# Lecture 15: Dynamical Systems and Poincaré Maps

## MATH 303 ODE and Dynamical Systems

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# Non-autonomous planar systems

We now consider non-autonomous planar systems of the form

$$x' = f(x, y, t),$$
$$y' = g(x, y, t),$$

where the functions $f, g$ are periodic in $t$ with the same period. Here we will assume that the period is $2\pi$ but the same arguments work for any other period $T$.

# A fundamental property

**Lemma.** Consider the system of differential equations

$$x' = f(x, y, t), \quad y' = g(x, y, t)$$

where $f, g$ are periodic in $t$ with period $2\pi$.

Let $(x_1(t), y_1(t))$ be the solution to the initial value problem $(x(0), y(0)) = (x_0, y_0)$.

Then the solution to the initial value problem $(x(2k\pi), y(2k\pi)) = (x_0, y_0),\ k \in \mathbb{Z}$ is given by $(x_2(t), y_2(t)) = (x_1(t - 2k\pi), y_1(t - 2k\pi))$.

**Proof.** The proof is a copy of the corresponding proof for the one-dimensional case.

We have $x_2(2k\pi) = x_1(2k\pi - 2k\pi) = x_1(0) = x_0$ so $x_2(t)$ indeed satisfies the given initial condition. Similarly, $y_2(t)$ satisfies the initial condition $y_2(2k\pi) = y_0$.

Moreover, $x_2(t)$ and $y_2(t)$ satisfy the system of differential equations since

$$x_2'(t) = \frac{d}{dt}[x_1(t - 2k\pi)] = x_1'(t - 2k\pi)\frac{d(t - 2k\pi)}{dt} = x_1'(t - 2k\pi)$$

$$= f(x_1(t - 2k\pi), y_1(t - 2k\pi), t - 2k\pi) = f(x_2(t), y_2(t), t - 2k\pi)$$

$$= f(x_2(t), y_2(t), t).$$

A similar computation shows that $y_2'(t) = g(x_2(t), y_2(t), t)$.

# Poincaré map for non-autonomous periodic systems

# Poincaré map

**Definition.** Consider the system of differential equations

$$x' = f(x, y, t), \quad y' = g(x, y, t)$$

where $f, g$ are periodic in $t$ with period $2\pi$. Then the associated **Poincaré map** $P : \mathbb{R}^2 \to \mathbb{R}^2$ sends the point $(x_0, y_0) \in \mathbb{R}^2$ to the point $P(x_0, y_0) = (x(2\pi), y(2\pi))$ where $(x(t), y(t))$ is the solution to the given system that satisfies the initial condition $(x(0), y(0)) = (x_0, y_0)$.

# Remark

With the notation of the previous definition we have

$$P^k(x_0, y_0) = (x(2k\pi), y(2k\pi)).$$

This means that the successive points $(x_k, y_k) = P^k(x_0, y_0)$ are the points that we would get if we would observe the full, continuous solution curve $(x(t), y(t))$ only at times $t = 2k\pi, k \in \mathbb{Z}$.

You can think of this as having a system that evolves in a dark room and that someone turns on the light periodically and records the system's state. Because of this interpretation, another name for this Poincaré map is **stroboscopic (频闪) map**.

# Mass on spring with external forcing

Consider the system

$$x' = y, \quad y' = -\omega^2 x + F\cos t.$$

For $\omega^2 \neq 1$ the general solution for $x$ is

$$x = A\cos(\omega t - \phi) + \frac{F}{\omega^2 - 1}\cos t,$$

while we also get

$$y = x' = -\omega A\sin(\omega t - \phi) - \frac{F}{\omega^2 - 1}\sin t.$$

**Remark.** The solution for $x$ can be obtained by converting the system to a second order equation for $x$ and using the methods for solving second order linear non-homogeneous equations with constant coefficients.

Considering the initial value problem $(x(0), y(0)) = (x_0, y_0)$ we find

$$x_0 = A\cos(\phi) + \frac{F}{\omega^2 - 1}, \quad y_0 = \omega A \sin(\phi).$$

Therefore, if we let $(x_1, y_1) = P(x_0, y_0) = (x(2\pi), y(2\pi))$, we find

$$x_1 = A\cos(2\pi\omega - \phi) + \frac{F}{\omega^2 - 1}, \quad y_1 = -\omega A \sin(2\pi\omega - \phi).$$

To better understand the dynamics of the map $P$ we define

$$u = x - \frac{F}{\omega^2 - 1}, \quad v = \frac{y}{\omega}.$$

Then

$$u_0 = A\cos(\phi),\ v_0 = A\sin(\phi),\ u_1 = A\cos(\phi - 2\pi\omega),\ v_1 = A\sin(\phi - 2\pi\omega).$$

To write an expression for $P$ in terms of the coordinates $(u, v)$ we use the expressions

$$u_0 = A\cos(\phi),\ v_0 = A\sin(\phi),\ u_1 = A\cos(\phi - 2\pi\omega),\ v_1 = A\sin(\phi - 2\pi\omega)$$

to get

$$u_1 = A\cos\phi\cos(2\pi\omega) + A\sin\phi\sin(2\pi\omega) = \cos(2\pi\omega)u_0 + \sin(2\pi\omega)v_0,$$

$$v_1 = A\sin\phi\cos(2\pi\omega) - A\cos\phi\sin(2\pi\omega) = -\sin(2\pi\omega)u_0 + \cos(2\pi\omega)v_0.$$

