Lecture 16: Linear Systems

MATH 303 ODE and Dynamical Systems

Konstantinos Efstathiou

Planar linear systems

We will now focus on a special class of planar systems that have the form

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t),$$
$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t)\,.$$

Here $a_{ij}(t)$, $f_i(t)$, $i, j \in \{1,2\}$ are continuous functions for t in some interval $I \subseteq \mathbb{R}$.

This is a planar non-autonomous system (it becomes autonomous if the functions $a_{ij}(t)$, $f_i(t)$ are constant).

The system

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t),$$
$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t),$$

can be written in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

From now on we will denote $\mathbf{x} = [x_1 \ x_2]^t$, $\mathbf{f}(t) = [f_1(t) \ f_2(t)]^t$, and $A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$.

With this notation, the system can be written as

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t).$$

Systems of this form are called **linear**.

n-dimensional linear systems

The matrix form of the planar linear system given by

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$$

suggests that we can generalize this equation to any dimension n, where $\mathbf{x} = [x_1\ x_2\ \dots\ x_n]^t$, $\mathbf{f}(t) = [f_1(t)\ f_2(t)\ \dots\ f_n(t)]^t$, and $A(t)$ is an $n \times n$ matrix with elements a_{ij}. We will be assuming that $\mathbf{f}(t)$ and $A(t)$ are continuous functions of t in an interval $I \subseteq \mathbb{R}$.

From now on we will be considering such linear systems in arbitrary dimension $n \geq 2$. Actually, what we discuss can be also applied to $n = 1$ but we have already discussed how to solve this case.

A **solution** to the linear system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ is a vector-valued function $\mathbf{x}(t)$ defined on $I \subseteq \mathbb{R}$ and taking values in $\mathbb{R}^n$.

If $\mathbf{f}(t) \equiv \mathbf{0}$ (that is, the constant zero vector) then the linear system is called **homogeneous**. Otherwise, the system is called **non-homogeneous**.

Linear systems from linear differential equations

Suppose that we have a linear differential equation of the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + p_{n-2}(t)y^{(n-2)} + \cdots + p_1(t)y' + p_0(t)y = g(t).$$

Then define

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

We then have the linear system

$$
\begin{aligned}
x_1' &= x_2 \\
x_2' &= x_3 \\
&\vdots \\
x_n' &= -p_{n-1}(t)x_n - p_{n-2}(t)x_{n-1} - p_1(t)x_2 - p_0(t)x_1 + g(t)
\end{aligned}
$$

This shows how linear differential equations give rise to equivalent linear systems.

Existence and Uniqueness

Theorem. If $A(t),\ \mathbf{f}(t)$ are continuous functions in an interval $I \subseteq \mathbb{R}$ and $t_0 \in I$ then for any initial vector $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a unique solution of $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ in I that satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

Linearity

We now focus on the homogeneous linear system

$$\mathbf{x}' = A(t)\mathbf{x}.$$

Such systems have the following important property.

Proposition. If $\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t)$ are solutions of $\mathbf{x}' = A(t)\mathbf{x}$ then any linear combination $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)$ is also a solution of $\mathbf{x}' = A(t)\mathbf{x}$.

Linear in-/dependence

Definition. The m vector valued functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_m(t)$ are **linearly dependent** in an interval I if there exist constants $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t) \equiv \mathbf{0}, \text{ for all } t \in I.$$

If they are not linearly dependent, they are called **linearly independent**.

Example

Let $\mathbf{x}_1(t) = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} 3e^t \\ 0 \\ 3e^t \end{bmatrix}$, $\mathbf{x}_3(t) = \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}$.

The given functions are linearly dependent since for all $t \in \mathbb{R}$ we have

$$3 \cdot \mathbf{x}_1(t) + (-1) \cdot \mathbf{x}_2(t) + 0 \cdot \mathbf{x}_3(t) = \mathbf{0}.$$

Wronskian

Consider n solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ of the linear system $\mathbf{x}' = A(t)\mathbf{x}$ and consider the $n \times n$ matrix $X(t)$ whose columns are the vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$. That is,

$$X(t) = [\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \ \ldots\ \ \mathbf{x}_n(t)] = \begin{bmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{bmatrix}.$$

Definition. The **Wronskian** $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t)$ of the n solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ is the determinant of $X(t)$, that is,

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det X(t).$$

Linear in-/dependence and Wronskian

Theorem. Suppose that $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are solutions of the linear system $\mathbf{x}' = A(t)\mathbf{x}$ in an interval I. Then the following statements are equivalent:

(a) The solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent in I.

(b) $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = 0$ for all $t \in I$.

(c) There is $t_0 \in I$ such that $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) = 0$.

Proof

First we check that (a) implies (b). If the vector valued functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent in I then we know from Linear Algebra that $\det X(t) = 0$. Therefore,

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det X(t) = 0 \text{ for all } t \in I.$$

Then it is clear that (b) implies (c).

Finally, we prove that (c) implies (a). The condition for (c) implies that $\det X(t_0) = 0$ which means that the vectors $\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \ldots, \mathbf{x}_n(t_0)$ are linearly dependent. Therefore, there are $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{0}.$$

Let

$$\mathbf{z}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t).$$

The function $\mathbf{z}(t)$ is a solution for the linear system $\mathbf{x}' = A(t)\mathbf{x}$ since it is a linear combination of the solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$. Moreover,

$$\mathbf{z}(t_0) = c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{0}.$$

Therefore, $\mathbf{z}(t)$ solves the initial value problem $\mathbf{x}' = A(t)\mathbf{x}$ with $\mathbf{z}(t_0) = \mathbf{0}$. From the uniqueness of solutions in I this implies that $\mathbf{z}(t) \equiv \mathbf{0}$ for all $t \in I$ and thus that $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent in I.

