$ (constant) and thus

$$r = 1 - s - i.$$

Therefore, we can consider only the planar system

$$s' = -a\,si, \quad i' = a\,si - k\,i,$$

defined for $s \geq 0,\ i \geq 0,\ s+i \leq 1$. We can introduce a fictitious time $\tau = at$ so that

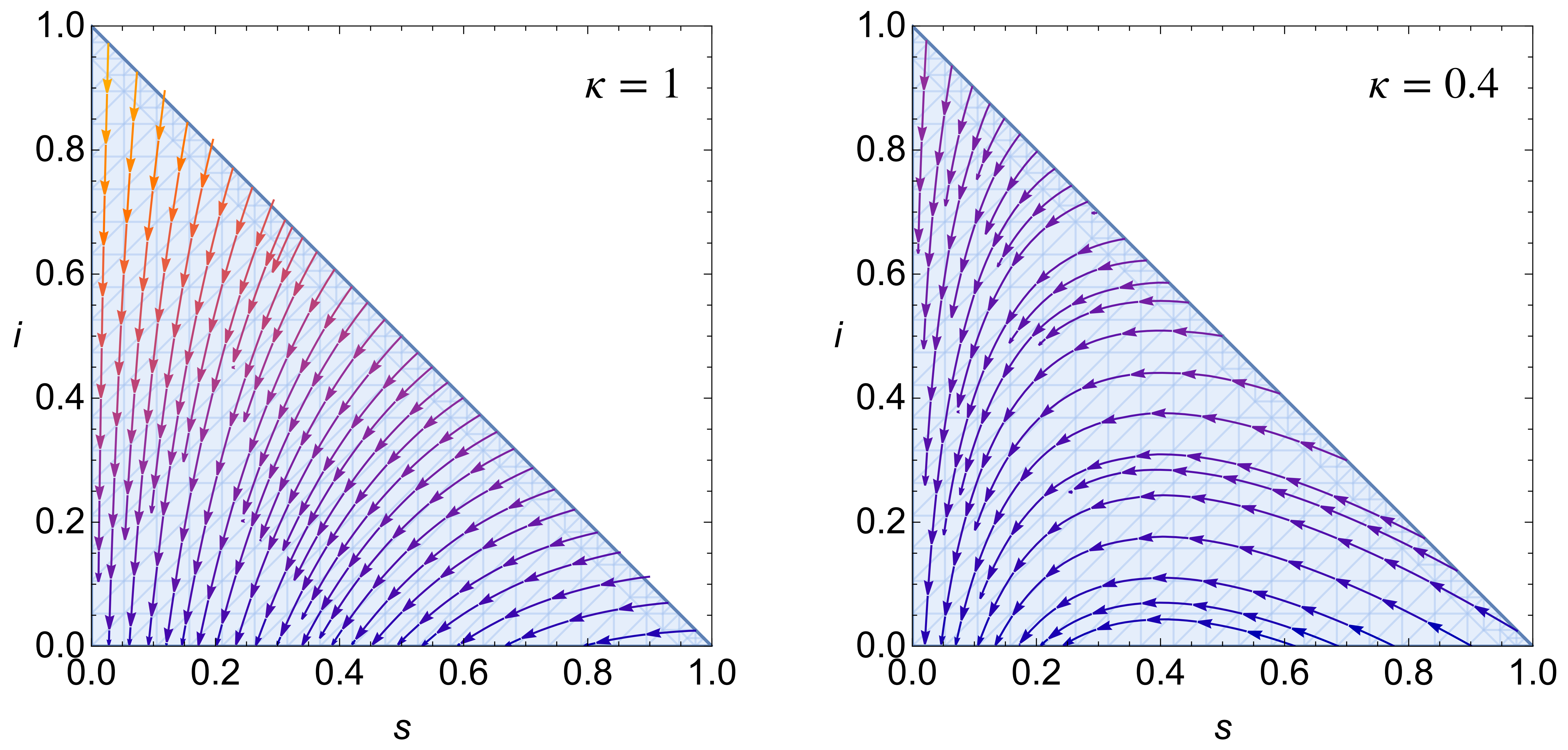
$$\frac{ds}{d\tau} = -si, \quad \frac{di}{d\tau} = si - \kappa i,$$

where $\kappa = k/a$. This shows that the only essential parameter if we ignore the time scale of the epidemic spread (this is allowed in mathematics but not in the real world) is κ.