Therefore,

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \cos(2\pi\omega) & \sin(2\pi\omega) \\ -\sin(2\pi\omega) & \cos(2\pi\omega) \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

It is even more convenient to define polar coordinates $(r, \theta)$ on the $(u, v)$ plane. Then the relations

$$u_0 = A\cos(\phi),\ v_0 = A\sin(\phi),\ u_1 = A\cos(\phi - 2\pi\omega),\ v_1 = A\sin(\phi - 2\pi\omega)$$

become in polar coordinates

$$r_0 = A,\ \theta_0 = \phi,\ r_1 = A,\ \theta_1 = \phi - 2\pi\omega.$$

Therefore, in coordinates $(r, \theta)$ the map $P$ becomes

$$r_1 = r_0, \quad \theta_1 = \theta_0 - 2\pi\omega.$$

This means that in the $(u, v)$ plane the map is a clockwise rotation by angle $2\pi\omega$.

**Remark.** The only **fixed point** of the Poincaré map in this example is $(u, v) = (0,0)$ or $(x, y) = (F/(\omega^2 - 1),0)$.

Each fixed point corresponds to a periodic orbit of period $2\pi$ and therefore, the system has only one such orbit.

Suppose now that $\omega = p/q$ where $p, q \in \mathbb{Z}, p, q \geq 1$ and $p, q$ have no common factors. Then every point $(x, y)$ is a **periodic point** of the Poincaré map with period $q$, since

$$r_q = r_0, \quad \theta_q = \phi - 2\pi\omega q = \phi - 2\pi p \equiv \phi.$$

Such periodic points of period $q$ correspond to periodic solution of the non-autonomous system with period $2\pi q$.

If $\omega$ is irrational then the Poincaré map has no periodic points (except the fixed point which is a periodic point of period 1).

# Forced Duffing equation

The **forced Duffing equation** is given by

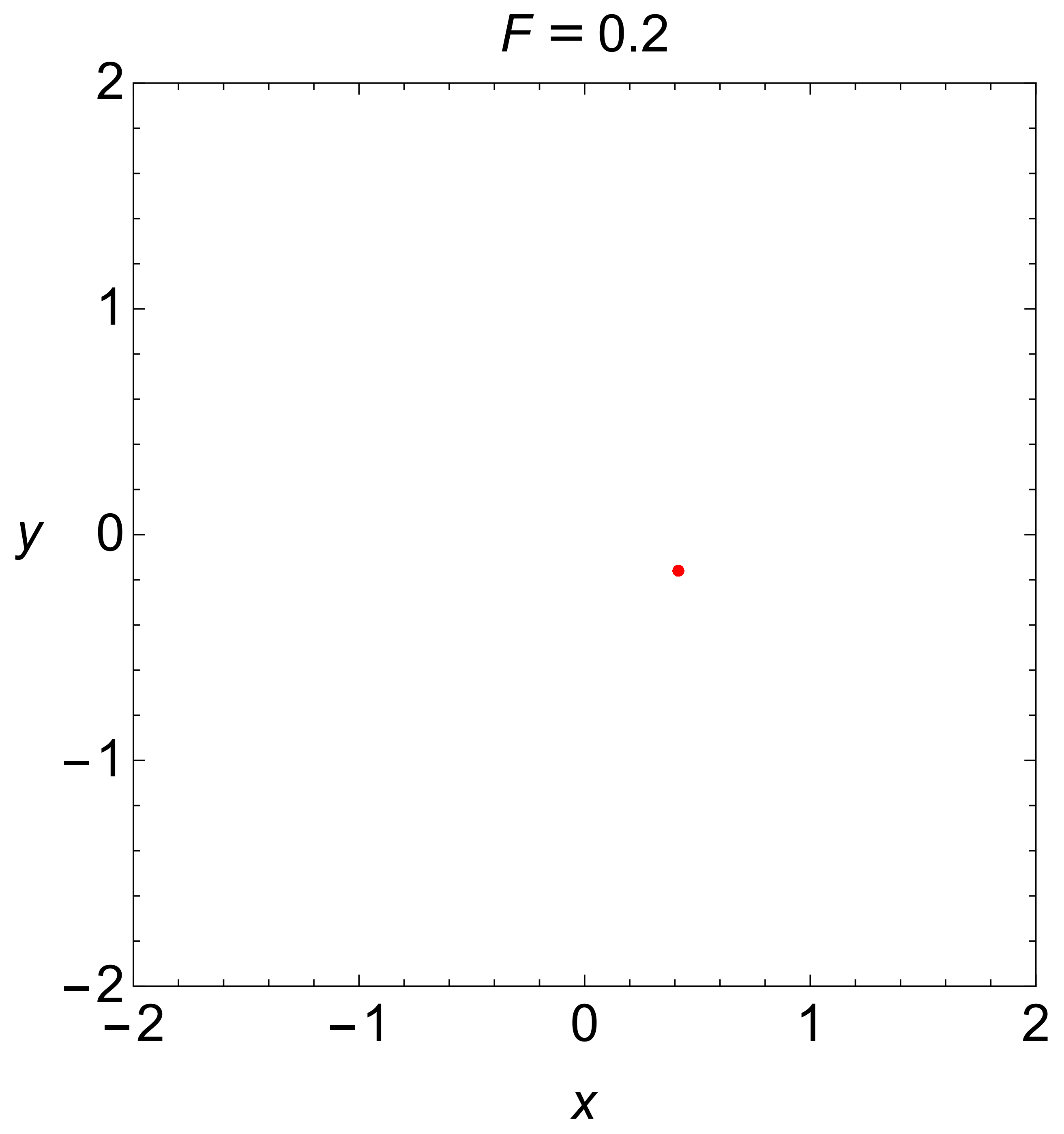
$$x' = y, \quad y' = -by + x - x^3 + F\sin(\gamma t).$$

Here, the period of the time-dependent term is $2\pi/\gamma$. For this reason we define the Poincaré map using $2\pi/\gamma$ instead of $2\pi$.

