Lecture 20: Almost Linear Systems

MATH 303 ODE and Dynamical Systems

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Planar systems and equilibria

We consider general **planar systems** of the form

$$x_1' = f_1(x_1, x_2)$$
$$x_2' = f_2(x_1, x_2)$$

We will also use vector notation to write the equation such systems by defining $\mathbf{x} = [x_1 \ \ x_2]^t$ and $\mathbf{f}(\mathbf{x}) = [f_1(x_1, x_2) \ \ f_2(x_1, x_2)]^t$, so we can write

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

Recall that an **equilibrium** is a point $\mathbf{x}_e = [x_{e,1} \ \ x_{e,2}]^t$ that satisfies $\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$, that is, $f_1(x_{e,1}, x_{e,2}) = f_2(x_{e,1}, x_{e,2}) = 0$.

Stability of equilibria in planar systems

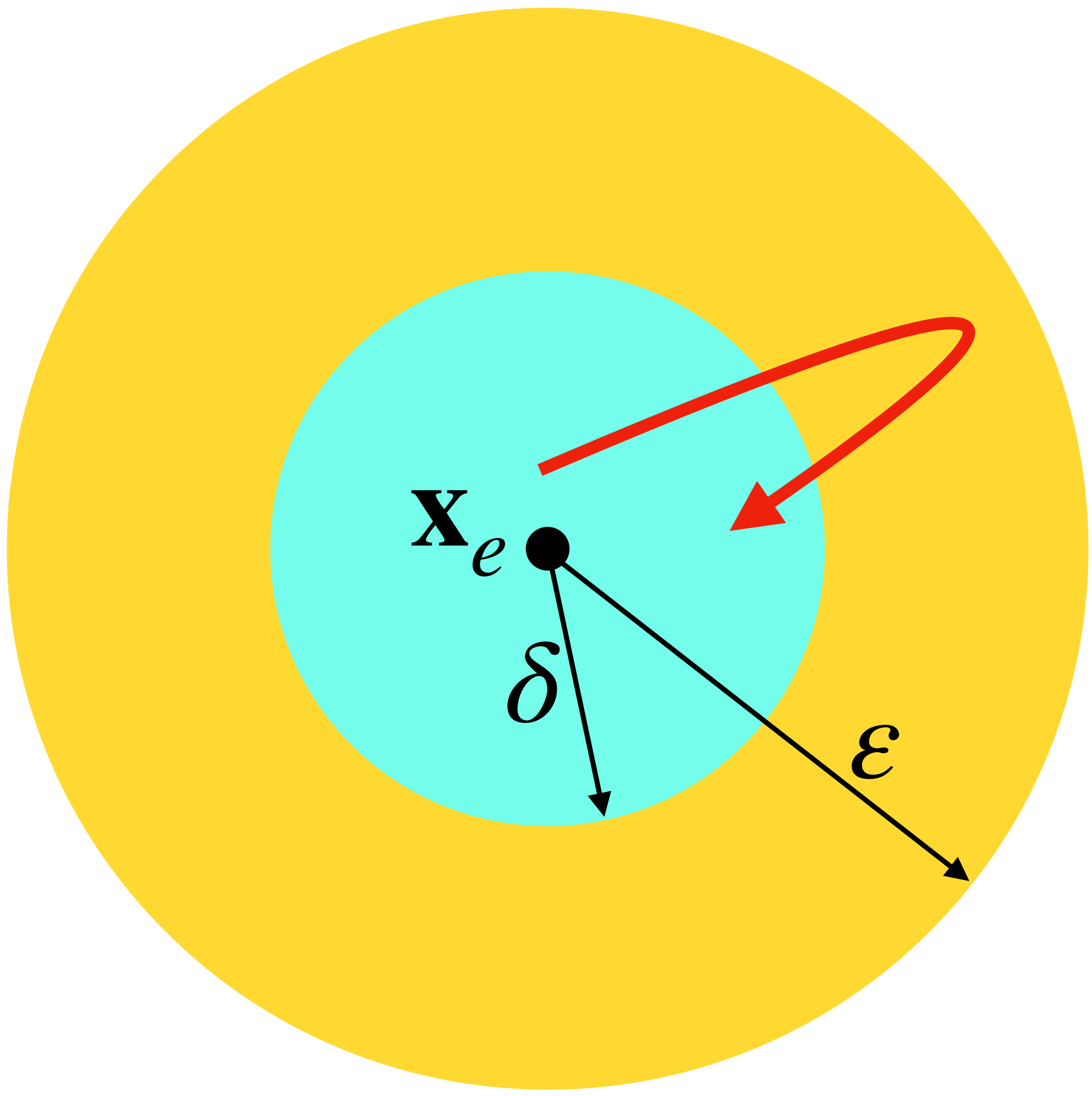
Definition. Given a point $\mathbf{x} \in \mathbb{R}^2$ define the **open disk** of radius $\delta > 0$ centered at $\mathbf{x}$ by $B_\delta(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 \; : \; \|\mathbf{y} - \mathbf{x}\| < \delta\}$.

