

Lecture 18: Matrix Exponential

MATH 303 ODE and Dynamical Systems

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Definition

For a real number a we define the exponential through the series

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \frac{1}{4!}a^4 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}a^k.$$

Analogously, for a $n \times n$ matrix A we define the **matrix exponential** e^A through the (matrix) series

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

Properties

1. The series defining e^A converges for all A .
2. $e^{\mathbb{O}} = \mathbb{I}$ where \mathbb{O} is the $n \times n$ zero matrix and \mathbb{I} is the $n \times n$ identity matrix.
3. $\det e^A = e^{\text{tr} A} > 0$, where $\text{tr} A$ denotes the trace of A . Recall that

$$\text{tr} A = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i, \text{ where } \lambda_i \text{ are the eigenvalues of } A.$$

4. If $AB = BA$ then $e^{A+B} = e^A e^B = e^B e^A$ (follows from the series definition and the formula for the Cauchy product of two series).
5. $e^A e^{-A} = \mathbb{I}$, that is, $(e^A)^{-1} = e^{-A}$ (follows easily from property 4).

More properties

6. If $A\mathbf{u} = r\mathbf{u}$ then $e^{A}\mathbf{u} = e^r\mathbf{u}$ (follows easily from the series definition).
7. $e^{U^{-1}AU} = U^{-1}e^AU$ (follows easily from the series definition and the fact that $(U^{-1}AU)^k = U^{-1}A^kU$).
8. $e^{\mathbb{1}t} = e^{t\mathbb{1}}$ (follows easily from the series definition).
9. $e^{A(t+s)} = e^{At}e^{As}$ (follows easily from property 4 in the previous slide).

Diagonal matrix

Consider a diagonal matrix $R = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then $R^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$. We have

$$e^R = \sum_{k=0}^{\infty} \frac{1}{k!} R^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} a^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} b^k \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Note that the same property holds for diagonal matrices of arbitrary size $n \times n$.

Remark. Recall that $e^{U^{-1}AU} = U^{-1}e^AU$. Therefore, if $A = U^{-1}RU$, where R is diagonal, then we have $e^A = e^{U^{-1}RU} = U^{-1}e^RU$.

What this has to do with linear systems?

Theorem. For a $n \times n$ constant matrix A we have

$$\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A.$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}(e^{At}) &= \frac{d}{dt} \left(\mathbb{I} + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \dots \right) \\ &= A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \dots \\ &= A \left(\mathbb{I} + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots \right) = Ae^{At}\end{aligned}$$

Theorem. The unique solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{A(t-t_0)} \mathbf{x}_0.$$

Proof. We have

$$\mathbf{x}'(t) = \frac{d}{dt} \left(e^{A(t-t_0)} \mathbf{x}_0 \right) = \frac{d}{dt} (e^{A(t-t_0)}) \mathbf{x}_0 = A e^{A(t-t_0)} \mathbf{x}_0 = A\mathbf{x}(t),$$

and

$$\mathbf{x}(t_0) = e^{A(t_0-t_0)} \mathbf{x}_0 = e^{\mathbf{0}} \mathbf{x}_0 = \mathbb{I} \mathbf{x}_0 = \mathbf{x}_0.$$

Therefore, $\mathbf{x}(t) = e^{A(t-t_0)} \mathbf{x}_0$ solves the given initial value problem.

Remark. In the particular case where $t_0 = 0$ we have $\mathbf{x}(t) = e^{At} \mathbf{x}_0$.

e^{At} is a fundamental matrix

Recall that we saw earlier that if $X(t)$ is a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ and we have the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ then the solution is

$$\mathbf{x}(t) = X(t)X(0)^{-1}\mathbf{x}_0.$$

Moreover, we just saw that

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

Since these relations hold for arbitrary \mathbf{x}_0 we conclude that

$$e^{At} = X(t)X(0)^{-1}.$$

This relation

$$e^{At} = X(t)X(0)^{-1}$$

shows that e^{At} is also a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ and, more specifically, it is the fundamental matrix which equals \mathbb{I} at $t = 0$.

Moreover, the relation

$$e^{At} = X(t)X(0)^{-1}$$

gives us a recipe for computing e^{At} . Find any fundamental matrix $X(t)$ and then compute $X(t)X(0)^{-1}$.

Generalized eigenvectors

Definition. A non-zero vector $\mathbf{u} \in \mathbb{C}^n$ that satisfies $(A - r\mathbb{I})^m \mathbf{u} = \mathbf{0}$ for some $r \in \mathbb{C}$ and a positive integer m is called a **generalized eigenvector of A associated with r** .