Fundamental solution & fundamental matrix

Definition. A collection of n linearly independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ of a linear system $\mathbf{x}' = A(t)\mathbf{x}$ is called a **fundamental solution** of the system.

The corresponding matrix $X(t) = [\mathbf{x}_1(t) \ \mathbf{x}_2(t) \ \ \ldots \ \ \mathbf{x}_n(t)]$ is called a **fundamental matrix**.

Theorem. If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ is a fundamental solution of the linear system $\mathbf{x}' = A(t)\mathbf{x}$ and $A(t)$ is continuous in $I \subseteq \mathbb{R}$ then the general solution of $\mathbf{x}' = A(t)\mathbf{x}$ has the form $\mathbf{y}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ for some real numbers $c_1, \ldots, c_n$.

Proof. Suppose that $\mathbf{y}(t)$ is any solution of $\mathbf{x}' = A(t)\mathbf{x}$ and for some $t_0 \in I$ let $\mathbf{y}(t_0) = \mathbf{y}_0$. Since the vectors $\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \ldots, \mathbf{x}_n(t_0)$ are n linearly independent vectors in $\mathbb{R}^n$, there are $c_1, \ldots, c_n$ such that

$$\mathbf{y}_0 = c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \cdots + c_n\mathbf{x}_n(t_0).$$

Define $\mathbf{u}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$. Then clearly $\mathbf{u}(t)$ solves the initial value problem $\mathbf{x}' = A(t)\mathbf{x}$ with $\mathbf{x}(t_0) = \mathbf{y}_0$. However, $\mathbf{y}(t)$ solves the same initial value problem and from uniqueness of solutions we conclude that

$$\mathbf{y}(t) = \mathbf{u}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \ \text{ for all } t \in I.$$

Remarks

1. The fundamental matrix $X(t)$ is invertible since $\det X(t) \neq 0$.

2. The relation $\mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$ can also be written in matrix form as

$$\mathbf{y}(t) = X(t)\mathbf{c},$$

where $\mathbf{c} = [c_1 \ \ \ldots \ \ c_n]^t$.

3. Since each column $\mathbf{x}_k(t)$ of $X(t)$ satisfies $\mathbf{x}_k'(t) = A(t)\mathbf{x}_k(t)$ we conclude that $X'(t) = A(t)X(t)$. This means that a fundamental matrix is a solution of the differential equation $X' = A(t)X$ where X is a $n \times n$ matrix valued function.

4. Consider the initial value problem $\mathbf{x}' = A(t)\mathbf{x}$ with $\mathbf{x}(t_0) = \mathbf{x}_0$. If $\mathbf{y}(t)$ is the (unique) solution to this problem then $\mathbf{y}(t) = X(t)\mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^n$. At $t = t_0$ we get $\mathbf{x}_0 = \mathbf{y}(t_0) = X(t_0)\mathbf{c}$.

Since the fundamental matrix is invertible we get $\mathbf{c} = X(t_0)^{-1}\mathbf{x}_0$. Therefore, the solution to the given initial problem is given by

$$\mathbf{y}(t) = X(t)X(t_0)^{-1}\mathbf{x}_0.$$

5. Let $Y(t)$ be another fundamental matrix for $\mathbf{x}' = A(t)\mathbf{x}$. Then we have

$$\mathbf{y}(t) = X(t)X(t_0)^{-1}\mathbf{x}_0 = Y(t)Y(t_0)^{-1}\mathbf{x}_0$$

which gives $[X(t)X(t_0)^{-1} - Y(t)Y(t_0)^{-1}]\mathbf{x}_0 = \mathbf{0}$.

Since this relation holds for all $\mathbf{x}_0 \in \mathbb{R}^n$ we conclude that $X(t)X(t_0)^{-1} = Y(t)Y(t_0)^{-1}$, or $X(t)^{-1}Y(t) = X(t_0)^{-1}Y(t_0) = C$, where C is a constant matrix. Therefore,

$$Y(t) = X(t)\,C.$$

Note that $\det C \neq 0$ (since $\det Y(t) \neq 0$ and $\det Y(t) = \det X(t) \det C$). Actually, the converse also holds. If C is a matrix with $\det C \neq 0$ and $X(t)$ is a fundamental matrix then $X(t)\,C$ is also a fundamental matrix.

6. Since $\mathbf{y}(t) = Y(t)Y(t_0)^{-1}\mathbf{x}_0$ it is clear that things can be simplified if we take a fundamental matrix $Y(t)$ such that $Y(t_0) = \mathbb{I}$ (the identity matrix). Then we will have

$$\mathbf{y}(t) = Y(t)\mathbf{x}_0.$$

If we know a fundamental matrix $X(t)$ then we can define $Y(t)$ with $Y(t_0) = \mathbb{I}$ by

$$Y(t) = X(t)X(t_0)^{-1}.$$

To check this, recall from the previous discussion, that such $Y(t)$ is a fundamental matrix and

$$Y(t_0) = X(t_0)X(t_0)^{-1} = \mathbb{I}.$$