Lecture 19: Planar Linear Systems

MATH 303 ODE and Dynamical Systems

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Planar linear systems

We now focus on the case of planar linear systems. In particular, we consider systems of the form

$$\mathbf{x}' = A\mathbf{x},$$

where A is a constant 2×2 matrix and $\mathbf{x} = [x_1 \ x_2]^t$.

To understand all possible dynamics of such systems we will consider different possible cases for the eigenvalues of A, determine the corresponding dynamics, and draw phase portraits.

Characteristic polynomial

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then the characteristic polynomial of A is

$$p(r) = \det(A - r\mathbb{I}) = \det \begin{bmatrix} a - r & b \\ c & d - r \end{bmatrix} = r^2 - (a + d)r + (ad - bc).$$

That is,

$$p(r) = r^2 - Tr + D,$$

where T is the trace of A, and D is its determinant.

Case 1. Real distinct eigenvalues

In this case the discriminant of $p(r)$ is $\Delta = T^2 - 4D > 0$, and there are two real eigenvalues $r_1 \neq r_2$ and corresponding linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2$.

Consider the matrix U whose columns are the two eigenvectors, that is,

$$U = [\mathbf{u}_1 \ \mathbf{u}_2].$$

Then we have $U\mathbf{e}_1 = \mathbf{u}_1$ and $U\mathbf{e}_2 = \mathbf{u}_2$. Moreover, we have

$$AU\mathbf{e}_1 = A\mathbf{u}_1 = r_1\mathbf{u}_1 \ \text{ and } \ AU\mathbf{e}_2 = A\mathbf{u}_2 = r_2\mathbf{u}_2.$$

The relations $AU\mathbf{e}_1 = r_1\mathbf{u}_1$ and $AU\mathbf{e}_2 = r_2\mathbf{u}_2$ imply that

$$AU = [r_1\mathbf{u}_1 \;\; r_2\mathbf{u}_2] = [\mathbf{u}_1 \;\; \mathbf{u}_2]\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = UR,$$

where we have defined $R = \mathrm{diag}(r_1, r_2)$. Therefore,

$$A = URU^{-1} \quad \text{or} \quad R = U^{-1}AU.$$

Using the properties of the matrix exponential we get

$$e^{At} = Ue^{Rt}U^{-1} = U\begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix}U^{-1}.$$

This shows that using the matrix U of eigenvectors we can diagonalize A through a similarity transformation and then easily compute the matrix exponential.

To understand the geometric meaning of this similarity transformation, define new coordinates $\mathbf{z} = [z_1 \ \ z_2]^t$ on the plane by

$$\mathbf{x} = U\mathbf{z}.$$

Then

$$\mathbf{z}' = U^{-1}\mathbf{x}' = U^{-1}A\mathbf{x} = U^{-1}AU\mathbf{z} = R\mathbf{z}.$$

This means that the linear coordinate transformation $\mathbf{x} = U\mathbf{z}$ has the effect of changing the linear system $\mathbf{x}' = A\mathbf{x}$ to the diagonal system $\mathbf{z}' = R\mathbf{z}$ which can be easily solved for $\mathbf{z}$ and then the solutions can be expressed in terms of $\mathbf{x} = U\mathbf{z}$.

Therefore, in the case of real distinct eigenvalues $r_1 \neq r_2$ it is sufficient to understand the dynamics of such diagonal systems with $R = \mathrm{diag}(r_1, r_2)$.

For the diagonal linear system we have the equations

$$z_1' = r_1 z_1, \quad z_2' = r_2 z_2,$$

with solutions

$$z_1(t) = z_1(0)e^{r_1 t}, \quad z_2(t) = z_2(0)e^{r_2 t}\ .$$

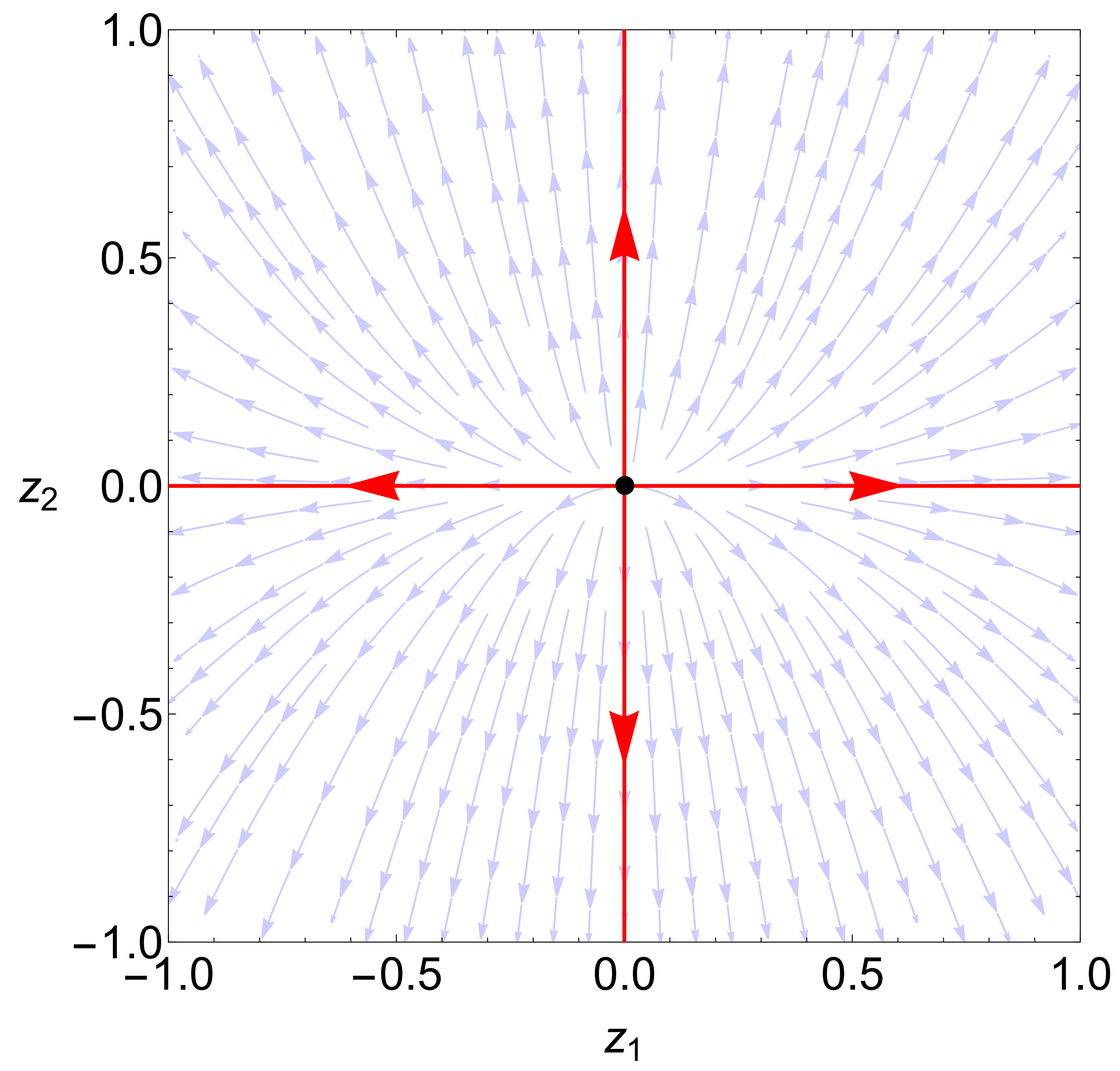
Then we can write $z_1(t)^{r_2} = C_1 e^{r_1 r_2 t}, \quad z_2(t)^{r_1} = C_2 e^{r_1 r_2 t}$. Therefore,

$$\frac{z_1(t)^{r_2}}{z_2(t)^{r_1}} = \frac{C_1}{C_2} = C.$$

The behavior and shape of the solutions will depend on the signs (positive, negative, zero) of r_1 and r_2. We distinguish several cases.

Case 1a. $0 < r_1 < r_2$

We have $\dfrac{z_1(t)^{r_2}}{z_2(t)^{r_1}} = C$ and therefore $z_2(t) = Kz_1(t)^{r_2/r_1}$, where $r_2/r_1 > 0$. We have the phase portrait in the z_1, z_2 plane shown at the right.



Notice that at the origin, the integral curves become tangent to the horizontal axis. This is because $r_2/r_1 > 1$.

Also note that all flow arrows point away from the origin which is an equilibrium.