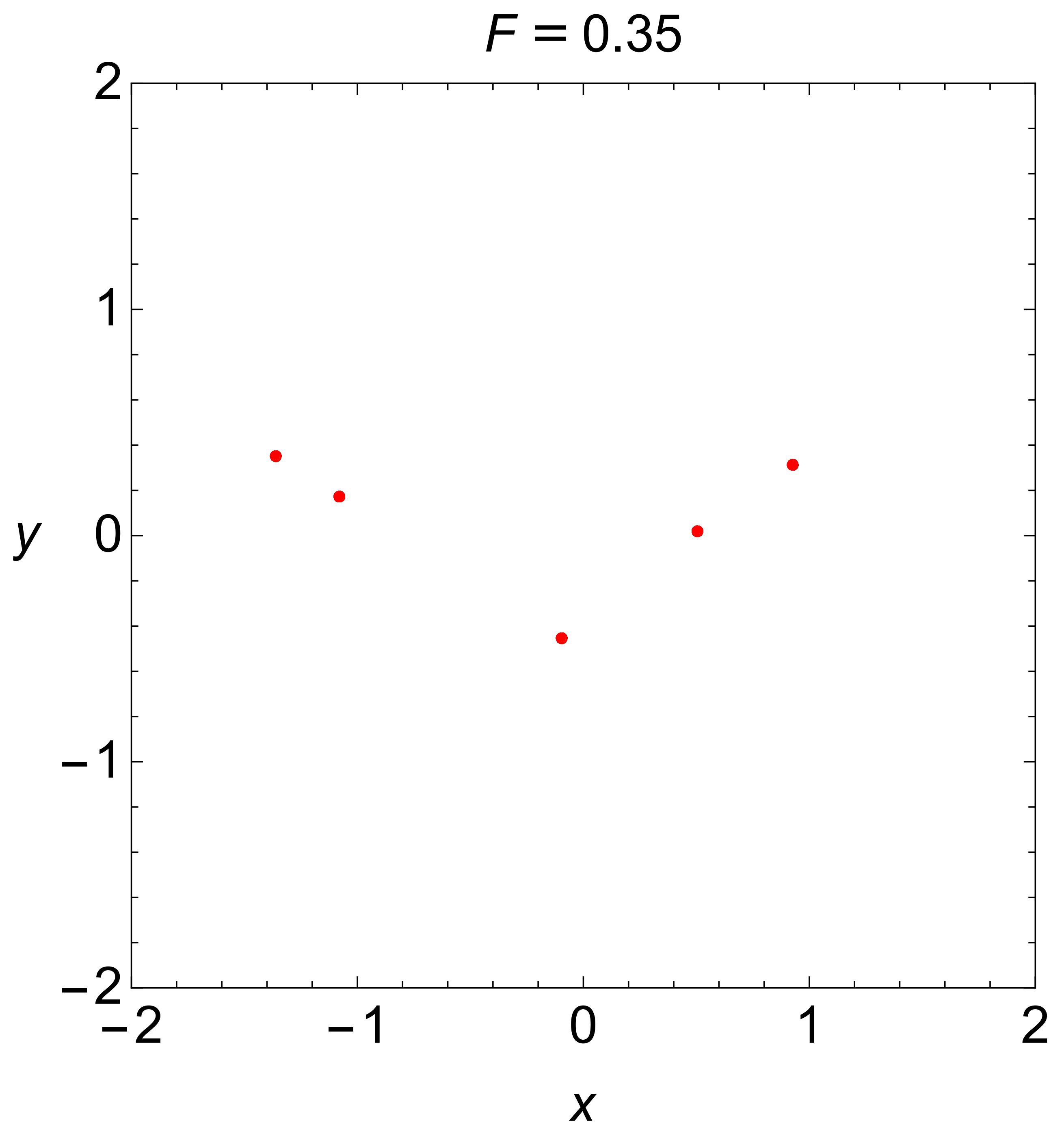
The following code computes the Poincaré map.

```
P[b_, F_, γ_][{x0_, y0_}] :=
 NDSolveValue[x'[t] == y[t] && y'[t] == -b y[t] + x[t] - x[t]^3 + F Sin[γ t] &&
   x[0] == x0 && y[0] == y0, {x[2 Pi / γ], y[2 Pi / γ]}, {t, 0, 2 Pi / γ}]
```