The related equation is

$$\frac{di}{ds} = \frac{\kappa}{s} - 1,$$

and can be trivially integrated to $i(s) = \kappa \ln s - s + C$.



Physics

Mass on a spring

An equation describing the motion of a mass $m > 0$ attached on a spring with spring constant $k > 0$ is given by

$$mx'' + bx' + kx = 0,$$

where the term $bx', b \geq 0$ represents friction (energy dissipation). Writing this as a planar system by letting $x' = y$ we get

$$x' = y, \quad y' = -\frac{1}{m}(by + kx) = -2\beta y - \kappa x,$$

where $2\beta = b/m \geq 0$ and $\kappa = k/m$.

Then the related equation is

$$\frac{dy}{dx} = -\frac{2\beta y + \kappa x}{y} = -2\beta - \kappa\frac{x}{y}.$$

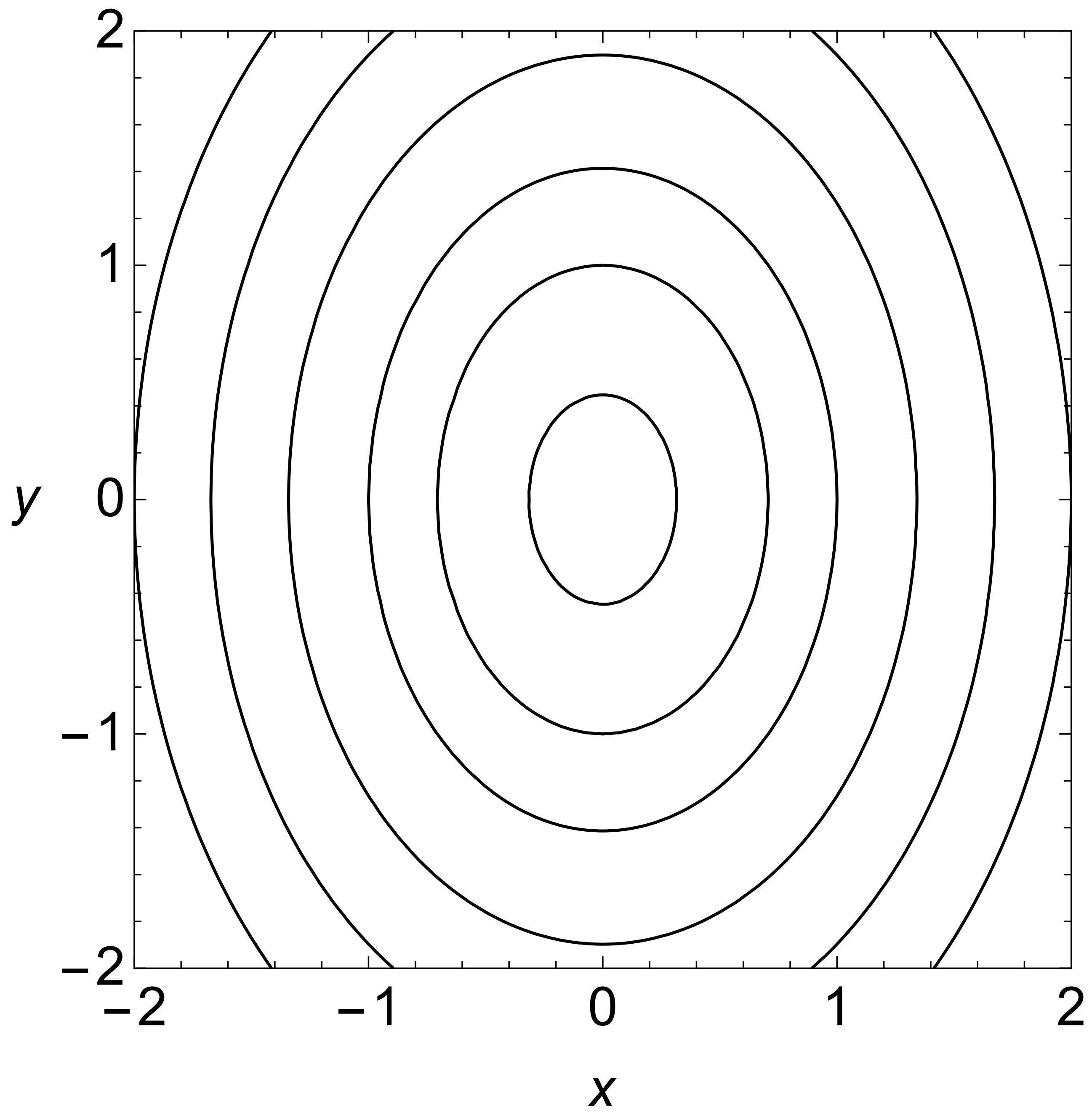
In the case $\beta = 0$ (no energy dissipation) we have