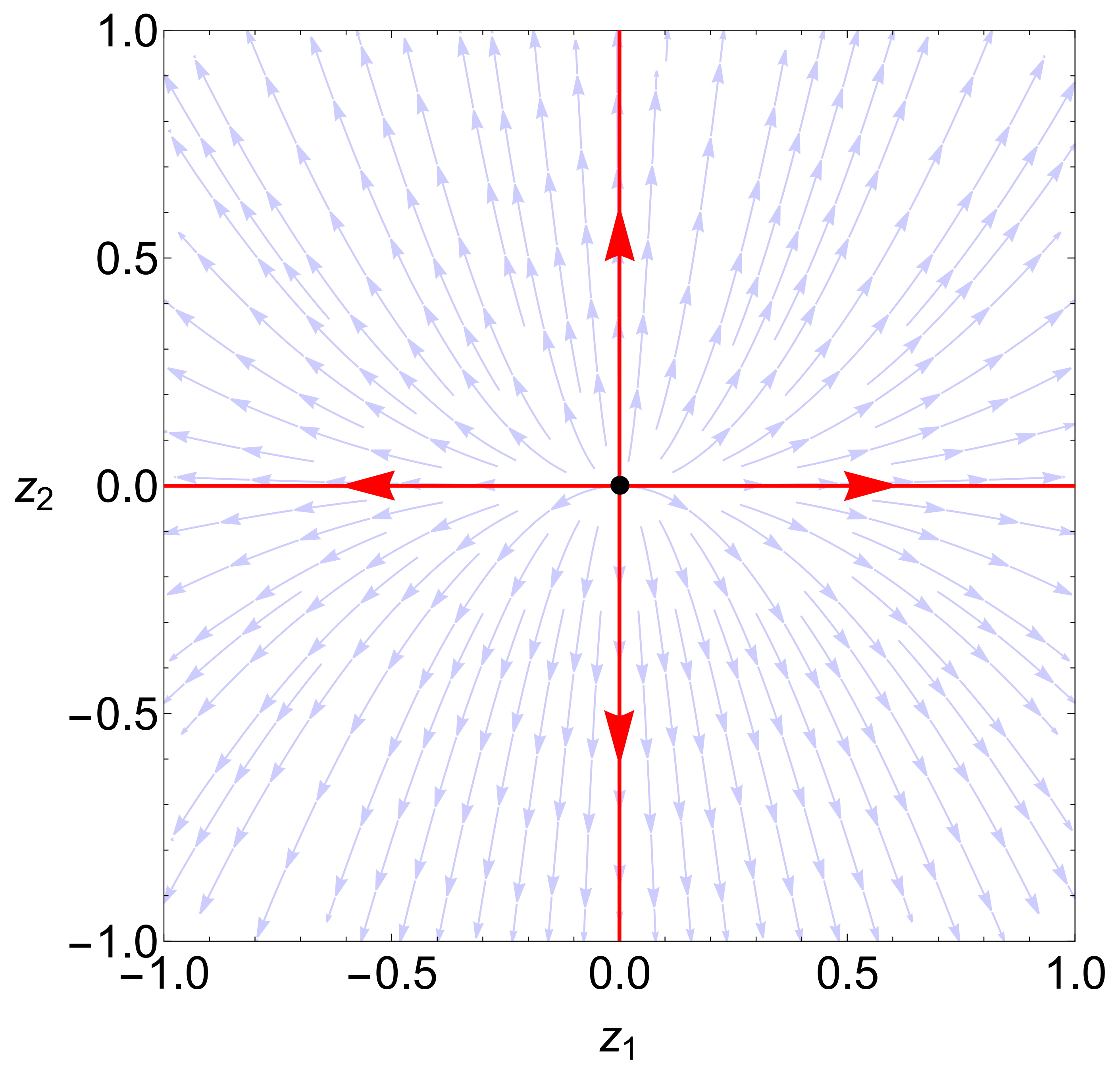
Such an equilibrium is called an **unstable node**.

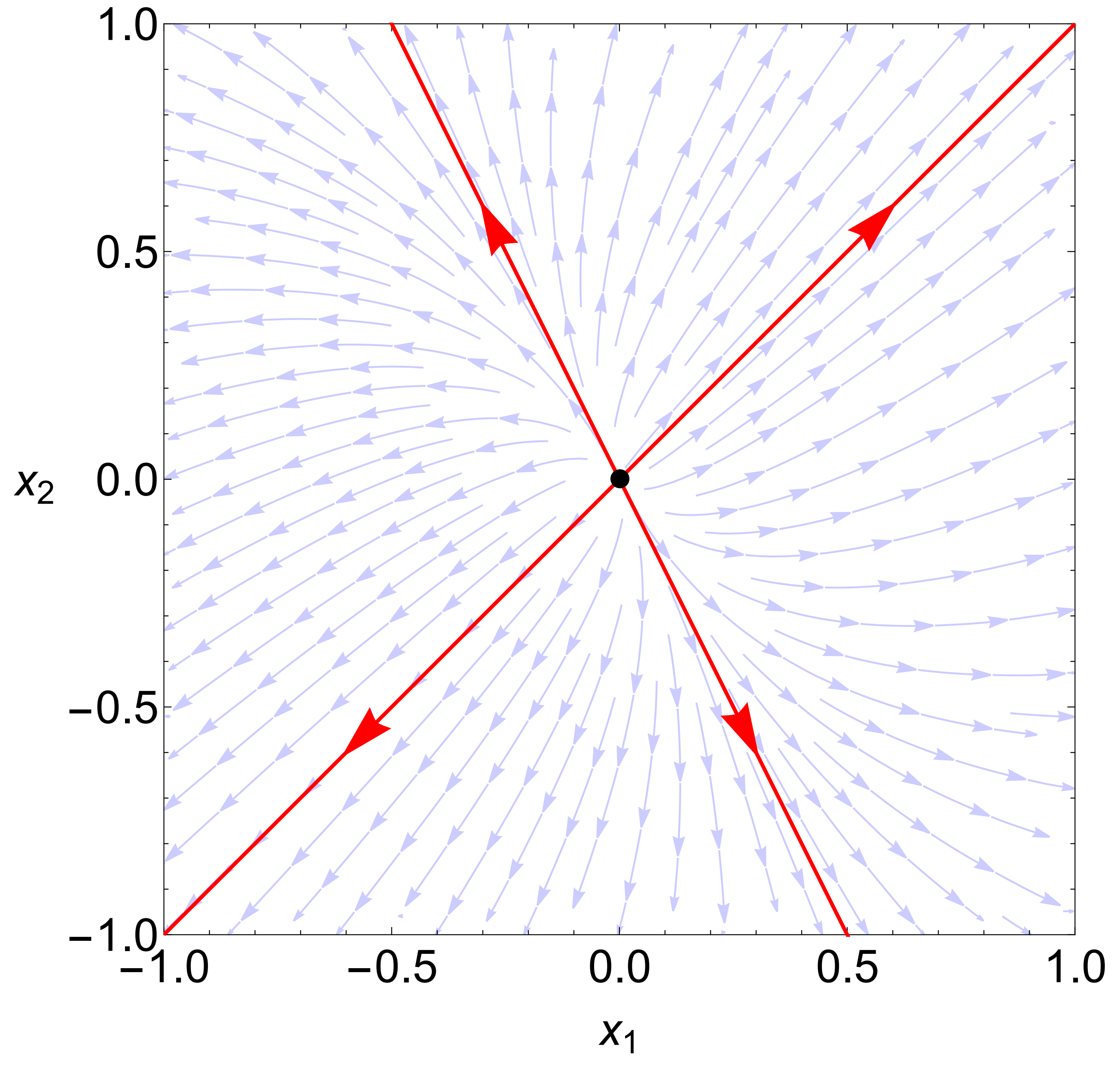
After drawing the phase portrait in the z_1, z_2 plane we can use the matrix $U = [\mathbf{u}_1 \;\; \mathbf{u}_2]$ to transform the phase portrait to the x_1, x_2 plane. Here, and in all the following examples, we have chosen

$$U = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}.$$

The unit vector $\mathbf{e}_1 = [1 \;\; 0]^t$ is sent by U to the vector $\mathbf{u}_1 = [-1 \;\; 2]^t$.