Here $\|\mathbf{y} - \mathbf{x}\|$ is the **Euclidean distance** in $\mathbb{R}^2$, that is,

$$\|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

Stable equilibria

Definition. An equilibrium $\mathbf{x}_e$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is **stable** if for every $\varepsilon > 0$ there is $\delta > 0$ such that any solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \in B_\delta(\mathbf{x}_e)$ satisfies $\mathbf{x}(t) \in B_\varepsilon(\mathbf{x}_e)$ for all $t \geq 0$.



Unstable & asymptotically stable equilibria

Definition. An equilibrium $\mathbf{x}_e$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is **unstable** if it is not stable, that is, if there is $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is a solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \in B_\delta(\mathbf{x}_e)$ which satisfies $\mathbf{x}(t_0) \notin B_\varepsilon(\mathbf{x}_e)$ for some $t_0 \geq 0$.

Definition. An equilibrium $\mathbf{x}_e$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is **asymptotically stable** if it is stable and if there is $\eta > 0$ such that for all solutions $\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \in B_\eta(\mathbf{x}_e)$ we have $\|\mathbf{x}(t) - \mathbf{x}_e\| \to 0$ as $t \to \infty$.

Poll

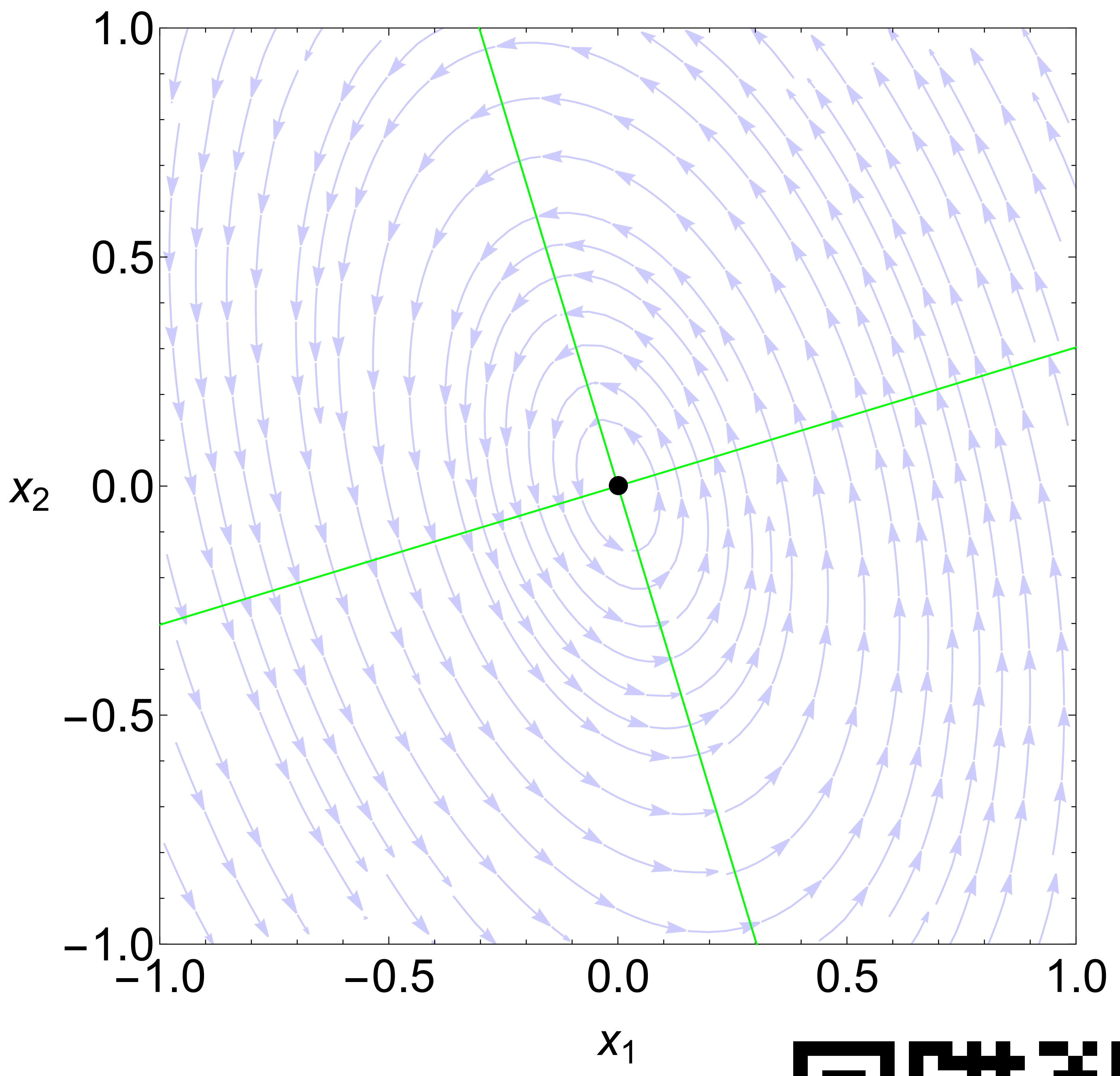
A **center** is:

