Lecture 17: Linear Systems with Constant Coefficients

MATH 303 ODE and Dynamical Systems

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Linear systems with constant coefficients

We consider systems of the form

$$\mathbf{x}' = A\mathbf{x}$$

where A is a constant $n \times n$ matrix and $\mathbf{x}$ takes values in $\mathbb{R}^n$.

Based on the results of the general theory we need to find a **fundamental solution** $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$.

Searching for a solution

We try solutions of the form

$$\mathbf{x}(t) = e^{rt}\mathbf{u}$$

where r is a number and $\mathbf{u}$ is a non-zero vector that must be determined.

Substituting into the system $\mathbf{x}' = A\mathbf{x}$ we get

$$(e^{rt}\mathbf{u})' = re^{rt}\mathbf{u} = Ae^{rt}\mathbf{u},$$

Therefore,

$$e^{rt}(A - r\mathbb{I})\mathbf{u} = \mathbf{0}.$$

Since $e^{rt} \neq 0$ we conclude that $r, \mathbf{u}$ must satisfy the equation

$$(A - r\mathbb{I})\mathbf{u} = \mathbf{0}.$$

Eigenvalue problem

The equation $(A - r\mathbb{I})\mathbf{u} = \mathbf{0}$ is the **eigenvalue equation** for the matrix A. That is, the solutions $r, \mathbf{u}$ are the eigenvalues and eigenvectors respectively of A.

For the equation $(A - r\mathbb{I})\mathbf{u} = \mathbf{0}$ to have non-zero solutions $\mathbf{u}$ we need that r is a root of the n-th degree **characteristic polynomial**

$$p(r) = \det(A - r\mathbb{I}),$$

that is, r is an eigenvalue of A. Then $\mathbf{u}$ is the corresponding eigenvector. Note that $\mathbf{u}$ is not unique but is determined only up to a multiplicative constant, that is, if $\mathbf{u}$ is an eigenvector for r, then $s\mathbf{u},\ s \neq 0$ is also an eigenvector for r.

Example 1

For $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ we have that $A - r\mathbb{I} = \begin{bmatrix} 2-r & -3 \\ 1 & -2-r \end{bmatrix}$.

Then $p(r) = \det(A - r\mathbb{I}) = r^2 - 1$.

Therefore, the eigenvalues are $r_1 = -1,\ r_2 = 1$.

For the eigenvalue $r_1 = -1$ we have the eigenvector $\mathbf{u} = [u_1\ u_2]^t$ that satisfies

$$(A + \mathbb{I})\mathbf{u} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The equations $\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ give $u_1 = u_2$. Then we can take the corresponding eigenvector to be $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the eigenvalue $r_2 = 1$ we have the eigenvector $\mathbf{u} = [u_1 \ u_2]^t$ that satisfies

$$(A - \mathbb{I})\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and gives $u_1 = 3u_2$. Then we can take the corresponding eigenvector to be $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Example 2

For $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ we have that $A - r\mathbb{I} = \begin{bmatrix} 1-r & -2 \\ 2 & 1-r \end{bmatrix}$.

Then $p(r) = \det(A - r\mathbb{I}) = r^2 - 2r + 5$.

Therefore, the eigenvalues are $r_1 = 1 - 2i,\ r_2 = 1 + 2i$.

For the eigenvalue $r_1 = 1 - 2i$ we have the eigenvector $\mathbf{u} = [u_1\ u_2]^t$ that satisfies

$$(A - r_1\mathbb{I})\mathbf{u} = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The equations $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ give $iu_1 = u_2$. Then we can take the corresponding eigenvector to be $\mathbf{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

For the eigenvalue $r_2 = 1 + 2i$ we have the eigenvector $\mathbf{u} = [u_1 \;\, u_2]^t$ that satisfies

$$(A - r_2\mathbb{I})\mathbf{u} = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and gives $-iu_1 = u_2$. Then we can take the corresponding eigenvector to be $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Remark

If r is a complex eigenvalue of a real matrix A then $\bar{r}$ (the complex conjugate of r) is also an eigenvalue.

Moreover, if $\mathbf{u}$ is an eigenvector for r then $\bar{\mathbf{u}}$ is an eigenvector for $\bar{r}$, since

$$A\bar{\mathbf{u}} = \overline{A\mathbf{u}} = \overline{ru} = \bar{r}\,\bar{\mathbf{u}}.$$

Poll

Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. The vector $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is...

Choose the correct answer at pollev.com/ke1.