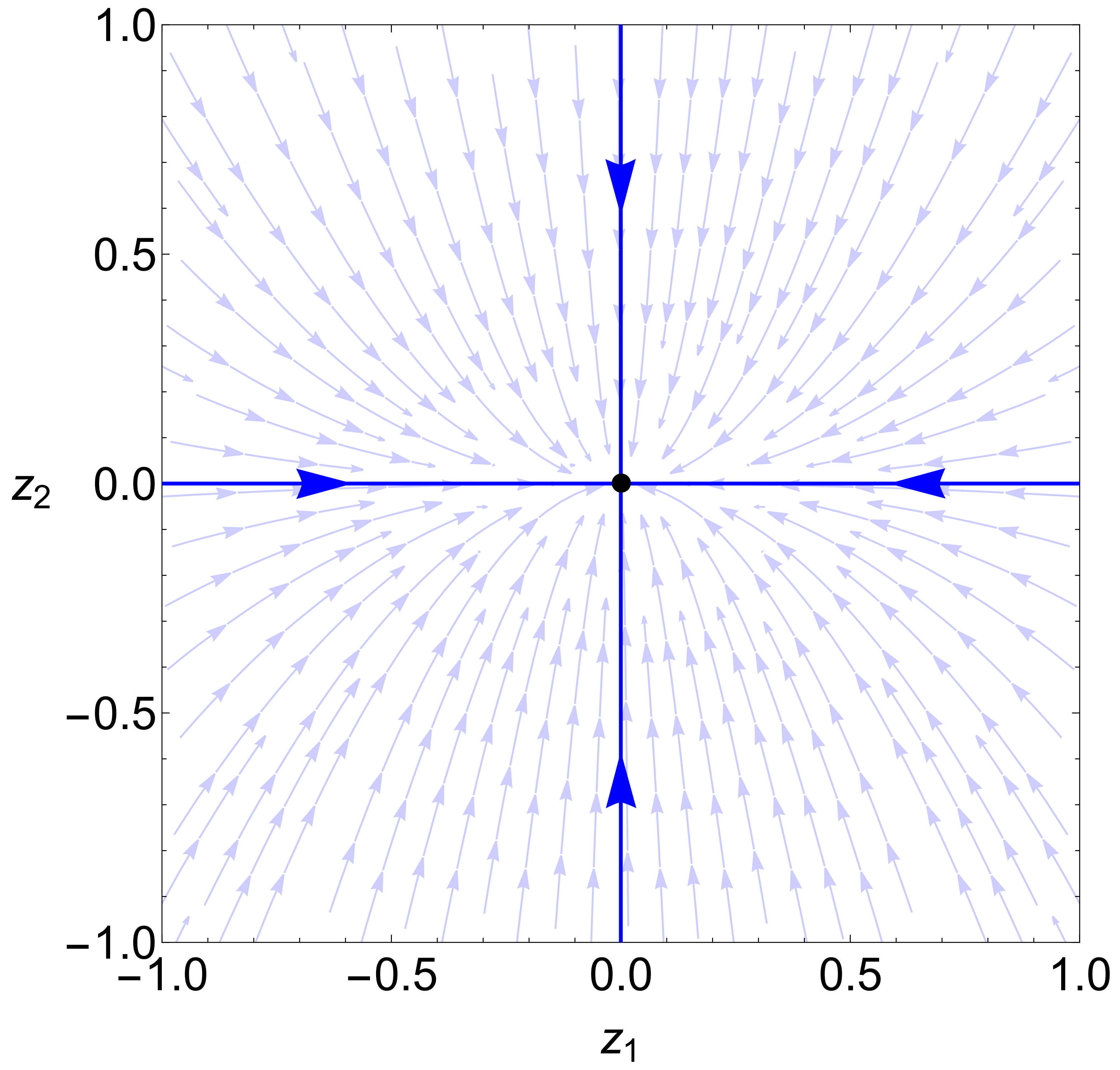
Similarly, $\mathbf{e}_2 = [0 \;\; 1]^t$ is sent by U to the vector $\mathbf{u}_2 = [1 \;\; 1]^t$