Remark. The case $m = 1$ corresponds to standard eigenvectors.

Remark. If \mathbf{u} is a generalized eigenvector associated with r then r is an eigenvalue of A . The reason for this is that if m is the smallest number for which $(A - r\mathbb{I})^m \mathbf{u} = \mathbf{0}$ (i.e., if we have $\mathbf{w} = (A - r\mathbb{I})^{m-1} \mathbf{u} \neq \mathbf{0}$) then

$$(A - r\mathbb{I})\mathbf{w} = (A - r\mathbb{I})[(A - r\mathbb{I})^{m-1} \mathbf{u}] = (A - r\mathbb{I})^m \mathbf{u} = \mathbf{0},$$

showing that \mathbf{w} is an eigenvector with eigenvalue r .

Generalized eigenvectors

Theorem. If the $n \times n$ matrix A has characteristic polynomial

$$p(r) = (r - r_1)^{m_1} \cdots (r - r_k)^{m_k}$$

with $m_1 + \cdots + m_k = n$, then there exist for each $j = 1, \dots, k$ linearly independent generalized eigenvectors $\mathbf{u}_{j,1}, \dots, \mathbf{u}_{j,m_j}$ with

$$(A - r_j I)^{m_j} \mathbf{u}_{j,\ell} = \mathbf{0} \text{ for } \ell = 1, \dots, m_j.$$

Moreover, the generalized eigenvectors $\{\mathbf{u}_{j,\ell}\}_{j=1,\dots,k;\ell=1,\dots,m_j}$ are linearly independent.

How to compute e^{At}

1. Find linearly independent generalized eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ by solving the corresponding equation for each root of the characteristic polynomial.
2. Compute the solutions $\mathbf{x}_j(t) = e^{At}\mathbf{u}_j$, $j = 1, \dots, n$. Here we will use the fact that for each generalized eigenvector \mathbf{u}_j there is a number k_j such that $(A - r_j I)^{k_j}\mathbf{u}_j = \mathbf{0}$.
3. Write the corresponding fundamental matrix $X(t) = [\mathbf{x}_1(t) \ \dots \ \mathbf{x}_n(t)]$.
4. Compute $e^{At} = X(t)X(0)^{-1}$.

Example

We will start with a non-trivial example to demonstrate how to use the previous theoretical results. Consider the linear system $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We want to compute e^{At} .

The characteristic polynomial is

$$p(r) = -(r - 1)^2 (r - 3).$$

Because $r_1 = r_2 = 1$ is an eigenvalue with multiplicity 2 we try to find generalized eigenvectors that satisfy $(A - \mathbb{1})^2 \mathbf{u} = \mathbf{0}$. We have

$$A - \mathbb{1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad (A - \mathbb{1})^2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

so we get the equation

$$(A - \mathbb{1})^2 \mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We only have the equation $u_1 + 2u_2 = 0$, which implies that u_3 is arbitrary and $u_1 = -2u_2$.

This means we can choose $u_1 = u_2 = 0, u_3 = 1$ and $u_1 = -2, u_2 = 1, u_3 = 0$ to get the linearly independent generalized eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

For the solution $\mathbf{x}_1(t)$ with $\mathbf{x}_1(0) = \mathbf{u}_1$ we have $\mathbf{x}_1(t) = e^{At}\mathbf{u}_1$.

We compute $\mathbf{x}_1(t)$ using the following trick. We know that \mathbf{u}_1 satisfies $(A - \mathbb{1})^2\mathbf{u}_1 = \mathbf{0}$. Therefore,

$$e^{At}\mathbf{u}_1 = e^{\mathbb{1}t+(A-\mathbb{1})t}\mathbf{u}_1 = e^{\mathbb{1}t}e^{(A-\mathbb{1})t}\mathbf{u}_1 = e^t e^{(A-\mathbb{1})t}\mathbf{u}_1.$$

But then we have

$$\begin{aligned} e^{(A-\mathbb{I})t}\mathbf{u}_1 &= \left(\mathbb{I} + t(A - \mathbb{I}) + \frac{t^2}{2}(A - \mathbb{I})^2 + \dots \right) \mathbf{u}_1 \\ &= \mathbf{u}_1 + t(A - \mathbb{I})\mathbf{u}_1 + \frac{t^2}{2}(A - \mathbb{I})^2\mathbf{u}_1 + \dots \\ &= \mathbf{u}_1 + t(A - \mathbb{I})\mathbf{u}_1 \end{aligned}$$

since $(A - \mathbb{I})^k\mathbf{u}_1 = \mathbf{0}$ for $k \geq 2$. What remains is to compute the last expression. We have

$$e^{(A-\mathbb{I})t}\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1(t) = e^{At}\mathbf{u}_1 = e^t e^{(A-\mathbb{1})t}\mathbf{u}_1 = e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