A. ... not an eigenvector of A.

B. ... an eigenvector of A with eigenvalue 1.

C. ... an eigenvector of A with eigenvalue 4.

D. ... an eigenvector of A with eigenvalue -1.

Linearly independent eigenvectors

Theorem. If the $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ with **real** corresponding eigenvalues $r_1, \ldots, r_n$ then the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{u}_1 + \cdots + c_n e^{r_n t}\mathbf{u}_n.$$

Proof. Based on the results for general linear systems, it is sufficient to show that the vector valued functions $\mathbf{x}_k(t) = e^{r_k t}\mathbf{u}_k$, $k = 1,\ldots,n$ are a fundamental solution. For this to hold it is sufficient to check that the Wronskian $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$ for some $t_0 \in \mathbb{R}$. Take $t_0 = 0$. Then

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](0) = \det[\mathbf{x}_1(0) \ \ \ldots \ \ \mathbf{x}_n(0)] = \det[\mathbf{u}_1 \ \ \ldots \ \ \mathbf{u}_n] \neq 0,$$

since the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent.

Example

Consider the linear system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$. Recall that the eigenvalues are -1 and 1 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Then the general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + 3c_2 e^{t} \\ c_1 e^{-t} + c_2 e^{t} \end{bmatrix}.$$

Real distinct eigenvalues

Theorem. If $r_1, \ldots, r_m$ (with $1 \leq m \leq n$) are real distinct eigenvalues of A then the corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent.

Corollary. If the matrix A has n real distinct eigenvalues $r_1, \ldots, r_n$ with corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ then a fundamental solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is $e^{r_1 t}\mathbf{u}_1, \ldots, e^{r_n t}\mathbf{u}_n$.

Complex eigenvalues

Consider now the case where one of the eigenvalues of A is complex, $r = \alpha + i\beta$, with eigenvector $\mathbf{u} = \mathbf{a} + i\mathbf{b}$. Recall that this means that then $\bar{r} = \alpha - i\beta$ is another eigenvalue with eigenvector $\overline{\mathbf{u}} = \mathbf{a} - i\mathbf{b}$.

Since $\mathbf{w}(t) = e^{rt}\mathbf{u}$ and $\overline{\mathbf{w}}(t) = e^{\bar{r}t}\overline{\mathbf{u}}$ are complex vector valued solutions of the linear system $\mathbf{x}' = A\mathbf{x}$ we can combine them to create real vector valued solutions:

$$\mathbf{x}_1(t) = \frac{1}{2}(\mathbf{w}(t) + \overline{\mathbf{w}}(t)) = \mathrm{Re}\mathbf{w}(t) \quad \text{and} \quad \mathbf{x}_2(t) = \frac{1}{2i}(\mathbf{w}(t) - \overline{\mathbf{w}}(t)) = \mathrm{Im}\mathbf{w}(t).$$

We have

$$\mathbf{w}(t) = e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b}) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{a} + i\mathbf{b})$$

$$= e^{\alpha t}(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}) + ie^{\alpha t}(\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b})$$

Therefore,

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}) \text{ and } \mathbf{x}_2(t) = e^{\alpha t}(\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}).$$

It can be proven that these two vector valued functions are linearly independent.

Example

Consider the linear system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Recall that one complex eigenvalue is $r = 1 + 2i$ with corresponding eigenvector $\mathbf{u} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Therefore, $\alpha = 1, \beta = 2, \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Using the relations we obtained earlier we find

$$\mathbf{x}_1(t) = e^t \left(\cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = e^t \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^t \left(\sin(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = e^t \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}.$$

Therefore the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = e^t \begin{bmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ c_1 \sin(2t) - c_2 \cos(2t) \end{bmatrix}.$$

Remark. It is rather difficult to remember the general expressions for $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$. What we do in practice is to write $\mathbf{w}(t) = e^{(1+2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and compute the real and imaginary parts of $\mathbf{w}(t)$.

Poll

Consider the linear system $\mathbf{x}' = A\mathbf{x}$ where A has eigenvalues $r_1 = -2, r_2 = -i, r_3 = i$ with corresponding eigenvectors $\mathbf{u}_1 = [1\ 0\ 0]^t$, $\mathbf{u}_2 = [0\ 1\ -i]^t$, $\mathbf{u}_3 = [0\ 1\ i]^t$. What is the general solution of the linear system?

Choose the correct answer at pollev.com/ke1.

A. $c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix}$

B. $c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \cos t \\ -\sin t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \sin t \\ \cos t \end{bmatrix}$

C. $c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \sin t \\ -\cos t \end{bmatrix}$

D. $c_1 \begin{bmatrix} e^{-2t} \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \cos t \\ \sin t \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ \sin t \\ -\cos t \end{bmatrix}$

$$-2 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad e^{-2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} = \vec{x}_1(t)$$

$$-i \quad \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$$i \quad \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \to e^{it}\begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} = (\cos t + i \sin t)\begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \cos t + i \sin t \\ i\cos t - \sin t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \cos t \\ -\sin t \end{bmatrix}}_{\vec{x}_2(t)} + i \underbrace{\begin{bmatrix} 0 \\ \sin t \\ \cos t \end{bmatrix}}_{\vec{x}_3(t)}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \cos t \\ -\sin t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \sin t \\ \cos t \end{bmatrix}$$

(B)