A. stable but not asymptotically stable

B. asymptotically stable

C. unstable

Choose the correct answer at pollev.com/ke1.



Examples

Asymptotically stable: stable node, stable spiral.

Stable but not asymptotically stable: center.

Unstable: unstable node, unstable spiral, saddle.

Almost linear systems

Recall that we write $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$.

We define the 2×2 **Jacobian matrix** of $\mathbf{f}$ at $\mathbf{x}$ by:

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x_1, x_2) & \dfrac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \dfrac{\partial f_2}{\partial x_1}(x_1, x_2) & \dfrac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix}.$$

Definition. Assume that $\mathbf{0}$ is an equilibrium of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and let $A = D\mathbf{f}(\mathbf{0})$. Moreover, assume that $\mathbf{f}$ is continuous in a neighborhood of $\mathbf{0}$, that $\det A \neq 0$, and that

$$\frac{||\mathbf{f}(\mathbf{x}) - A\mathbf{x}||}{||\mathbf{x}||} \to 0 \text{ as } ||\mathbf{x}|| \to 0.$$

Then the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is called an **almost linear system**.

Remark. If all the partial derivatives $\partial f_i / \partial x_j$ are continuous in a neighborhood of $(0,0)$ then the condition $||\mathbf{f}(\mathbf{x}) - A\mathbf{x}||/||\mathbf{x}|| \to 0$ as $||\mathbf{x}|| \to 0$ is satisfied. Actually, this condition is the definition of the **derivative** $D\mathbf{f}(\mathbf{0})$ and then one shows that if the derivative exists then it must be the Jacobian matrix given in the previous slide.

Proposition. The system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is an almost linear system if $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $\mathbf{f}$ and the partial derivatives $\partial f_i / \partial x_j$ are continuous in a neighborhood of $\mathbf{0}$, and $\det A \neq 0$, where $A = D\mathbf{f}(\mathbf{0})$.

Suppose now that $\mathbf{x}_e$ is an equilibrium of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. Then let $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$. We have

$$\mathbf{y}' = \mathbf{x}' = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_e + \mathbf{y}) =: \mathbf{g}(\mathbf{y}).$$

Definition. The system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is an **almost linear system at $\mathbf{x}_e$** if the system $\mathbf{y}' = \mathbf{g}(\mathbf{y})$, where $\mathbf{g}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_e + \mathbf{y})$, is an almost linear system.