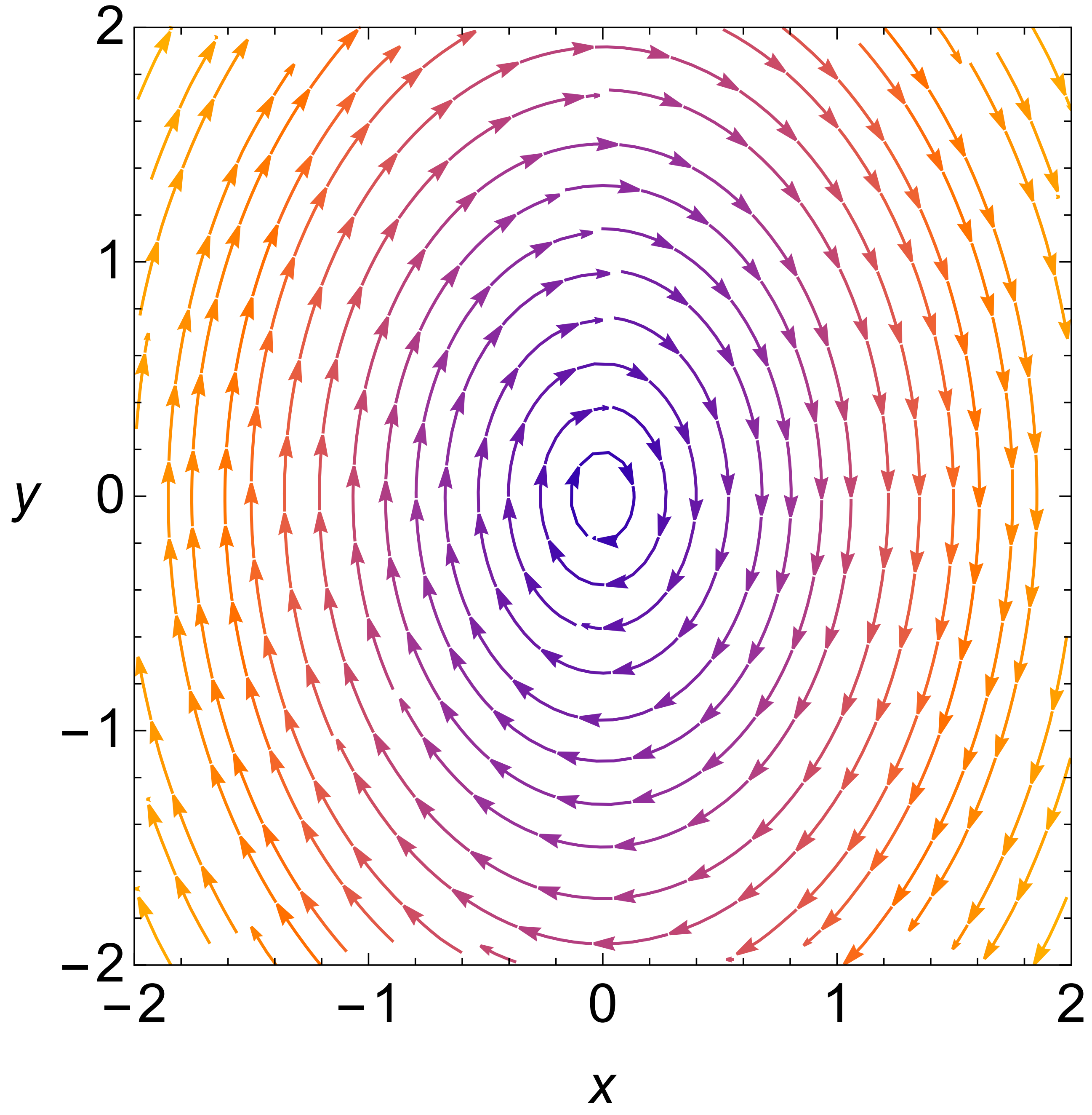
$$\frac{dy}{dx} = -\kappa\frac{x}{y},$$

which gives

$$\frac{1}{2}y^2 + \frac{\kappa}{2}x^2 = C.$$

Therefore, in this case the integral curves are ellipses. Note that if we use $\sqrt{\kappa}x, y$ as coordinates then the ellipses become circles.