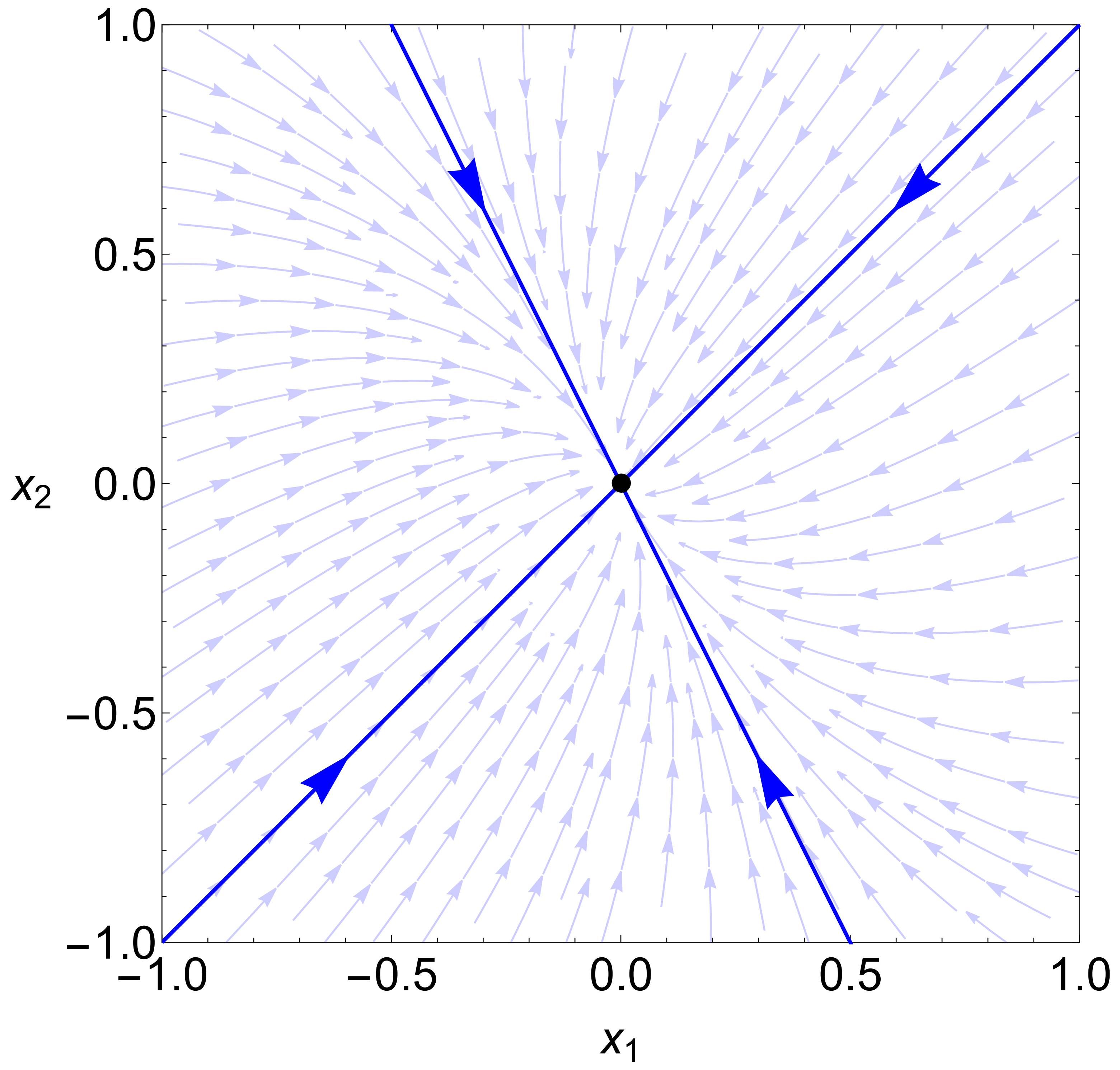




Case 1b. $r_2 < r_1 < 0$

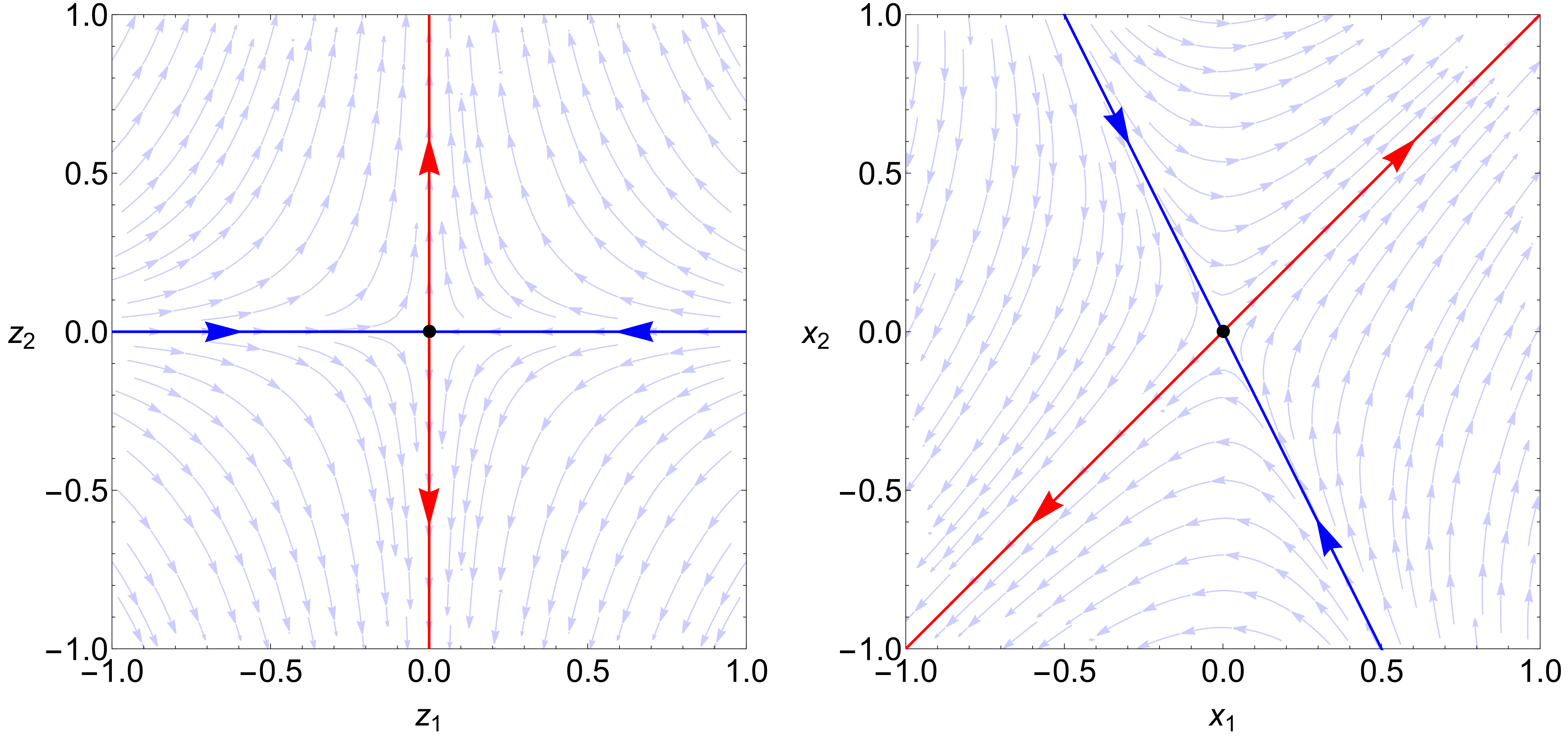
The situation here is exactly the same as in the previous case, except for the directions of the flow arrows. Therefore, we have the following phase portraits. The equilibrium in this case is called a **stable node**.





