# Lecture 18: Matrix Exponential

## MATH 303 ODE and Dynamical Systems

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# Definition

For a real number $a$ we define the exponential through the series

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \frac{1}{4!}a^4 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}a^k.$$

Analogously, for a $n \times n$ matrix $A$ we define the **matrix exponential** $e^A$ through the (matrix) series

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

# Properties

1. The series defining $e^A$ converges for all $A$.

2. $e^{\mathbb{O}} = \mathbb{I}$ where $\mathbb{O}$ is the $n \times n$ zero matrix and $\mathbb{I}$ is the $n \times n$ identity matrix.

3. $\det e^A = e^{\operatorname{tr} A} > 0$, where $\operatorname{tr} A$ denotes the trace of $A$. Recall that

   $$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$ where $\lambda_i$ are the eigenvalues of $A$.

4. If $AB = BA$ then $e^{A+B} = e^A e^B = e^B e^A$ ( follows from the series definition and the formula for the Cauchy product of two series ).

5. $e^A e^{-A} = \mathbb{I}$, that is, $(e^A)^{-1} = e^{-A}$ ( follows easily from property 4 ).

# More properties

6. If $A\mathbf{u} = r\mathbf{u}$ then $e^{A}\mathbf{u} = e^{r}\mathbf{u}$ ( follows easily from the series definition ).
7. $e^{U^{-1}AU} = U^{-1}e^{A}U$ ( follows easily from the series definition and the fact that $(U^{-1}AU)^{k} = U^{-1}A^{k}U$ ).
8. $e^{\square t} = e^{t}\square$ ( follows easily from the series definition ).
9. $e^{A(t+s)} = e^{At}e^{As}$ ( follows easily from property 4 in the previous slide ).

# Diagonal matrix

Consider a diagonal matrix $R = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then $R^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$. We have

$$e^R = \sum_{k=0}^{\infty} \frac{1}{k!} R^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} b^k \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Note that the same property holds for diagonal matrices of arbitrary size $n \times n$.

**Remark.** Recall that $e^{U^{-1}AU} = U^{-1} e^A U$. Therefore, if $A = U^{-1} R U$ is diagonal then we have $e^A = e^{U^{-1}RU} = U^{-1} e^R U$.

# What this has to do with linear systems?

**Theorem.** For a $n \times n$ constant matrix $A$ we have

$$\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A.$$

**Proof.** We have

$$\begin{aligned}\frac{d}{dt}(e^{At}) &= \frac{d}{dt}\left(\mathbb{I} + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots\right)\\ &= A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \cdots\\ &= A\left(\mathbb{I} + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = Ae^{At}\end{aligned}$$

**Theorem.** The unique solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{A(t-t_0)}\,\mathbf{x}_0.$$

**Proof.** We have

$$\mathbf{x}'(t) = \frac{d}{dt}\left(e^{A(t-t_0)}\mathbf{x}_0\right) = \frac{d}{dt}(e^{A(t-t_0)})\mathbf{x}_0 = Ae^{A(t-t_0)}\mathbf{x}_0 = A\mathbf{x}(t),$$

and

$$\mathbf{x}(t_0) = e^{A(t_0-t_0)}\mathbf{x}_0 = e^{\mathbb{O}}\mathbf{x}_0 = \mathbb{I}\mathbf{x}_0 = \mathbf{x}_0.$$

Therefore, $\mathbf{x}(t) = e^{A(t-t_0)}\,\mathbf{x}_0$ solves the given initial value problem.

**Remark.** In the particular case where $t_0 = 0$ we have $\mathbf{x}(t) = e^{At}\mathbf{x}_0$.

# $e^{At}$ is a fundamental matrix

Recall that we saw earlier that if $X(t)$ is a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ and we have the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ then the solution is

$$\mathbf{x}(t) = X(t)X(0)^{-1}\mathbf{x}_0.$$

Moreover, we just saw that

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

Since these relations hold for arbitrary $\mathbf{x}_0$ we conclude that

$$e^{At} = X(t)X(0)^{-1}.$$

This relation

$$e^{At} = X(t)X(0)^{-1}$$

shows that $e^{At}$ is also a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ and, more specifically, it is the fundamental matrix which equals $\mathbb{I}$ at $t = 0$.

Moreover, the relation

$$e^{At} = X(t)X(0)^{-1}$$

gives us a recipe for computing $e^{At}$. Find any fundamental matrix $X(t)$ and then compute $X(t)X(0)^{-1}$.