We can work similarly to find

$$\mathbf{x}_2(t) = e^{At}\mathbf{u}_2 = e^t e^{(A-\mathbb{1})t}\mathbf{u}_2.$$

Since $(A - \mathbb{1})^k \mathbf{u}_2 = \mathbf{0}$ for $k \geq 2$ we will get, exactly as for \mathbf{u}_1 , that

$$e^{(A-\mathbb{1})t}\mathbf{u}_2 = \mathbf{u}_2 + t(A - \mathbb{1})\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ t \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_2(t) = e^{At}\mathbf{u}_2 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_2 = e^t \begin{bmatrix} -2 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} -2e^t \\ e^t \\ te^t \end{bmatrix}.$$

Finally, for $r_3 = 3$ which has multiplicity 1, we search for a standard eigenvector. We find the eigenvector

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore, the corresponding solution is

$$\mathbf{x}_3(t) = e^{At}\mathbf{u}_3 = e^{3t}\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2e^{3t} \\ e^{3t} \end{bmatrix}.$$

The corresponding fundamental matrix is

$$X(t) = [\mathbf{x}_1(t) \ \mathbf{x}_2(t) \ \mathbf{x}_3(t)] = \begin{bmatrix} 0 & -2e^t & 0 \\ 0 & e^t & 2e^{3t} \\ e^t & te^t & e^{3t} \end{bmatrix}.$$

Moreover, $X(0) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $X(0)^{-1} = \frac{1}{4} \begin{bmatrix} -1 & -2 & 4 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ and we find

$$e^{At} = X(t)X(0)^{-1} = \frac{1}{4} \begin{bmatrix} 4e^t & 0 & 0 \\ -2e^t + 2e^{3t} & 4e^{3t} & 0 \\ -e^t + e^{3t} - 2te^t & -2e^t + 2e^{3t} & 4e^t \end{bmatrix}.$$

Example

Consider the planar linear system $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We want to compute e^{At} .

The characteristic polynomial is $p(r) = (r - 1)^2$ with $r_1 = r_2 = 1$. Here we can look for generalized eigenvectors as solutions of $(A - \mathbb{I})^2\mathbf{u} = \mathbf{0}$.

We have $(A - \mathbb{I})^2 = \mathbf{0}$ and therefore all non-zero vectors are generalized eigenvectors. Choose $\mathbf{u}_1 = [1 \ 0]^t$ and $\mathbf{u}_2 = [0 \ 1]^t$.

We have

$$e^{At}\mathbf{u}_1 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_1 = e^t (\mathbf{u}_1 + t(A - \mathbb{I})\mathbf{u}_1) = e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

Similarly,

$$e^{At}\mathbf{u}_2 = e^t e^{(A-\mathbb{I})t}\mathbf{u}_2 = e^t (\mathbf{u}_2 + t(A - \mathbb{I})\mathbf{u}_2) = e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} te^t \\ e^t \end{bmatrix}$$

We then consider the fundamental matrix

$$X(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix},$$

and we observe that $X(0) = \mathbb{I}$. Therefore, $e^{At} = X(t)X(0)^{-1} = X(t)$.

Remark. Note that this example is sufficiently simple so that e^{At} can be computed directly through the Taylor series. The crucial fact here is that

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Remark. Another way to obtain the same result is to observe that $(A - \mathbb{1})^2 = \mathbb{0}$. This can be verified by a simple computation. However, the way to realize that this relation holds is from the fact that the characteristic polynomial of A is $p(r) = (r - 1)^2$ and it is known that a matrix is a "root" of its characteristic polynomial, that is, $p(A) = (A - \mathbb{1})^2 = \mathbb{0}$. Then

$$e^{At} = e^t e^{(A - \mathbb{1})t} = e^t \left(\mathbb{1} + t(A - \mathbb{1}) + \frac{t^2}{2}(A - \mathbb{1})^2 + \dots \right) = e^t (\mathbb{1} + t(A - \mathbb{1})).$$

Therefore,

$$e^{At} = e^t (\mathbb{1} + t(A - \mathbb{1})) = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$