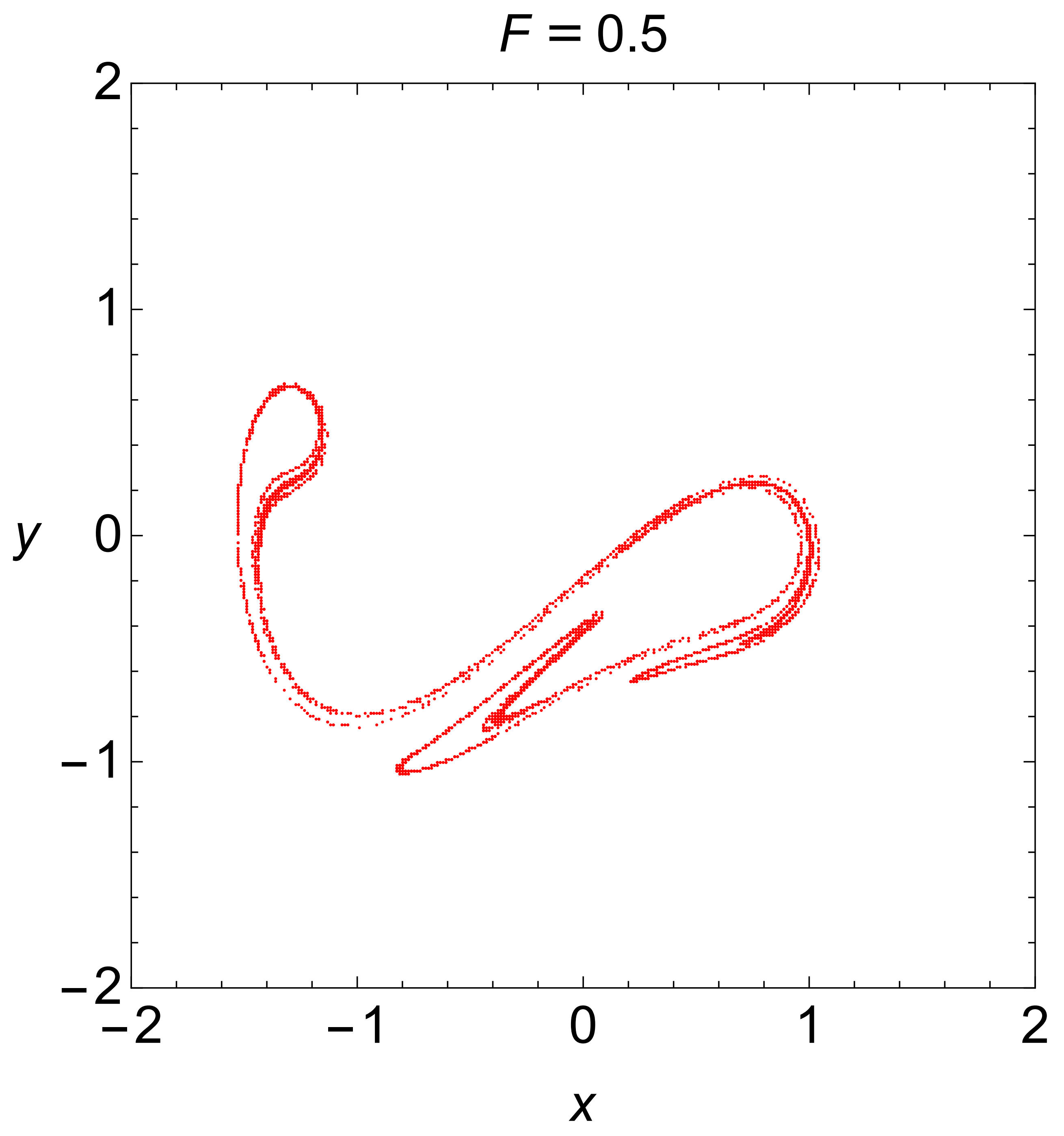
Below we show a single trajectory of $P$ with initial point $(0,0)$, different values of $F$, and $b = 0.3, \gamma = 1.2$.