When $\beta > 0$ it is possible to solve the resulting equation for dy/dx but the solutions are complicated and difficult to analyze. It is more productive to go back to the original second order equation

$$x'' + 2\beta x' + \kappa x = 0.$$

The solutions will depend on the sign of the discriminant $4(\beta^2 - \kappa)$.

If $0 < \beta^2 < \kappa$ then the general solution is

$$x = e^{-\beta t}\left(c_1 \cos(\omega_\beta t) + c_2 \sin(\omega_\beta t)\right) = Ae^{-\beta t}\cos(\omega_\beta t - \phi)$$

where $\omega_\beta = \sqrt{\kappa - \beta^2}$. This expression represents an oscillation with period $2\pi/\omega_\beta$ where the amplitude $Ae^{-\beta t}$ dies off as time goes by.

Then

$$y = x' = -A\beta e^{-\beta t}\cos(\omega_\beta t - \phi) - Ae^{-\beta t}\sin(\omega_\beta t - \phi) = -\beta x - Ae^{-\beta t}\sin(\omega_\beta t - \phi)$$

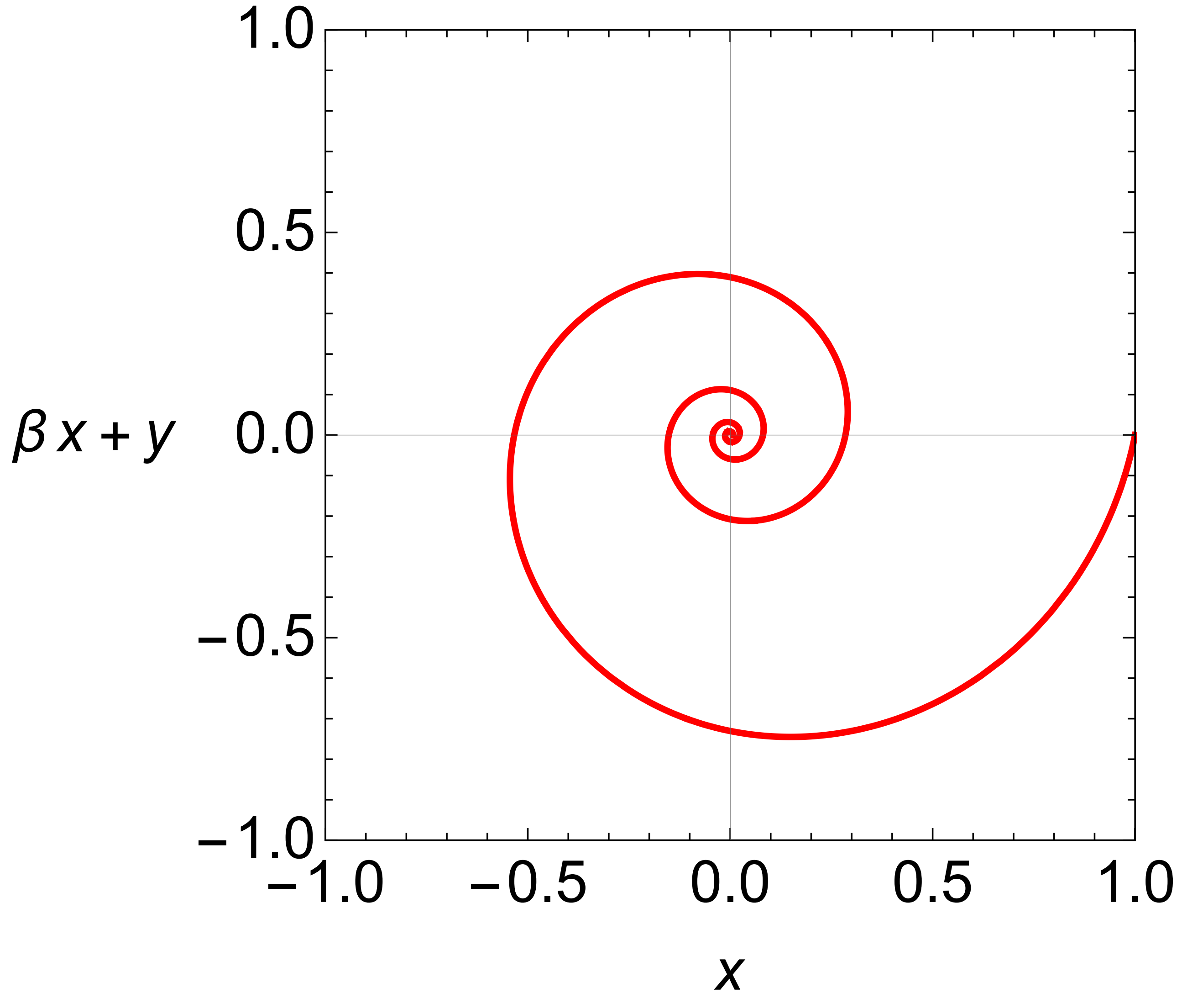
If we use as coordinates $(x, y + \beta x)$ then we have

