Lecture 21: Energy Method

MATH 303 ODE and Dynamical Systems

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Mechanical systems

We consider planar systems of the form

$$x' = y$$
$$y' = f(x)$$

Remark. Such systems appear commonly in physics (classical mechanics) and they are often called **mechanical systems**. However, even if we get a system of this type that does not arise from a physics problem we can still use the same techniques to analyze its phase portrait.

Potential and energy conservation

Given the function $f(x)$ we define the **potential** $U(x)$ as an antiderivative of $-f(x)$.

Then the **energy** function

$$E(x, y) = \frac{1}{2}y^2 + U(x)$$

is a conserved quantity.

This can be shown as follows:

$$\frac{dE}{dt} = y\frac{dy}{dt} + \frac{dU}{dx}\frac{dx}{dt} = yf(x) + (-f(x))y = 0.$$

Level sets

The fact that $E(x, y)$ is a conserved quantity allows us to easily derive the phase portrait of the original system.

Suppose that we fix the value of $E(x, y)$ to a value h and we want to draw the corresponding level set $E(x, y) = h$.

Solving the equation $h = \frac{1}{2}y^2 + U(x)$ for y we get

$$y = \pm\sqrt{2(h - U(x))}.$$

Remarks

1. The expression for y makes sense only when $h - U(x) \geq 0$. Therefore, the graph of y is defined only over the subsets of the x-axis where $U(x) \leq h$.

2. Since we have $y = \pm\sqrt{2(h - U(x))}$ the level sets consists of two parts, one above the x-axis ($y \geq 0$) and one below the x-axis ($y \leq 0$), and each one of these parts is the reflection of the other.

3. The intersections of the level set $E(x, y) = h$ with the x-axis are the points $(x,0)$ where $E(x,0) = U(x) = h$.

4. The absolute value $|y|$ on a given level set is larger for larger values of $h - U(x)$.

Example

Consider the system

$$x' = y,$$
$$y' = -a^2 x, \quad a > 0.$$

In this case, $f(x) = -a^2 x$ and therefore we can take $U(x) = \frac{1}{2}a^2x^2$. The energy is