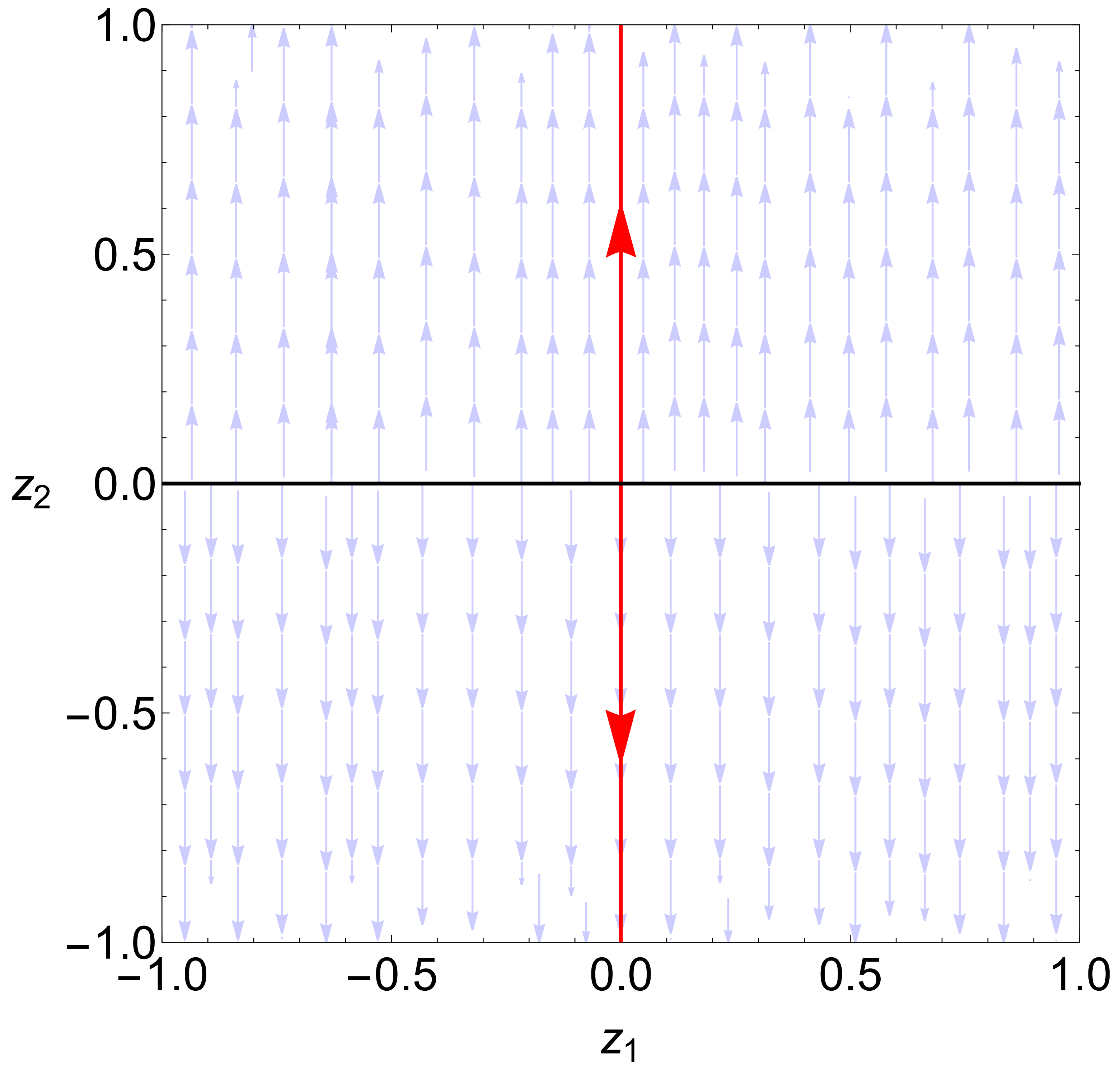
Case 1c. $r_1 < 0 < r_2$

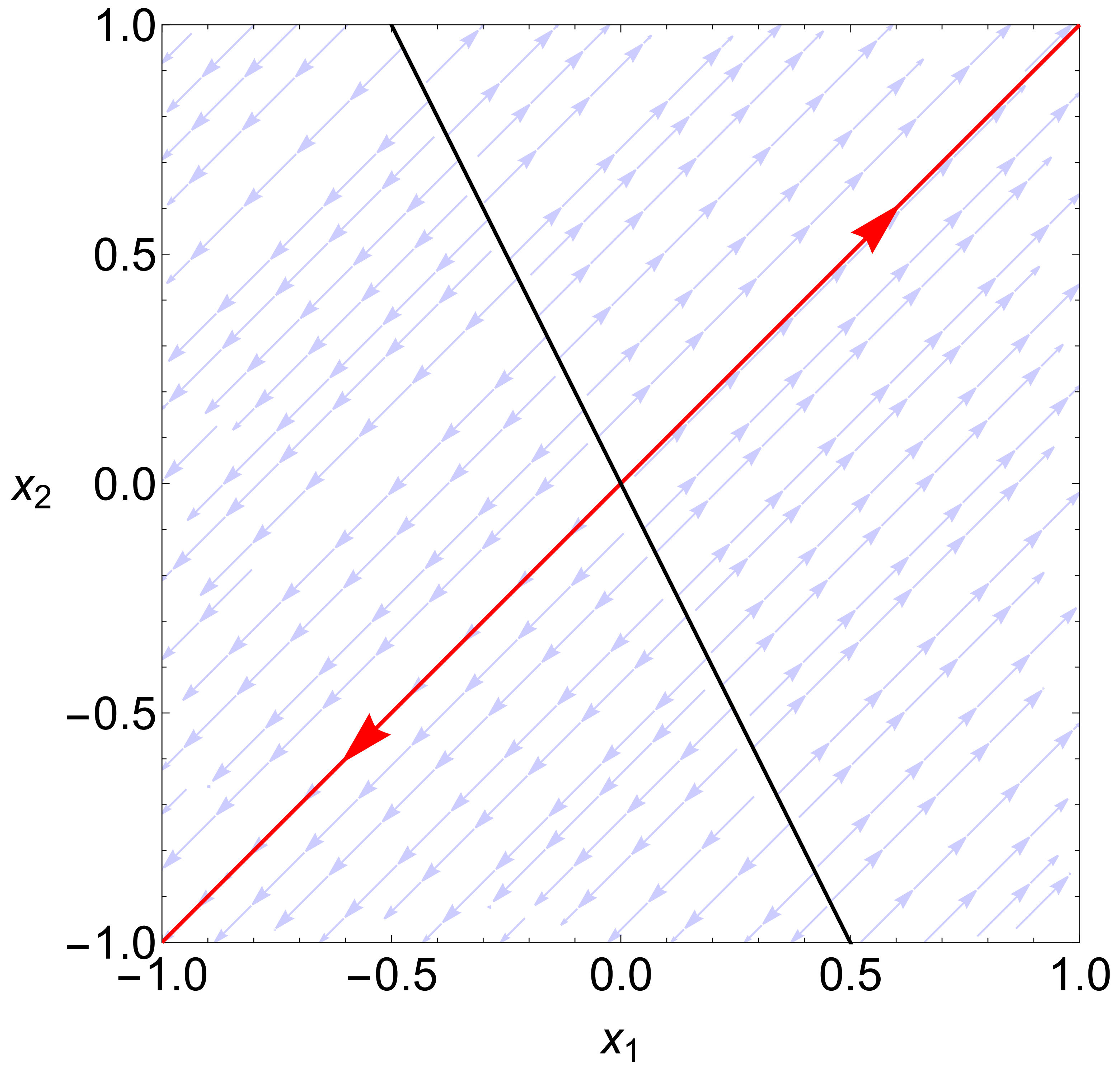
We have $z_1(t)^{r_2}/z_2(t)^{r_1} = C$ and therefore $z_1(t)^{r_2} z_2(t)^{-r_1} = C$, where $-r_1, r_2 > 0$. Therefore, the integral curves on the z_1, z_2 plane are (generalized) hyperbolas. The equilibrium in this case is called a **saddle**. Notice the existence of a stable direction (blue) and an unstable direction (red).



Case 1d. $0 = r_1 < r_2$

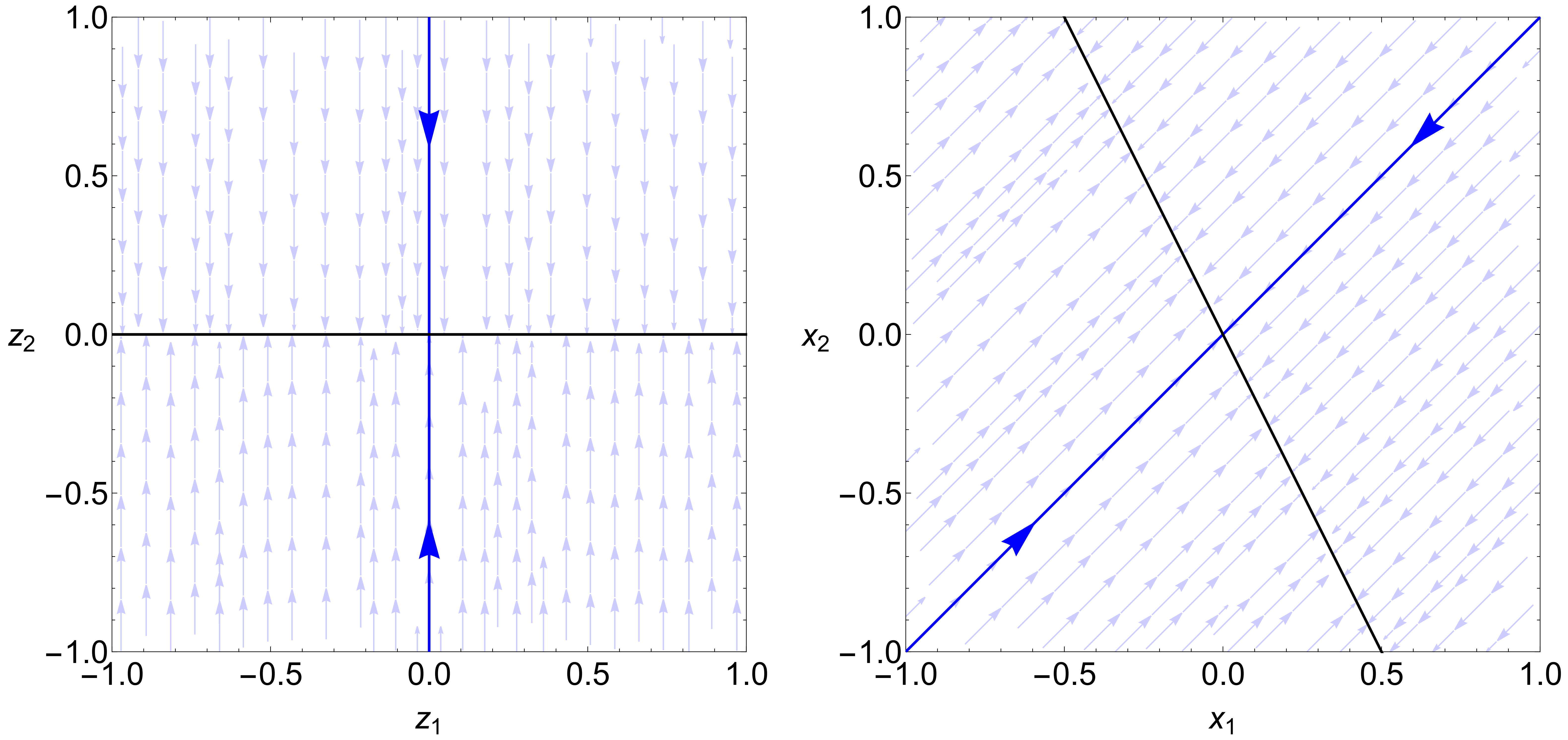
In this case the system is $z_1' = 0,\ z_2' = r_2 z_2$ with solutions $z_1(t) = z_1(0),\ z_2(t) = z_2(0)e^{r_2 t}$. The phase portrait is as follows.





Case 1e. $r_2 < r_1 = 0$

In this case the system is $z_1' = 0,\ z_2' = r_2 z_2$ with solutions $z_1(t) = z_1(0),\ z_2(t) = z_2(0)e^{r_2 t}$. The phase portrait is as follows.



Case 2. Complex conjugate eigenvalues

In this case the discriminant of $p(r)$ is $\Delta = T^2 - 4D < 0$. There are two complex conjugate eigenvalues $\alpha \pm i\beta$ and corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$ such that $\mathbf{a},\ \mathbf{b}$ are linearly independent.

Then we have

$$A(\mathbf{a} + i\mathbf{b}) = (\alpha + i\beta)(\mathbf{a} + i\mathbf{b}) = (\alpha\mathbf{a} - \beta\mathbf{b}) + i(\alpha\mathbf{b} + \beta\mathbf{a}).$$

Comparing real and imaginary parts we find

$$A\mathbf{a} = \alpha\mathbf{a} - \beta\mathbf{b} \ \text{ and } \ A\mathbf{b} = \beta\mathbf{a} + \alpha\mathbf{b}.$$

Consider the matrix U whose columns are $\mathbf{a}$, $\mathbf{b}$, that is,

$$U = [\mathbf{a} \ \mathbf{b}].$$

Then we have $U\mathbf{e}_1 = \mathbf{a}$ and $U\mathbf{e}_2 = \mathbf{b}$. Moreover, we have

$$AU\mathbf{e}_1 = A\mathbf{a} = \alpha\mathbf{a} - \beta\mathbf{b} \ \text{ and } \ AU\mathbf{e}_2 = A\mathbf{b} = \beta\mathbf{a} + \alpha\mathbf{b}.$$

Therefore,

$$AU = [\alpha\mathbf{a} - \beta\mathbf{b} \ \ \beta\mathbf{a} + \alpha\mathbf{b}] = [\mathbf{a} \ \ \mathbf{b}]\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = UR.$$

Here we have

$$e^{Rt} = e^{\alpha t}\begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix}.$$

Therefore,

$$e^{At} = Ue^{Rt}U^{-1} = U\left(e^{\alpha t}\begin{bmatrix}\cos(\beta t) & \sin(\beta t)\\ -\sin(\beta t) & \cos(\beta t)\end{bmatrix}\right)U^{-1}.$$

To understand the dynamics in the z_1, z_2 plane, where $\mathbf{x} = U\mathbf{z}$, we use polar coordinates $r = (z_1^2 + z_2^2)^{1/2}$ and $\theta = \arctan(z_2/z_1)$.