$$(x, y + \beta x) = Ae^{-\beta t}(\cos(\omega_\beta t - \phi), -\sin(\omega_\beta t - \phi)).$$

In the expression

$$(x, y + \beta x) = Ae^{-\beta t}(\cos(\omega_\beta t - \phi), -\sin(\omega_\beta t - \phi))$$

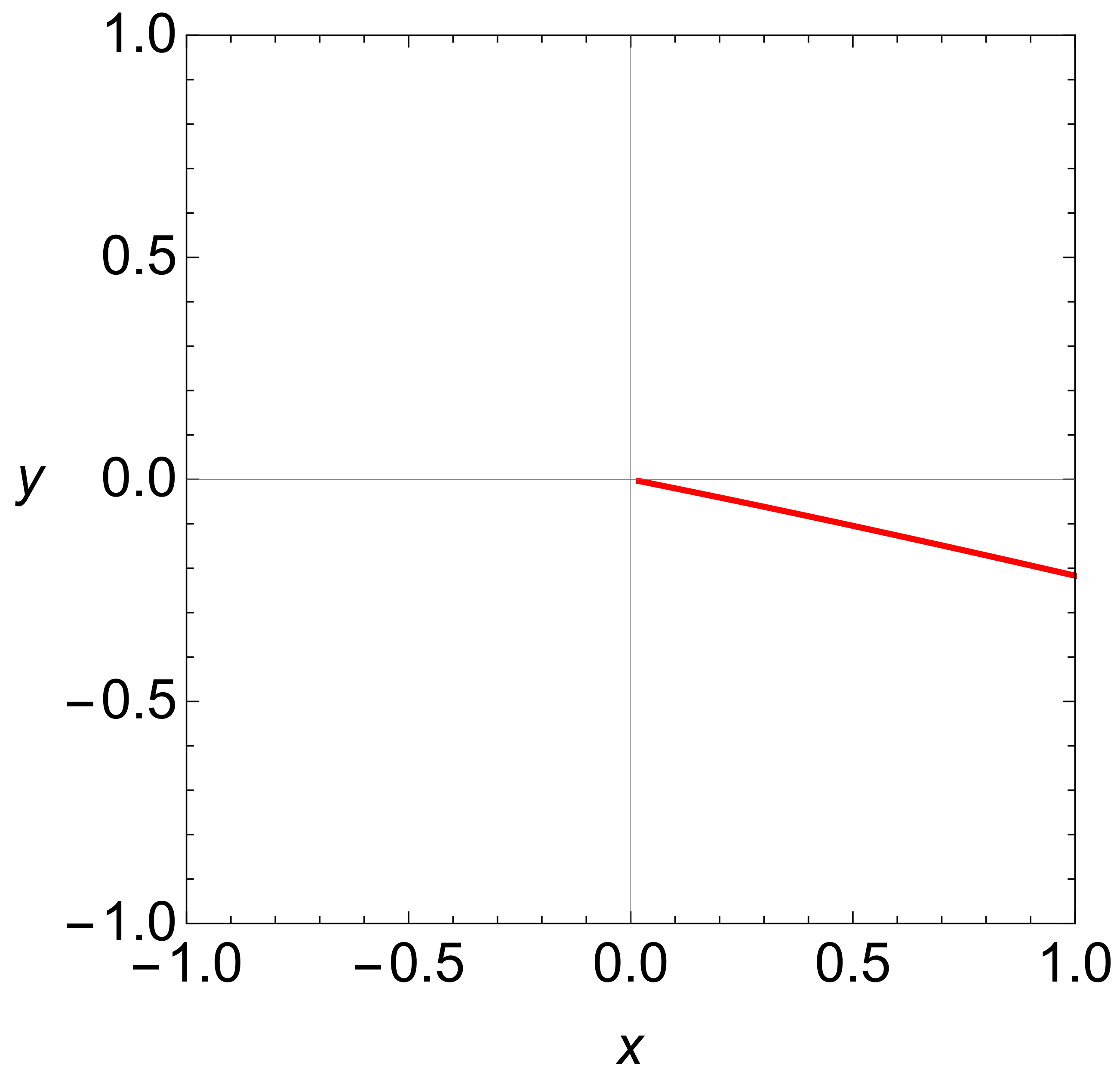
the terms in the brackets represent clockwise rotation with period $2\pi/\omega_\beta$ in the $(x, y + \beta x)$ plane while the exponential $e^{-\beta t}$ shows that the solution curve will spiral toward the origin as it rotates in the clockwise direction around it. This case is called **underdamped**.