$$E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}a^2x^2.$$

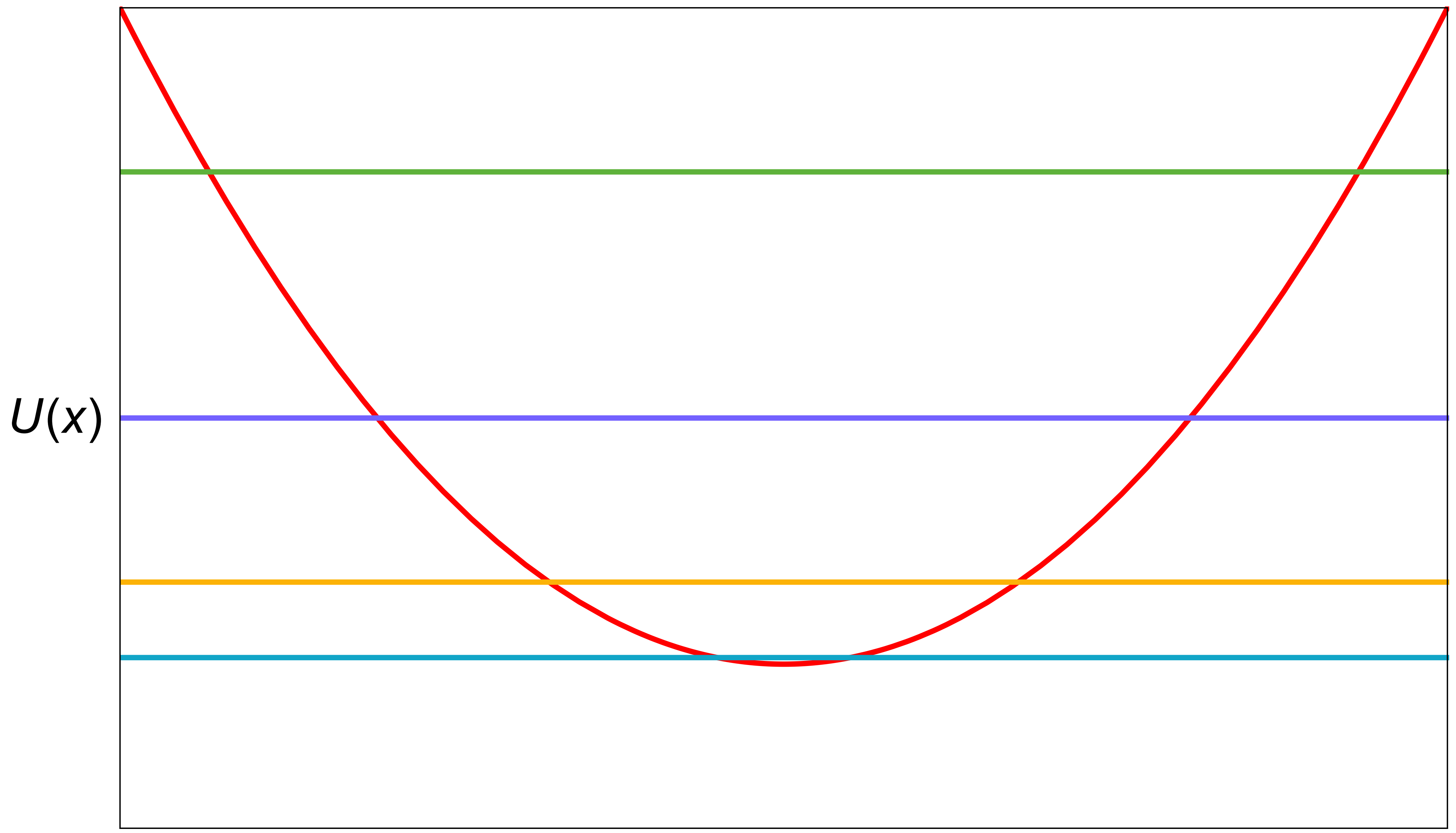
Clearly, the level sets are ellipses for $h > 0$, a single point for $h = 0$, and are not defined for $h < 0$. We will temporarily ignore this fact and try to use the remarks from the previous slide to draw the phase portrait.