Then we have

$$\theta' = \frac{z_1 z_2' - z_2 z_1'}{z_1^2 + z_2^2} = \frac{z_1(-\beta z_1 + \alpha z_2) - z_2(\alpha z_1 + \beta z_2)}{z_1^2 + z_2^2} = -\beta,$$

and

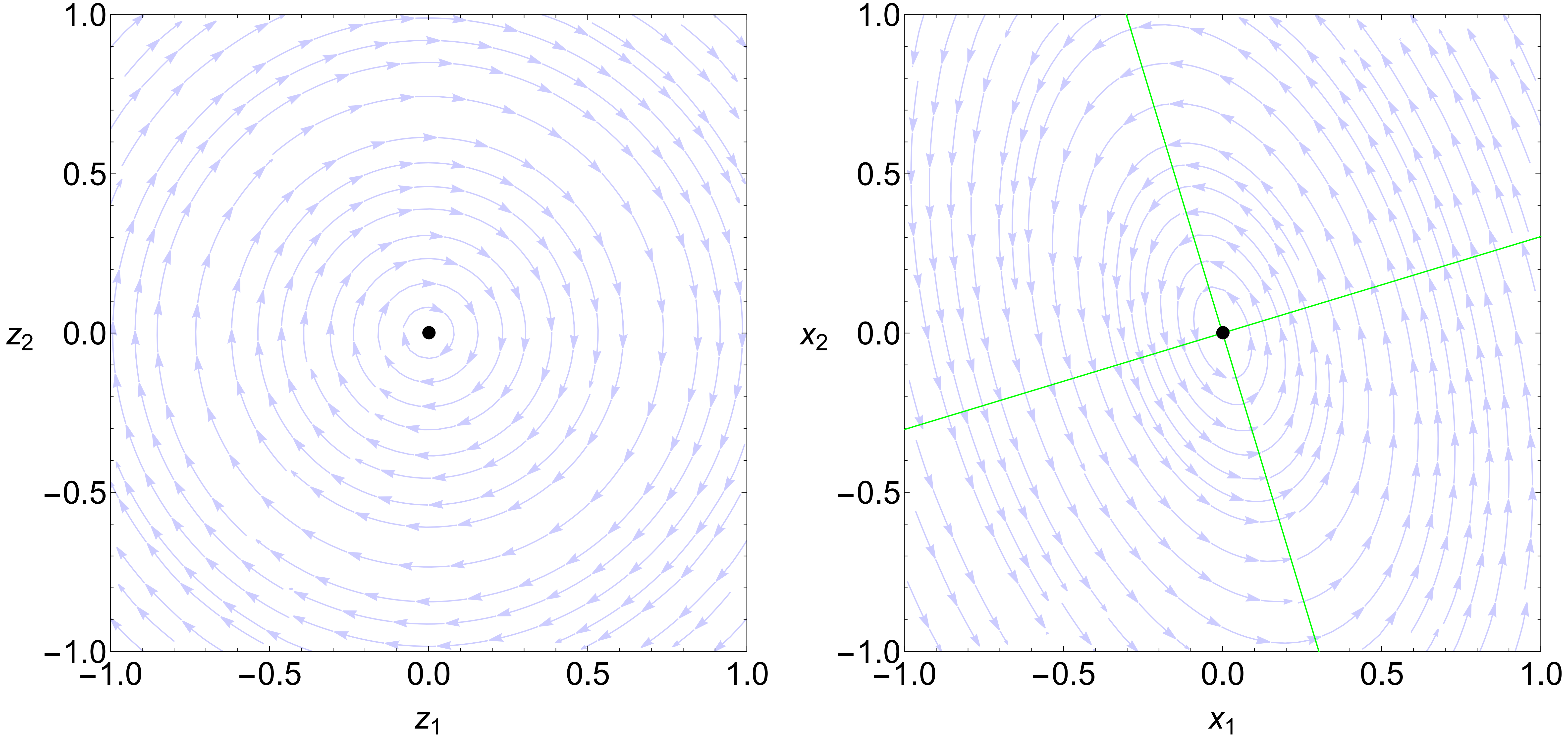
$$r' = \frac{z_1 z_1' + z_2 z_2'}{(z_1^2 + z_2^2)^{1/2}} = \frac{z_1(\alpha z_1 + \beta z_2) + z_2(-\beta z_1 + \alpha z_2)}{(z_1^2 + z_2^2)^{1/2}} = \alpha(z_1^2 + z_2^2)^{1/2} = \alpha r.$$

This means that in the radial direction we have the solution $r(t) = r(0)e^{\alpha t}$ which goes away from the origin ($r = 0$) when $\alpha > 0$, approaches the origin when $\alpha < 0$, and remains constant (so the solutions move on circles) when $\alpha = 0$.

Moreover, assuming $\beta > 0$, θ decreases at a constant rate $-\beta$, and we have $\theta(t) = \theta(0) - \beta t$. The decrease corresponds to clockwise rotation.

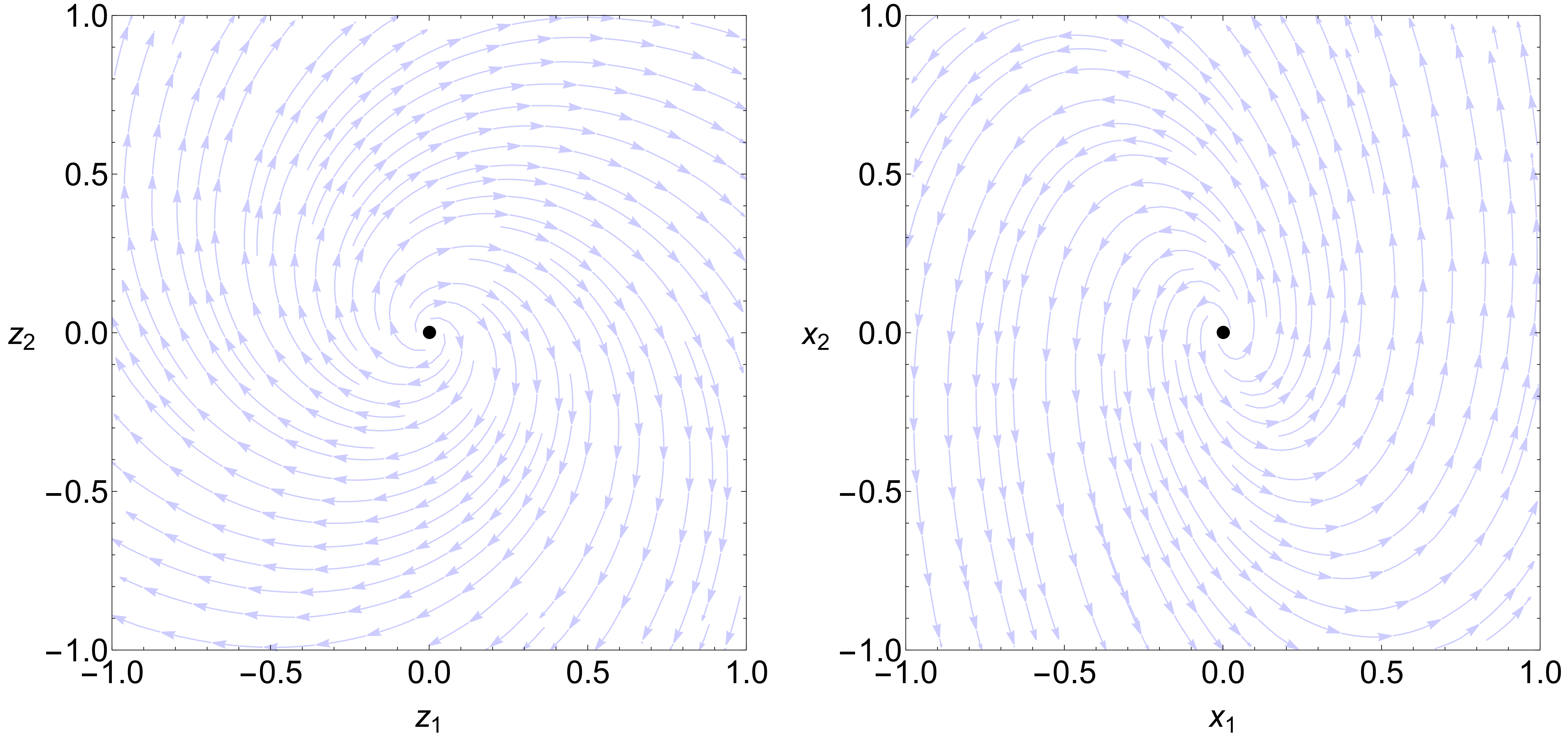
Case 2c. $\alpha = 0$

In the case $\alpha = 0$ all solutions, except for the equilibrium at the origin, are closed curves (periodic solution) with period $T = 2\pi/\beta$. On the z_1, z_2 plane they are circles while on the x_1, x_2 plane they are transformed to ellipses. The equilibrium is called a **center.** The semiaxes of the ellipses are the eigenvectors of the matrix UU^t.