If $\beta^2 > \kappa$ then the general solution is

$$x = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t},$$

where $\lambda_{1,2} = \beta \pm \sqrt{\beta^2 - \kappa} > 0.$



Then $y = x' = -\lambda_1 c_1 e^{-\lambda_1 t} - \lambda_2 c_2 e^{-\lambda_2 t}$. Here we observe that there is no rotation around the origin. This case is called **overdamped**.

Lastly, consider the energy

$$E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}\kappa x^2,$$

which is a conserved quantity for $\beta = 0$. In the general case we find

$$\frac{dE}{dt} = yy' + \kappa xx' = -\beta y^2 - \kappa xy + \kappa xy = -\beta y^2 \leq 0.$$

This implies that the energy is a decreasing function when $\beta > 0$ and therefore the distance from the origin decreases, as we also found through the explicit solutions to the second order equation.

Simple pendulum

An equation describing the motion of a pendulum of mass $m > 0$ attached to a string of length ℓ in the gravitational field g is given by

$$\theta'' = -\frac{g}{\ell}\sin\theta,$$

where θ represents the angle between the pendulum and the vertical direction. Writing this as a planar system by letting $x = \theta,\ y = \theta'$ we get

$$x' = y, \quad y' = -\gamma \sin x,$$

where $\gamma = g/\ell$.