Note that $\mathbf{g}(\mathbf{0}) = \mathbf{f}(\mathbf{x}_e) = \mathbf{0}$ and $D\mathbf{g}(\mathbf{0}) = D\mathbf{f}(\mathbf{x}_e)$.

Hyperbolic linear systems

Definition. A matrix A is called **hyperbolic** if all the eigenvalues of A have non-zero real part.

Definition. An equilibrium $\mathbf{x}_e$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is called **hyperbolic** if $D\mathbf{f}(\mathbf{x}_e)$ is a hyperbolic matrix.

Main Theorem

Theorem. Assume that the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is an almost linear system at $\mathbf{x}_e$, and $\mathbf{x}_e$ is hyperbolic. Then there is (non-linear) coordinate transformation $\mathbf{y} = \varphi(\mathbf{x})$ near $\mathbf{x}_e$ so that $\varphi(\mathbf{x}_e) = \mathbf{0}$ and the dynamics of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is transformed to the dynamics of the system $\mathbf{y}' = A\mathbf{y}$, where $A = D\mathbf{f}(\mathbf{x}_e)$.

Remark. This theorem is known as the **Hartman-Grobman theorem** or the **Linearization theorem**.

Example: Duffing equation

Consider the system

$$x' = y$$
$$y' = -2y + x - x^3$$

The equilibria are $(-1,0),\ (0,0),\ (1,0)$. Note that $f(x, y) = y$ and $g(x, y) = -2y + x - x^3$ are continuous on $\mathbb{R}^2$ and so are all their first partial derivatives.

The Jacobian matrix at arbitrary (x, y) is given by

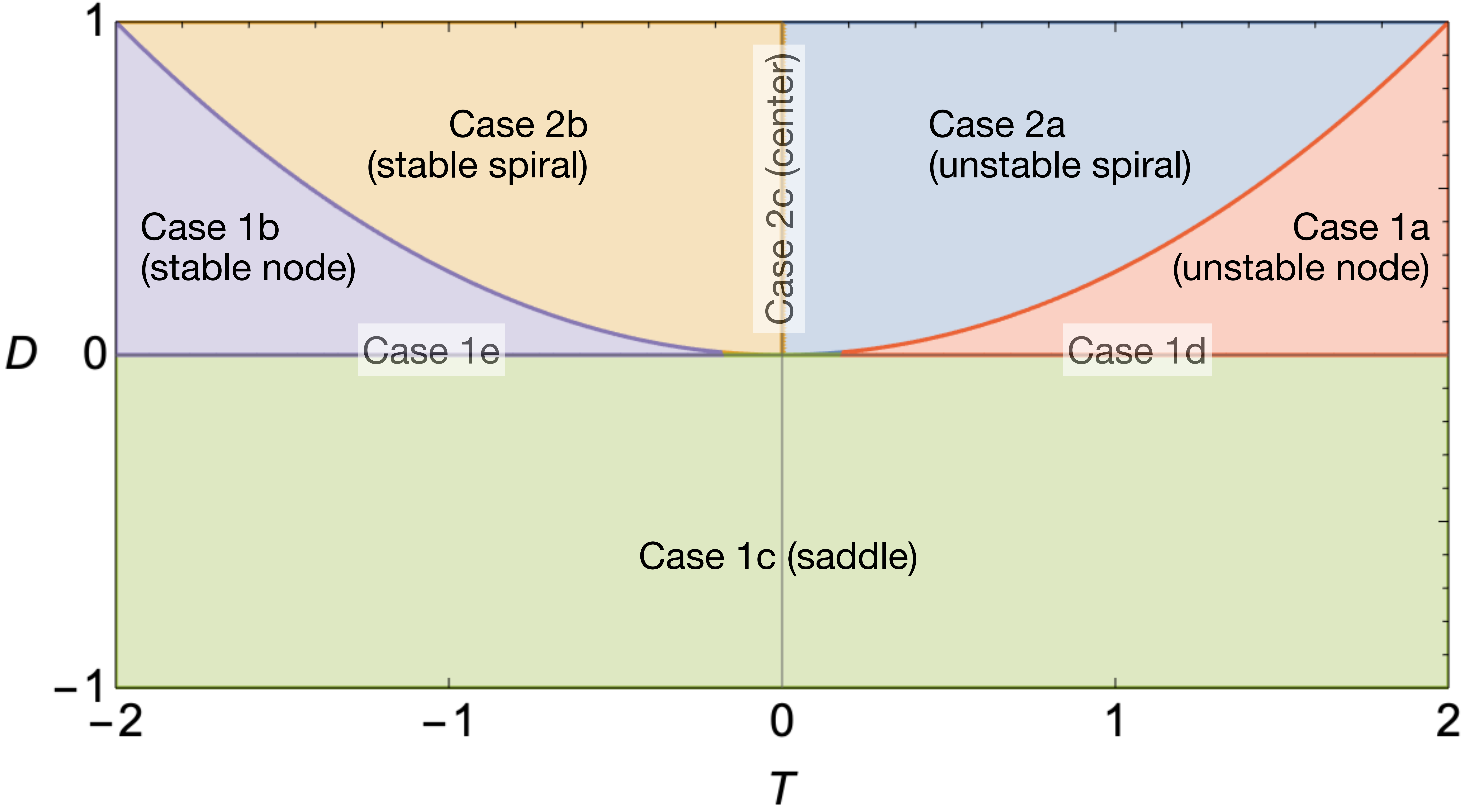
$$D\mathbf{f}(x, y) = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -2 \end{bmatrix}.$$