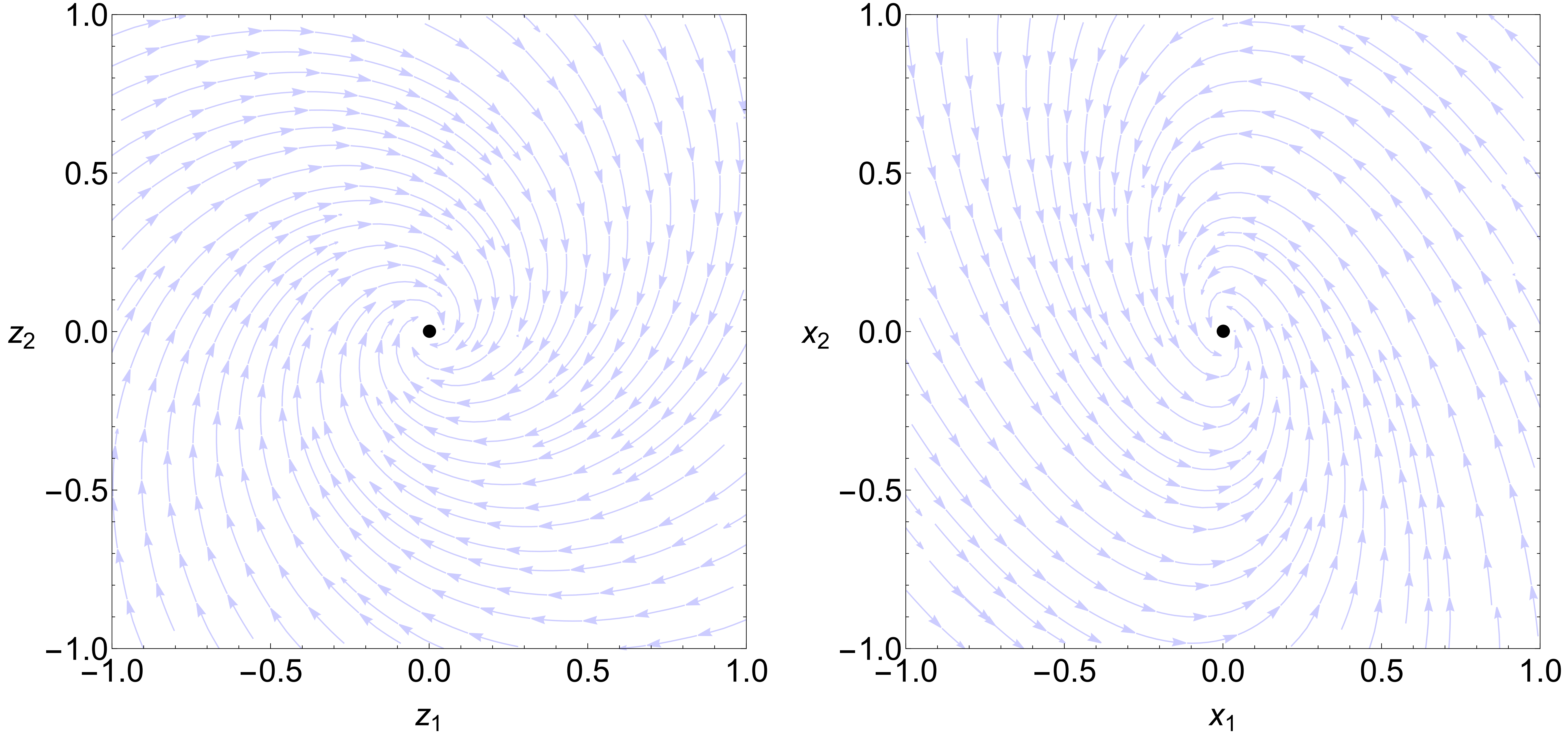
Case 2a. $\alpha > 0$

In this case the solutions rotate around the origin while moving away from it. The equilibrium is called an **unstable spiral**. The phase portraits are shown below.



Case 2b. $\alpha < 0$

In this case the solutions rotate around the origin while moving toward it. The equilibrium is called a **stable spiral**. The phase portraits are shown below.



Rotation direction

In all cases I have drawn the direction of the solutions in the z_1, z_2 plane as clockwise. The reason for this is that if we choose $\beta > 0$ (and for consistency, let's always choose $\beta > 0$) then the rotation is always clockwise since, as we saw earlier, we have $\theta' = -\beta < 0$.

In the x_1, x_2 plane I have drawn all rotations to be counterclockwise. However, this doesn't have to be the case. The direction of the rotation in the x_1, x_2 plane depends on the sign of $\det U$.

If $\det U > 0$ then the rotation in the x_1, x_2 plane has the same direction as the rotation in the z_1, z_2 plane, that is, clockwise.

If $\det U < 0$ then the rotation in the x_1, x_2 plane has the opposite direction from the rotation in the z_1, z_2 plane, that is, counterclockwise.

The deeper reason for this is that the sign of $\det U$ tells us if the corresponding coordinate transformation $\mathbf{x} = U\mathbf{z}$ keeps ($\det U > 0$) or reverses ($\det U < 0$) the orientation on the plane.

For a more computational proof, we work as follows. Let $\phi = \arctan(x_2/x_1)$. Then

$$\phi' = \frac{x_1 x_2' - x_2 x_1'}{x_1^2 + x_2^2} = \cdots = -\frac{\beta}{\det U}\frac{(a_1x_2 - a_2x_1)^2 + (b_1x_2 - b_2x_1)^2}{x_1^2 + x_2^2}.$$

From the expression for ϕ' and assuming that $\beta > 0$ we find that $\phi' < 0$ (clockwise rotation) if $\det U > 0$ and $\phi' > 0$ (counterclockwise rotation) if $\det U < 0$.