Fixed point



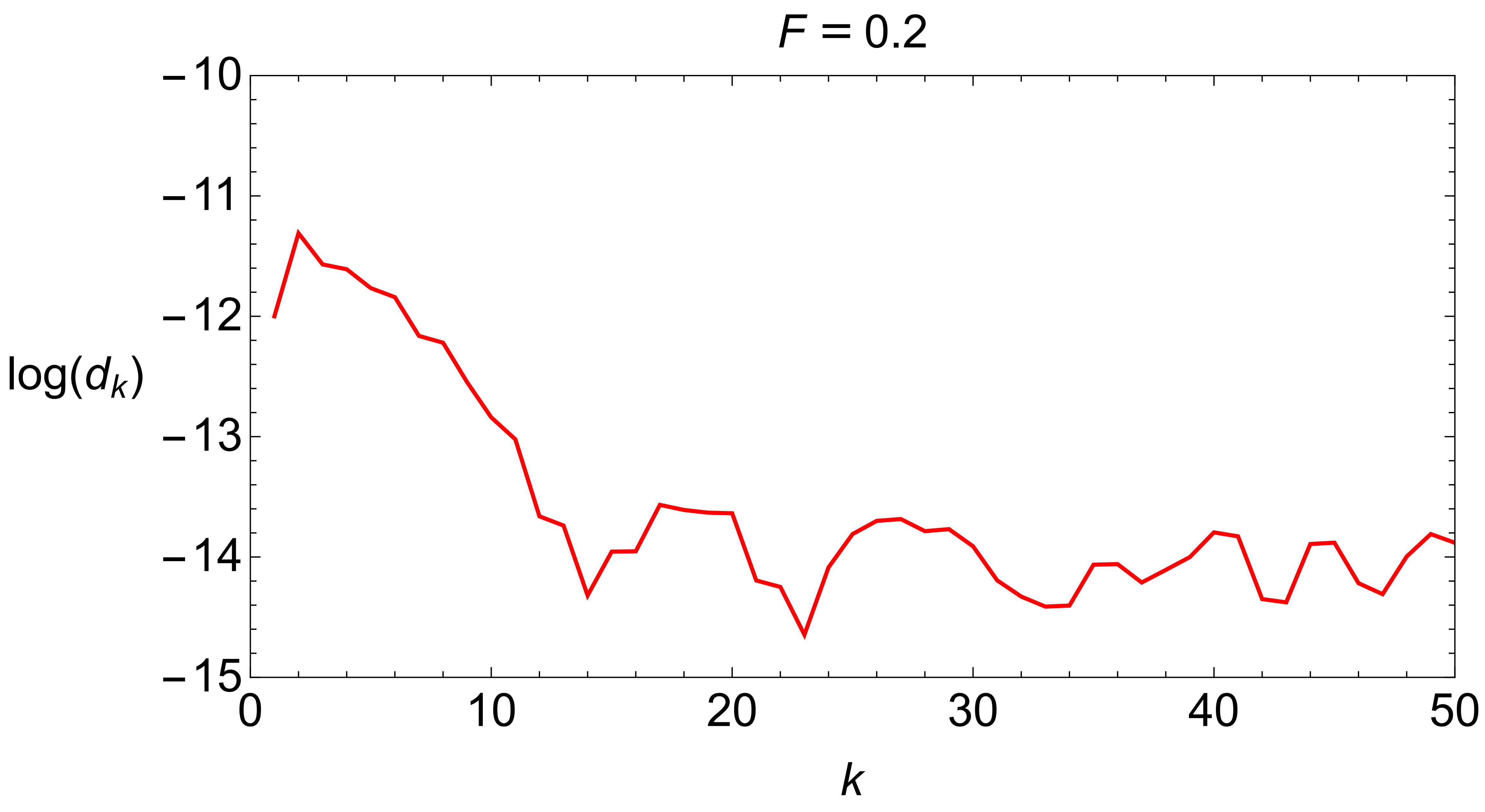

Period 5

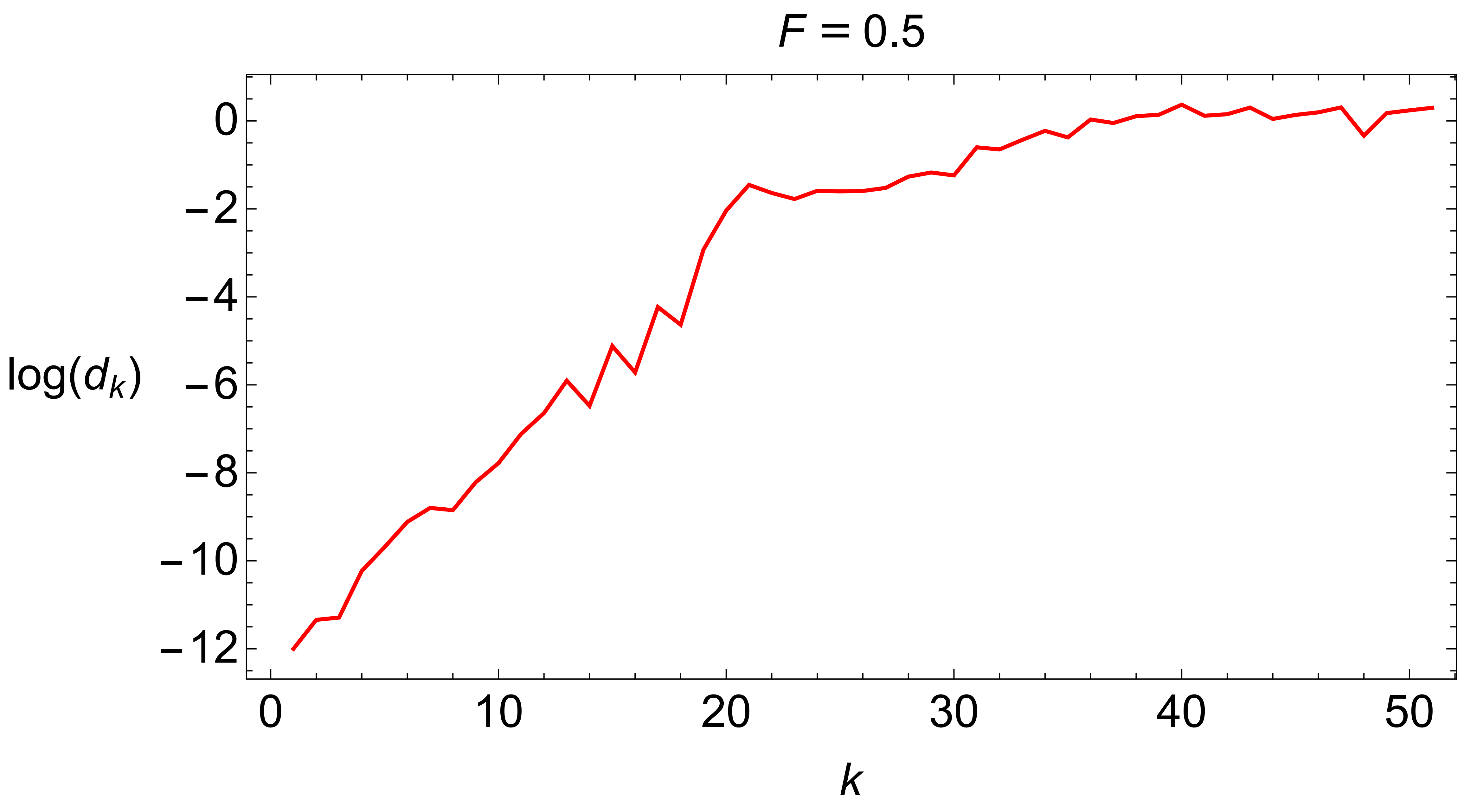



Chaotic attractor

# Chaos

Chaos is associated to **sensitive dependence on initial conditions**. This means that even if we choose two initial conditions very close to each other, the two trajectories will deviate from each other exponentially fast. In the picture below we consider two initial conditions $(0,0)$ and $(0,10^{-12})$ for the forced Duffing system from the previous slide. Then we denote by $d_k$ the distance between the two trajectories at "time" $k$ and we plot $\log d_k$ as a function of $k$. The plots below are for $F = 0.2$ and $F = 0.5$.