# Generalized eigenvectors

**Definition.** A non-zero vector $\mathbf{u} \in \mathbb{C}^n$ that satisfies $(A - r\mathbb{I})^m\mathbf{u} = \mathbf{0}$ for some $r \in \mathbb{C}$ and a positive integer $m$ is called a **generalized eigenvector of** $A$ **associated with** $r$.

**Remark.** The case $m = 1$ corresponds to standard eigenvectors.

**Remark.** If $\mathbf{u}$ is a generalized eigenvector associated with $r$ then $r$ is an eigenvalue of $A$. The reason for this is that if $m$ is the smallest number for which $(A - r\mathbb{I})^m\mathbf{u} = \mathbf{0}$ ( i.e., if we have $\mathbf{w} = (A - r\mathbb{I})^{m-1}\mathbf{u} \neq \mathbf{0}$ ) then

$$(A - r\mathbb{I})\mathbf{w} = (A - r\mathbb{I})[(A - r\mathbb{I})^{m-1}\mathbf{u}] = (A - r\mathbb{I})^m\mathbf{u} = \mathbf{0},$$

showing that $\mathbf{w}$ is an eigenvector with eigenvalue $r$.

# Generalized eigenvectors

**Theorem.** If the $n \times n$ matrix $A$ has characteristic polynomial

$$p(r) = (r - r_1)^{m_1} \ \cdots \ (r - r_k)^{m_k}$$

with $m_1 + \cdots + m_k = n$, then there exist for each $j = 1,\ldots,k$ linearly independent generalized eigenvectors $\mathbf{u}_{j,1}, \ldots, \mathbf{u}_{j,m_j}$ with

$$(A - r_j \mathbb{I})^{m_j} \mathbf{u}_{j,\ell} = \mathbf{0} \ \text{ for } \ \ell = 1,\ldots, m_j.$$

Moreover, the generalized eigenvectors $\{\mathbf{u}_{j,\ell}\}_{j=1,\ldots,k;\ell=1,\ldots,m_j}$ are linearly independent.

# How to compute $e^{At}$

1. Find linearly independent generalized eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ by solving the corresponding equation for each root of the characteristic polynomial.

2. Compute the solutions $\mathbf{x}_j(t) = e^{At}\mathbf{u}_j$, $j = 1,\ldots,n$. Here we will use the fact that for each generalized eigenvector $\mathbf{u}_j$ there is a number $k_j$ such that $(A - r_j I)^{k_j}\mathbf{u}_j = \mathbf{0}$.

3. Write the corresponding fundamental matrix $X(t) = [\mathbf{x}_1(t) \;\; \ldots \;\; \mathbf{x}_n(t)]$.

4. Compute $e^{At} = X(t)X(0)^{-1}$.

# Example

We will start with a non-trivial example to demonstrate how to use the previous theoretical results. Consider the linear system $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We want to compute $e^{At}$.

The characteristic polynomial is

$$p(r) = -(r-1)^2(r-3).$$

Because $r_1 = r_2 = 1$ is an eigenvalue with multiplicity 2 we try to find generalized eigenvectors that satisfy $(A - \mathbb{I})^2\mathbf{u} = \mathbf{0}$. We have

$$A - \mathbb{I} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } (A - \mathbb{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

so we get the equation

$$(A - \mathbb{I})^2\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We only have the equation $u_1 + 2u_2 = 0$, which implies that $u_3$ is arbitrary and $u_1 = -2u_2$.

This means we can choose $u_1 = u_2 = 0,\ u_3 = 1$ and $u_1 = -2,\ u_2 = 1,\ u_3 = 0$ to get the linearly independent generalized eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

For the solution $\mathbf{x}_1(t)$ with $\mathbf{x}_1(0) = \mathbf{u}_1$ we have $\mathbf{x}_1(t) = e^{At}\mathbf{u}_1$.

We compute $\mathbf{x}_1(t)$ using the following trick. We know that $\mathbf{u}_1$ satisfies $(A - \mathbb{1})^2\mathbf{u}_1 = \mathbf{0}$. Therefore,

$$e^{At}\mathbf{u}_1 = e^{\mathbb{1}t + (A-\mathbb{1})t}\mathbf{u}_1 = e^{\mathbb{1}t}e^{(A-\mathbb{1})t}\mathbf{u}_1 = e^{t}e^{(A-\mathbb{1})t}\mathbf{u}_1.$$