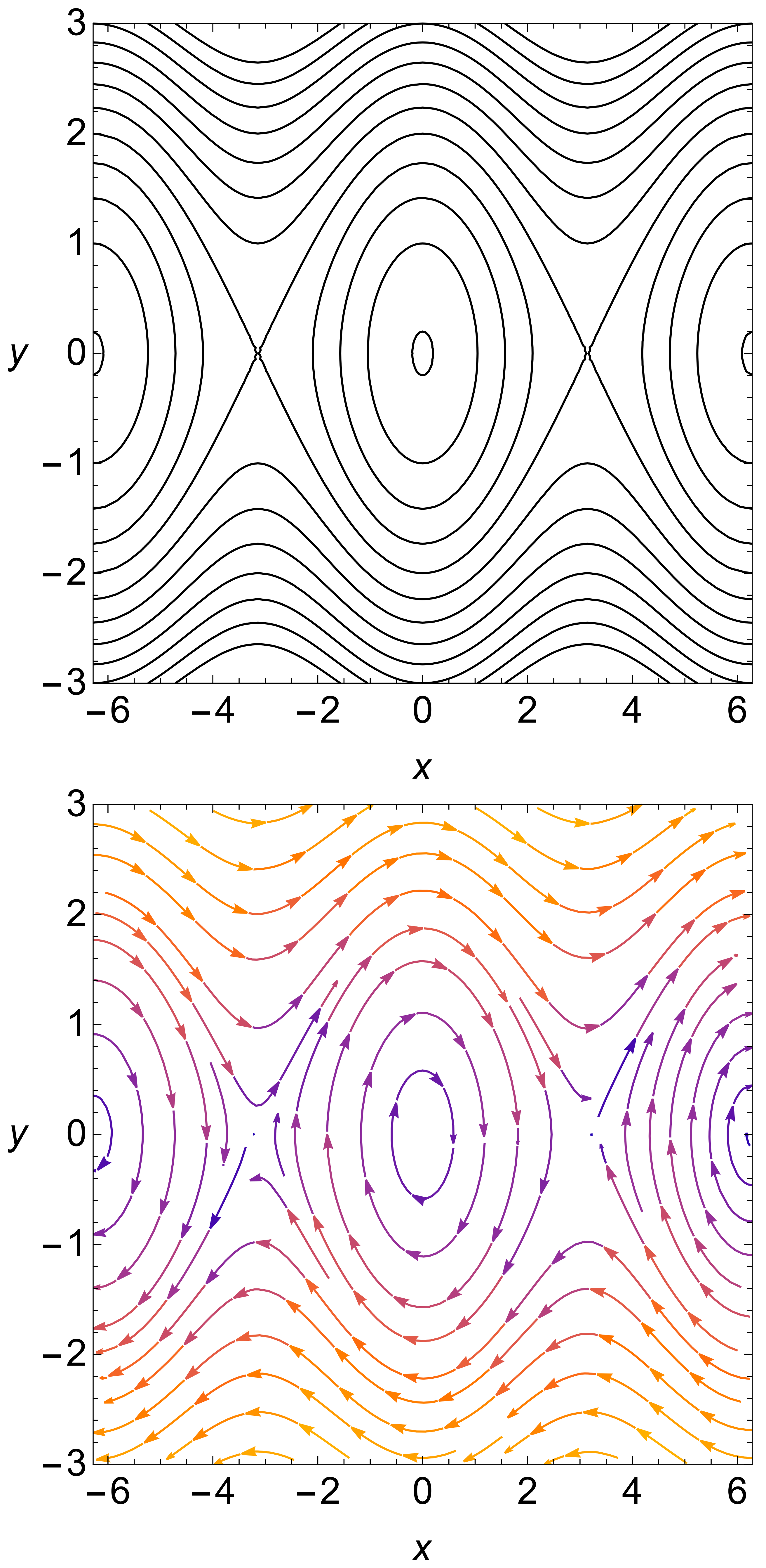
Then the related equation is

$$\frac{dy}{dx} = -\frac{\gamma \sin x}{y}.$$

Separation of variables and integration gives

$$\frac{1}{2}y^2 - \gamma \cos x = C.$$

The integral curves in this case are shown at the right.



Rigid body

The equations for the rate of change of the components of the angular momentum for a free rigid body are

$$x' = yz, \quad y' = -2xz, \quad z' = xy.$$

This looks like a system in $\mathbb{R}^3$. However, the Physics of the problem tells us that the total angular momentum $x^2 + y^2 + z^2$ must be conserved.

We can check that

$$(x^2 + y^2 + z^2)' = 2xx' + 2yy' + 2zz' = 2xyz - 4xyz + 2xyz = 0.$$

This means that each sphere $x^2 + y^2 + z^2 = L^2$ is invariant.

Are there more quadratic conserved quantities of the form $ax^2 + by^2 + cz^2$?

If yes, then we would have

$$2axx' + 2byy' + 2czz' = 2axyz - 4bxyz + 2cxyz = 2(a - 2b + c)xyz = 0,$$

that is, $2b = a + c$. From here we also get $2(b - c) = a - c$.

One solution is clearly $a = b = c$ giving the conserved quantity $L^2 = x^2 + y^2 + z^2$. Another solution is $a = 2b,\ c = 0$ giving the conserved quantity $F = 2x^2 + y^2$.

For any other such conserved quantity we have

$$\begin{aligned} ax^2 + by^2 + cz^2 &= (a - c)x^2 + (b - c)y^2 + c(x^2 + y^2 + z^2) \\ &= (b - c)F + cL^2 . \end{aligned}$$

Since L^2 and F are conserved quantities, the integral curves lie on the intersections of their level sets.