Therefore,

$$A_0 = D\mathbf{f}(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } A_1 = D\mathbf{f}(-1,0) = D\mathbf{f}(1,0) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

We have $\det A_0 = -1 < 0$. Therefore, $(0,0)$ is a saddle in the linearization and from the Linearization theorem it is a saddle also for the non-linear system.

To find the stable and unstable directions of the saddle we compute the eigenvectors of A_0. These are $\mathbf{u}_1 = [1-\sqrt{2} \quad 1]^t$ with eigenvalue $r_1 = -1-\sqrt{2}$ and $\mathbf{u}_2 = [1+\sqrt{2} \quad 1]^t$ with eigenvalue $r_2 = -1+\sqrt{2}$.



Case 2b
(stable spiral)
Case 2c (center)
Case 2a
(unstable spiral)
Case 1b
(stable node)
Case 1a
(unstable node)
Case 1e
Case 1d
Case 1c (saddle)
D
T
1
0
−1
−2
−1
0
1
2

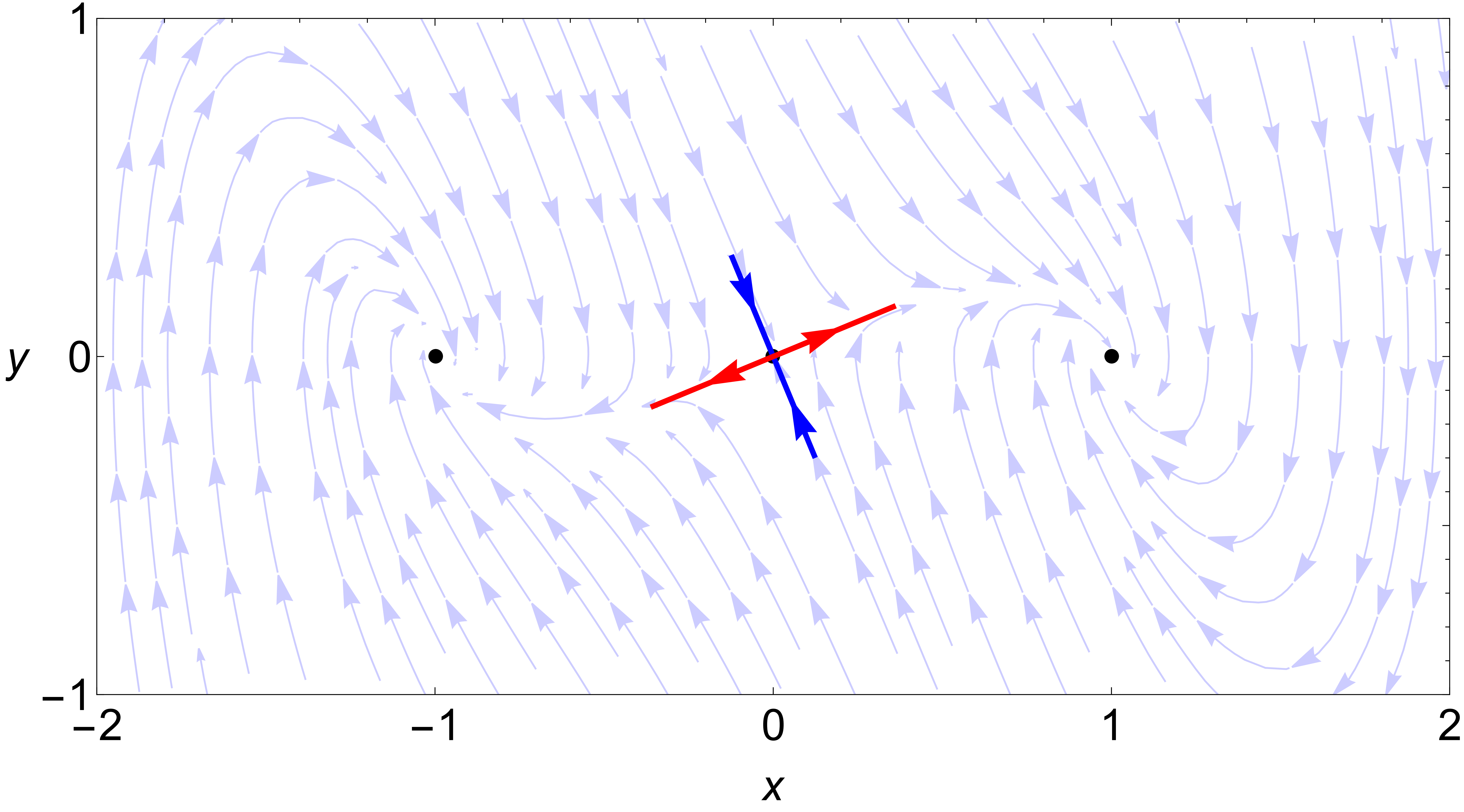
For $(\pm 1,0)$ we have $\det A_1 = 2 > 0$ and $\operatorname{tr} A_1 = -2 < 0$. Therefore in the linearization these points are **stable spirals** since $\det A_1 > \frac{1}{4}(\operatorname{tr} A_1)^2$. Since the equilibria are hyperbolic they will be also stable spirals in the non-linear system.

To find the rotation direction of the spirals we need to compute the real and imaginary part of the eigenvectors of A_1.