But then we have

$$
\begin{aligned}
e^{(A-\mathbb{I})t}\mathbf{u}_1 &= \left( \mathbb{I} + t(A-\mathbb{I}) + \frac{t^2}{2}(A-\mathbb{I})^2 + \cdots \right)\mathbf{u}_1 \\
&= \mathbf{u}_1 + t(A-\mathbb{I})\mathbf{u}_1 + \frac{t^2}{2}(A-\mathbb{I})^2\mathbf{u}_1 + \cdots \\
&= \mathbf{u}_1 + t(A-\mathbb{I})\mathbf{u}_1
\end{aligned}
$$

since $(A-\mathbb{I})^k\mathbf{u}_1 = \mathbf{0}$ for $k \geq 2$. What remains is to compute the last expression. We have

$$
e^{(A-\mathbb{I})t}\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

Therefore,

$$\mathbf{x}_1(t) = e^{At}\mathbf{u}_1 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_1 = e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

We can work similarly to find

$$\mathbf{x}_2(t) = e^{At}\mathbf{u}_2 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_2.$$

Since $(A-\mathbb{I})^k\mathbf{u}_2 = \mathbf{0}$ for $k \geq 2$ we will get, exactly as for $\mathbf{u}_1$, that

$$e^{(A-\mathbb{I})t}\mathbf{u}_2 = \mathbf{u}_2 + t(A-\mathbb{I})\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ t \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_2(t) = e^{At}\mathbf{u}_2 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_2 = e^t \begin{bmatrix} -2 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} -2e^t \\ e^t \\ te^t \end{bmatrix}.$$

Finally, for $r_3 = 3$ which has multiplicity 1, we search for a standard eigenvector. We find the eigenvector

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore, the corresponding solution is

$$\mathbf{x}_3(t) = e^{At}\mathbf{u}_3 = e^{3t}\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2e^{3t} \\ e^{3t} \end{bmatrix}.$$

The corresponding fundamental matrix is

$$X(t) = [\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \mathbf{x}_3(t)] = \begin{bmatrix} 0 & -2e^t & 0 \\ 0 & e^t & 2e^{3t} \\ e^t & te^t & e^{3t} \end{bmatrix}.$$

Moreover, $X(0) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $X(0)^{-1} = \frac{1}{4}\begin{bmatrix} -1 & -2 & 4 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ and we find

$$e^{At} = X(t)X(0)^{-1} = \frac{1}{4}\begin{bmatrix} 4e^t & 0 & 0 \\ -2e^t + 2e^{3t} & 4e^{3t} & 0 \\ -e^t + e^{3t} - 2te^t & -2e^t + 2e^{3t} & 4e^t \end{bmatrix}.$$

# Example

Consider the planar linear system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We want to compute $e^{At}$.

The characteristic polynomial is $p(r) = (r-1)^2$ with $r_1 = r_2 = 1$. Here we can look for generalized eigenvectors as solutions of $(A - \mathbb{I})^2\mathbf{u} = \mathbf{0}$.

We have $(A - \mathbb{I})^2 = \mathbb{O}$ and therefore all non-zero vectors are generalized eigenvectors. Choose $\mathbf{u}_1 = [1\ \ 0]^t$ and $\mathbf{u}_2 = [0\ \ 1]^t$.

We have

$$e^{At}\mathbf{u}_1 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_1 = e^t \left(\mathbf{u}_1 + t(A-\mathbb{I})\mathbf{u}_1\right) = e^t \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

Similarly,

$$e^{At}\mathbf{u}_2 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_2 = e^t \left(\mathbf{u}_2 + t(A-\mathbb{I})\mathbf{u}_2\right) = e^t \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} te^t \\ e^t \end{bmatrix}$$

We then consider the fundamental matrix

$$X(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix},$$

and we observe that $X(0) = \mathbb{I}$. Therefore, $e^{At} = X(t)X(0)^{-1} = X(t)$.

**Remark.** Note that this example is sufficiently simple so that $e^{At}$ can be computed directly through the Taylor series. The crucial fact here is that $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

**Remark.** Another way to obtain the same result is to observe that $(A - \mathbb{I})^2 = \mathbb{O}$. This can be verified by a simple computation. However, the way to realize that this relation holds is from the fact that the characteristic polynomial of $A$ is $p(r) = (r - 1)^2$ and it is known that a matrix is a "root" of its characteristic polynomial, that is, $p(A) = (A - \mathbb{I})^2 = \mathbb{O}$. Then

$$e^{At} = e^t e^{(A-\mathbb{I})t} = e^t \left( \mathbb{I} + t(A - \mathbb{I}) + \frac{t^2}{2}(A - \mathbb{I})^2 + \cdots \right) = e^t \big( \mathbb{I} + t(A - \mathbb{I}) \big).$$

Therefore,

$$e^{At} = e^{t}\left(\mathbb{I} + t(A - \mathbb{I})\right) = e^{t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}.$$