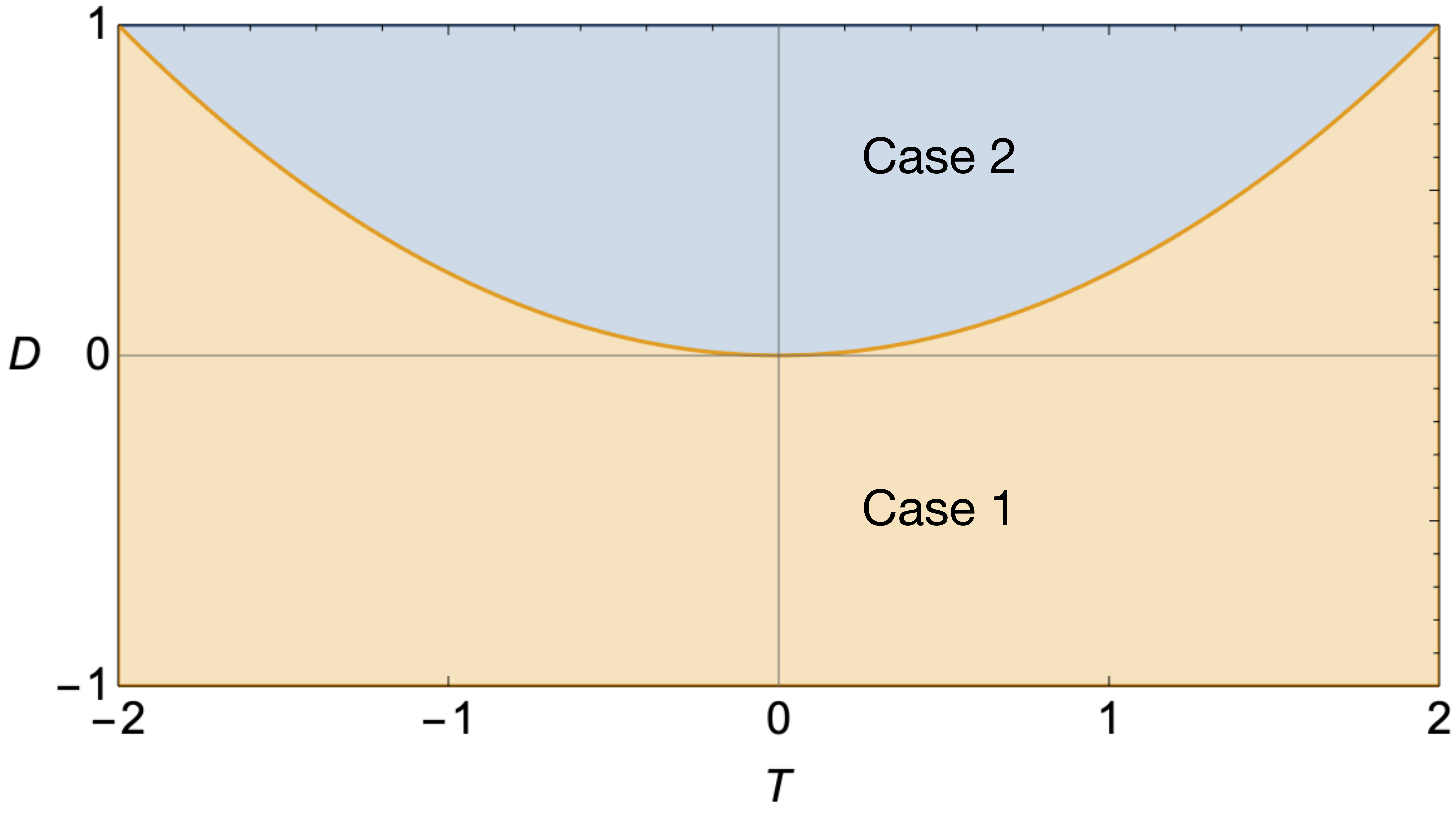
In the examples, I have chosen a matrix U with $\det U = -3$ and this led to the change of rotation direction that can be seen in the pictures.

Classification on the T, D plane

Recall that the characteristic polynomial of A is $p(r) = r^2 - Tr + D$, where T is the trace of A, and D is its determinant.

We have **Case 1** when $\Delta = T^2 - 4D > 0$, i.e., when $D < \frac{1}{4}T^2$.

We have **Case 2** when $\Delta = T^2 - 4D < 0$, i.e., when $D > \frac{1}{4}T^2$.



Case 2
Case 1
1
0
−1
D
−2
−1
0
1
2
T

In **Case 1** the real eigenvalues are

$$\frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

If we assume $\Delta > 0,\ D > 0,\ T > 0$ then we find that both eigenvalues are positive. This is **Case 1a (unstable node)**.

If we assume $\Delta > 0,\ D > 0,\ T < 0$ then we find that both eigenvalues are negative. This is **Case 1b (stable node)**.

If we assume $\Delta > 0,\ D < 0$ then we find that one eigenvalue is negative and the other one is positive. This is **Case 1c (saddle)**.

If we assume $\Delta > 0,\ D = 0,\ T > 0$ then we find that one eigenvalue is positive and the other one is zero. This is **Case 1d**. If we assume $\Delta > 0,\ D = 0,\ T < 0$ then we find that one eigenvalue is negative and the other one is zero. This is **Case 1e**.

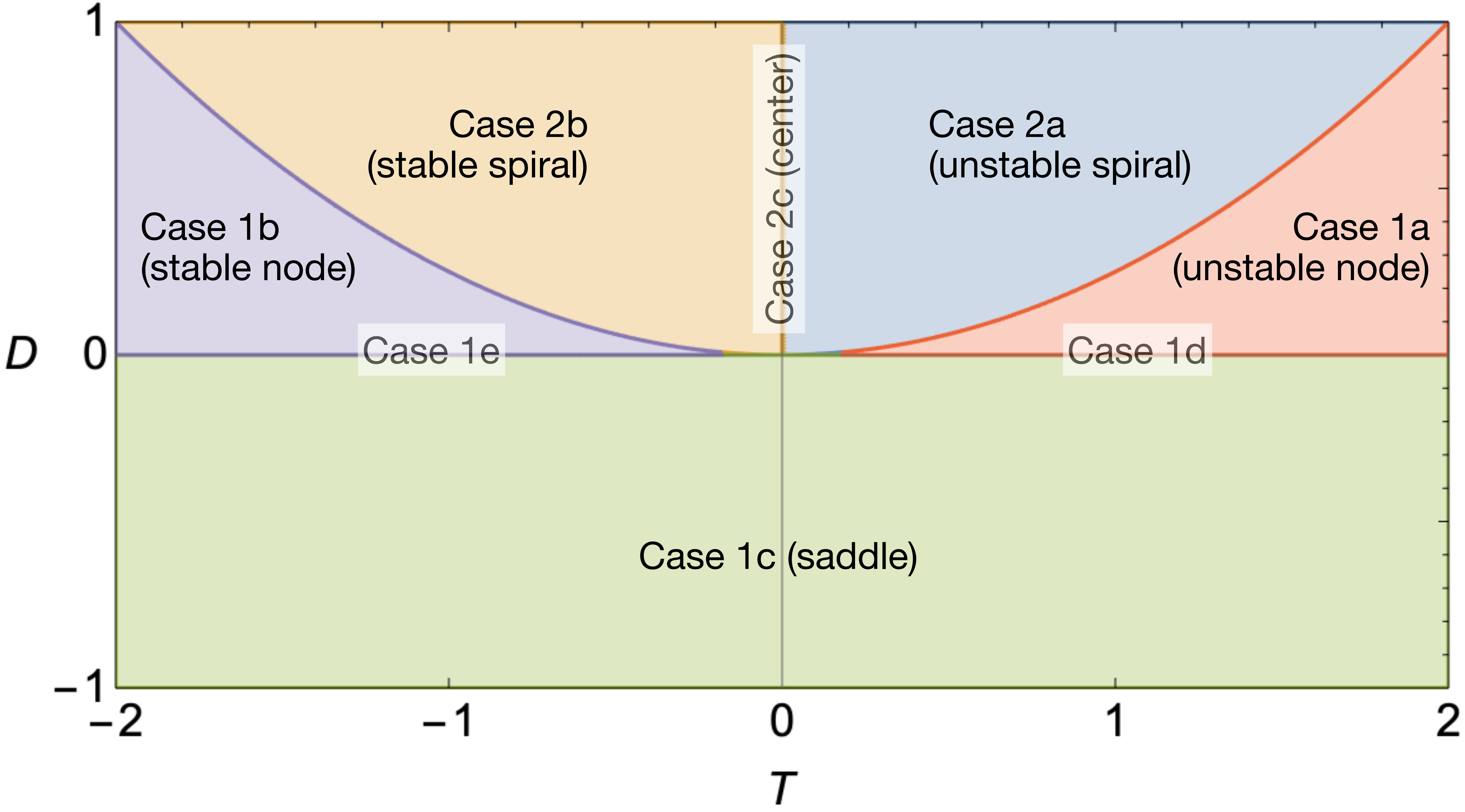
In **Case 2** the complex eigenvalues are $\dfrac{T \pm i\sqrt{4D - T^2}}{2}$. Therefore, $\alpha = T/2$.

If we assume $\Delta < 0,\ T > 0$ then we find that the real part is positive. This is **Case 2a (unstable spiral)**.

If we assume $\Delta < 0,\ T < 0$ then we find that the real part is negative. This is **Case 2b (stable spiral)**.

If we assume $\Delta < 0,\ T = 0$ then we find that the real part is zero. This is **Case 2c (center)**.

We can summarize this classification in the following diagram. The importance of the diagram is that it allows us to directly classify a system in terms of its stability if we know T, D and it also allows us to see how we can change the parameters of a system to obtain different types of equilibria.



Case 2b
(stable spiral)
Case 2c (center)
Case 2a
(unstable spiral)
Case 1b
(stable node)
Case 1a
(unstable node)
Case 1e
Case 1d
Case 1c (saddle)
D
T
1
0
−1
−2
−1
0
1
2