The eigenvalues are $-1 \pm i$ and the eigenvector for $-1+i$ is $[1 \quad -1+i]^t$.
Therefore,

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, the transformation matrix $U = [\mathbf{a} \ \mathbf{b}]$ has $\det U = 1 > 0$ and therefore the rotation must be clockwise.



1
0
−1
y
−2
−1
0
1
2
x

Example: Lotka-Volterra model

The Lotka-Volterra population model is given by

$$x' = Ax - Bxy, \quad y' = -Cy + Dxy,$$

where $A, B, C, D > 0$. The equilibria are $(0,0)$ and $\left(\frac{C}{D}, \frac{A}{B}\right)$. The Jacobian matrix is

$$\begin{bmatrix} A - By & -Bx \\ Dy & -C + Dx \end{bmatrix}.$$

Evaluated at the equilibrium $\left(\frac{C}{D}, \frac{A}{B}\right)$ this gives the matrix $M = \begin{bmatrix} 0 & -BC/D \\ DA/B & 0 \end{bmatrix}$.

Here we have $\operatorname{tr} M = 0$ and $\det M = AC > 0$. This shows that in the linear approximation the equilibrium is a center. However, this is a non-hyperbolic equilibrium. Therefore, the Linearization theorem does not tell us whether it is also a center (that is, surrounded by periodic solutions) for the non-linear system.

As we discussed earlier, and as shown in the phase portrait below, the equilibrium is a center also for the non-linear system but we cannot deduce this from the Linearization theorem.