We observe that for $F = 0.5$ and $k \leq 20$ we have a linear increase

$$\log d_k = ak + b,$$

that is,

$$d_k \sim e^{ak}$$

showing the exponential divergence of the two trajectories.

The essence of this phenomenon is that **the exact state of the system is fundamentally unpredictable even though the system is deterministic**.

However, the chaotic attractors produced for the two initial conditions will look identical.

# Poincaré map for autonomous systems

# Hénon-Heiles system

We consider the Hénon-Heiles system in $\mathbb{R}^4$ given by

$$
\begin{aligned}
x_1' &= y_1 \\
y_1' &= -x_1 - 2x_1x_2 \\
x_2' &= y_2 \\
y_2' &= -x_1^2 + x_2^2 - x_2
\end{aligned}
$$

The system has the conserved quantity

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2 + 2x_1^2x_2 - \frac{2}{3}x_2^3).$$

This can be verified by checking that $dH/dt = 0$. $H$ represents the mechanical energy of a system and in this context is called the Hamiltonian function.

A Poincaré map can be defined in the following way.

Fix the hyperplane $\Sigma$ in $\mathbb{R}^4$ defined by $x_1 = 0$.

On this hyperplane we have $H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_2^2 - \frac{2}{3}x_2^3)$. If we consider an initial condition on $\Sigma$ then that corresponds to a value $h$ for $H$ that does not change along the corresponding solution $(x_1(t), y_1(t), x_2(t), y_2(t))$.

Therefore, when the solution reaches again $\Sigma$ it will be at a point corresponding to $H = h$. If we keep track only at the intersections of the solution with $\Sigma$ where $y_1 > 0$ then it is enough to know $(x_2, y_2)$ – and the constant $h$ – to be able to determine

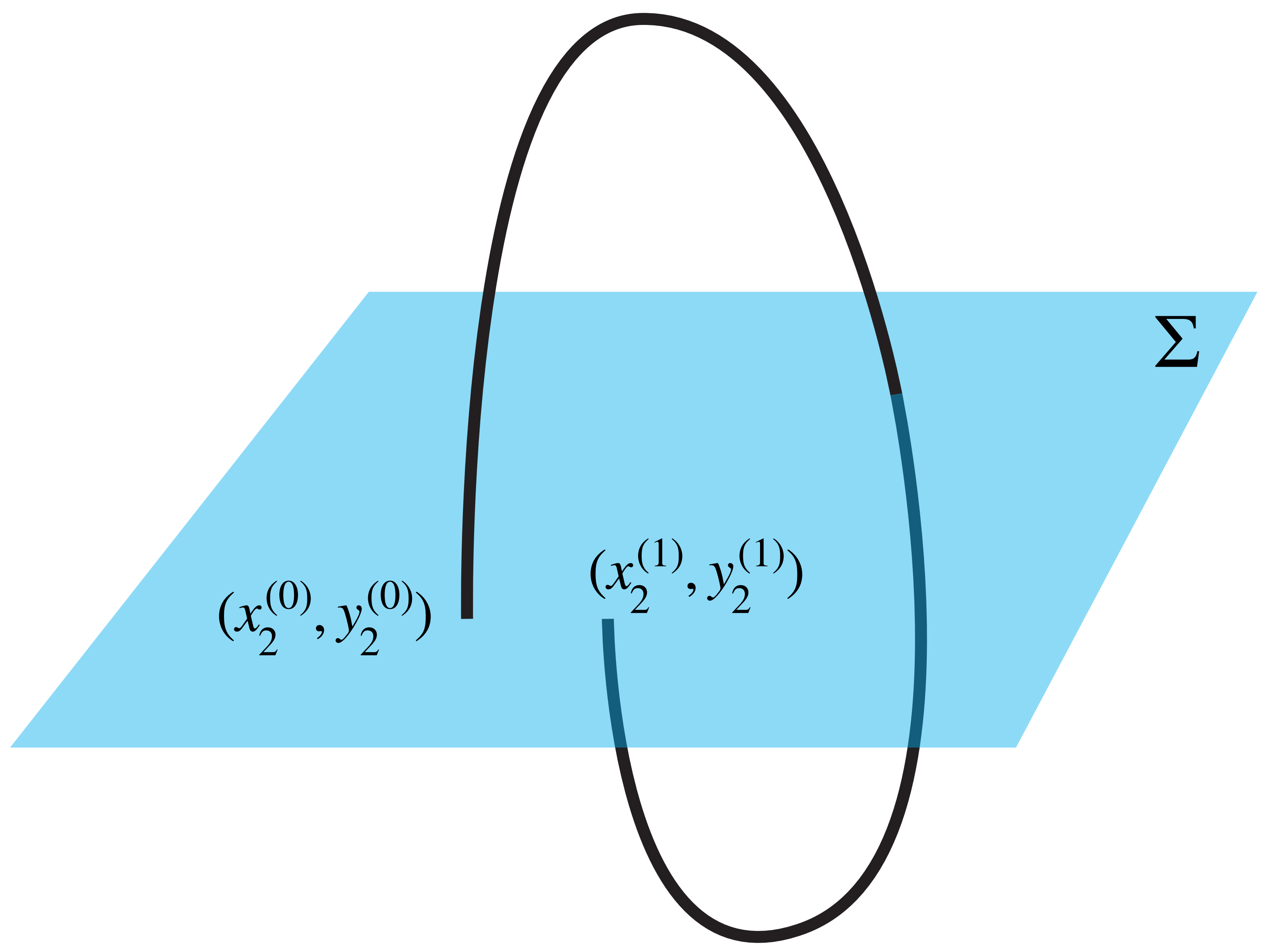
$$y_1 = \sqrt{2h - y_2^2 - x_2^2 + \frac{2}{3}x_2^3} \quad \text{and} \quad x_1 = 0.$$

So, suppose that we fix $h$ and consider a point $(x_2^{(0)}, y_2^{(0)})$.

Then define a point $(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}) \in \mathbb{R}^4$ by

$$x_1^{(0)} = 0 \text{ and } y_1^{(0)} = \sqrt{2h - (x_2^{(0)})^2 + \frac{2}{3}(x_2^{(0)})^3 - (y_2^{(0)})^2}.$$

This point is defined so that it is on $\Sigma$ and the corresponding value of $H$ equals $h$.




Then consider the solution of the system in $\mathbb{R}^4$ with initial condition $(x_1^{(0)}, y_1^{(0)}, x_2^{(0)}, y_2^{(0)}) \in \mathbb{R}^4$ and track the solution $(x_1(t), y_1(t), x_2(t), y_2(t))$ until the first time $t_0 > 0$ when we have $x_1(t_0) = 0$ with $y_1(t_0) > 0$.

Then the Poincaré map $P : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

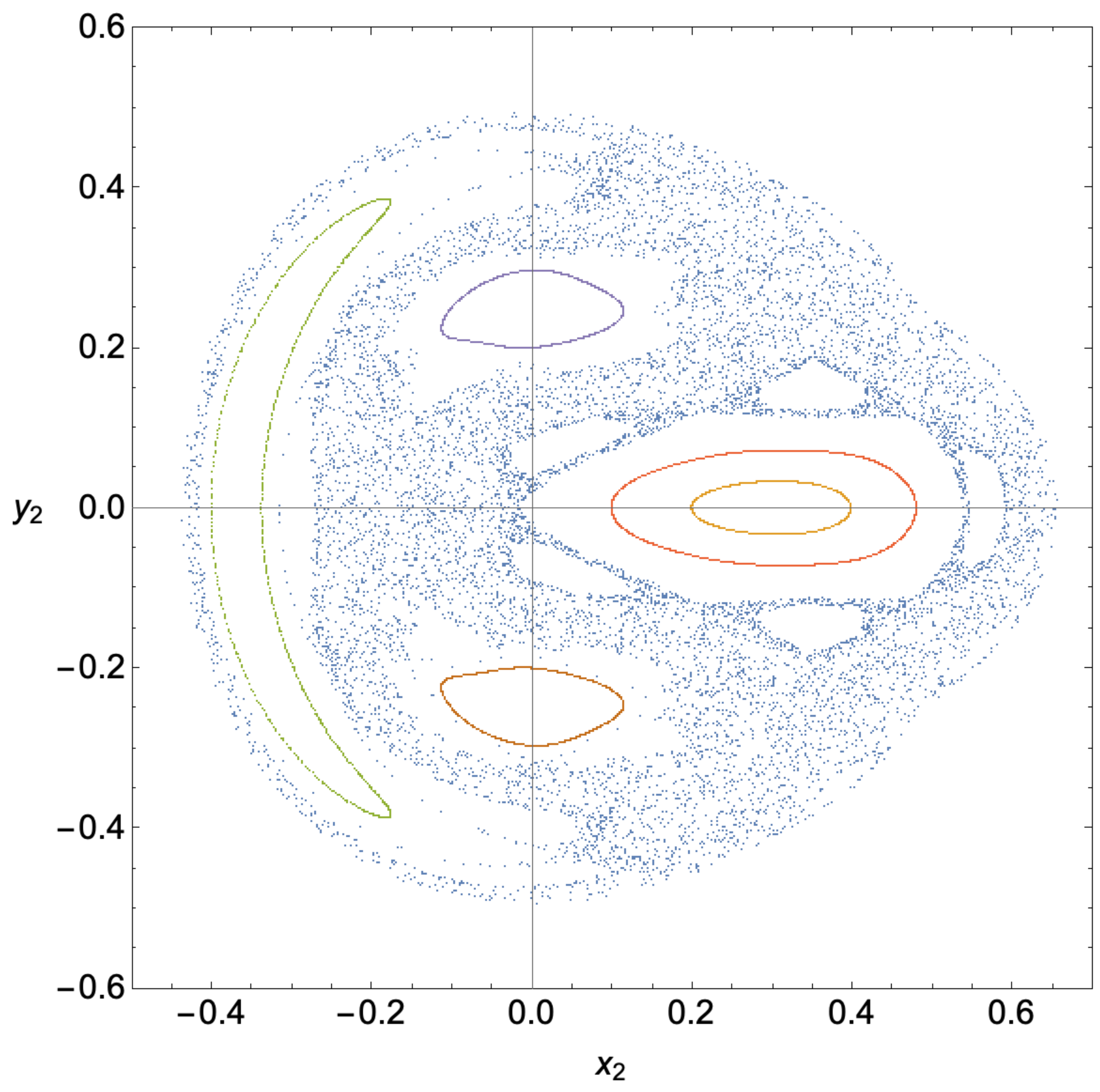
$$(x_2^{(1)}, y_2^{(1)}) = P(x_2^{(0)}, y_2^{(0)}) = (x_2(t_0), y_2(t_0)).$$

The Mathematica code for defining and computing this Poincaré map is shown below.

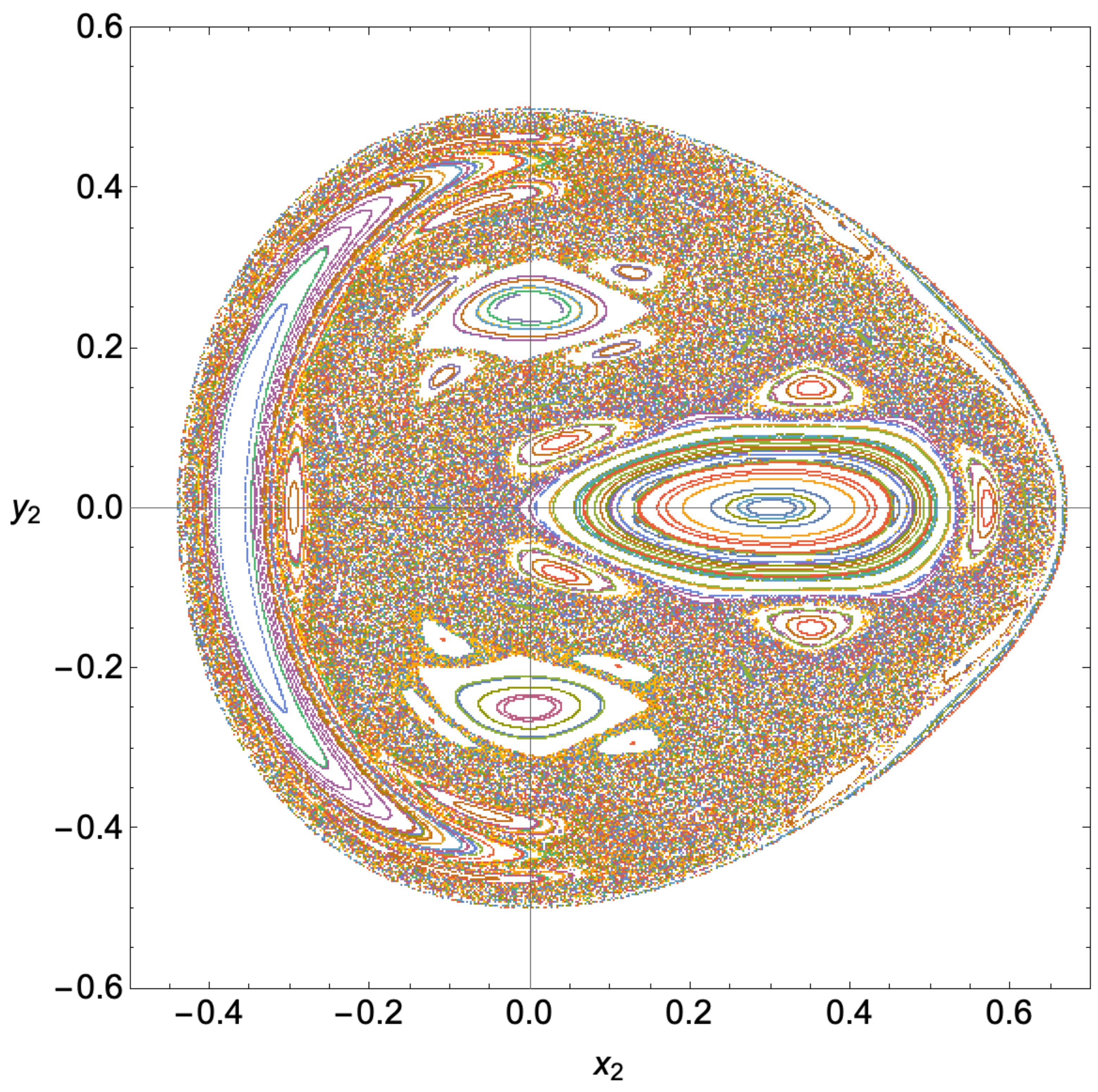
```
poincareMap[h_][{x0_, y0_}] := Module[{stopTime},
  NDSolveValue[
    x1'[t] == y1[t]
    && y1'[t] == -x1[t] - 2 x1[t] × x2[t]
    && x2'[t] == y2[t]
    && y2'[t] == -x1[t]^2 - x2[t] + x2[t]^2
    && x1[0] == 0
    && y1[0] == Sqrt[2 h - y0^2 - x0^2 + 2/3 x0^3]
    && x2[0] == x0
    && y2[0] == y0
    && WhenEvent[x1[t] == 0 && y1[t] > 0, stopTime = t; "StopIntegration"],
   {x2[stopTime], y2[stopTime]},
   {t, 0, Infinity}]]
```

Several trajectories of the Poincaré map are shown with different colors for $h = 0.125$. We observe a chaotic orbit, and "islands" at the centers of which we have fixed points or periodic points.

The fixed points and the periodic points of the Poincaré map correspond to periodic orbits of the system in $\mathbb{R}^4$ but we do not know the period of such orbits unless we compute it numerically.




Chaotic orbit with 10000 points. Some organized orbits with 1000 points for each.




200 random initial conditions with 1000 points for each corresponding orbit.