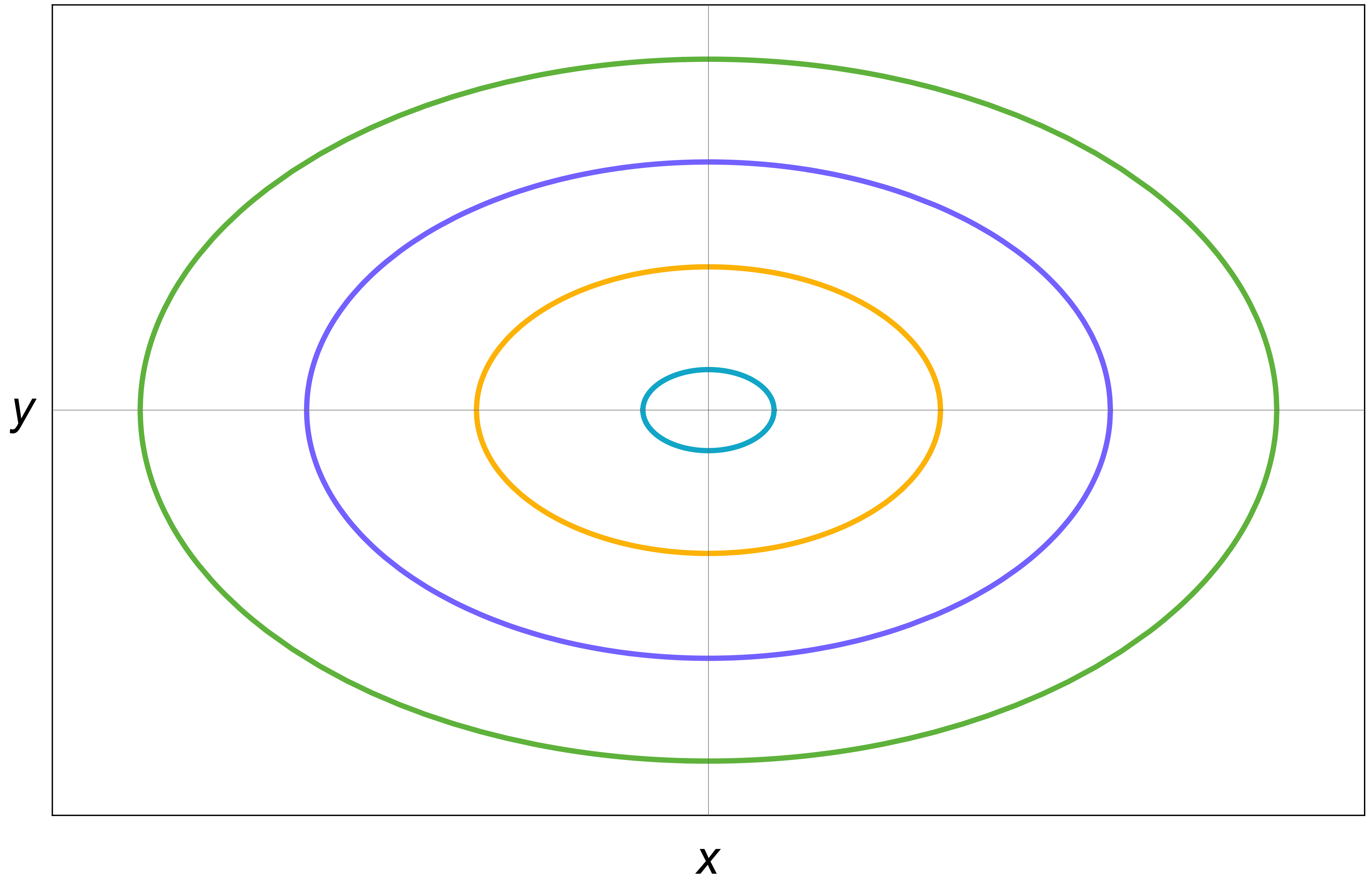
At the right we have drawn the graph of the potential $U(x)$ together with different values of h (top) and the corresponding level curves of $E(x, y)$ (bottom).

Note that for each value of h the corresponding level curve is defined only in the region where $U(x)$ is below h.

All level curves are symmetric under reflections through the x-axis (changing y to $-y$).

Also, the level curves are furthest away from the x-axis at $x = 0$ since at that point the distance between $U(x)$ and h is maximal.





One more remark

Recall that the level set $E(x, y) = h$ intersects the x-axis at points x_i with $U(x_i) = h$. We will show that if $U'(x_i) \neq 0$ then the corresponding level curve (if defined on one side of x_i) has "infinite slope", that is, it is vertical there.

Consider the upper side of the level curve so that $y = \sqrt{2(h - U(x))}$. We find

$$\frac{dy}{dx} = -\frac{U'(x)}{\sqrt{2(h - U(x))}}.$$

Suppose that for $x \leq x_i$ we have $U(x) \leq U(x_i) = h$ and $U'(x_i) > 0$. Then

$$\lim_{x \to x_i^-} \frac{dy}{dx} = -\lim_{x \to x_i^-} \frac{U'(x)}{\sqrt{2(h - U(x))}} = -\infty.$$

Example

Consider the system

$$x' = y,$$
$$y' = b^2 x, \quad b > 0.$$

In this case, $f(x) = b^2 x$ and therefore we can take $U(x) = -\frac{1}{2}b^2x^2$. The energy is

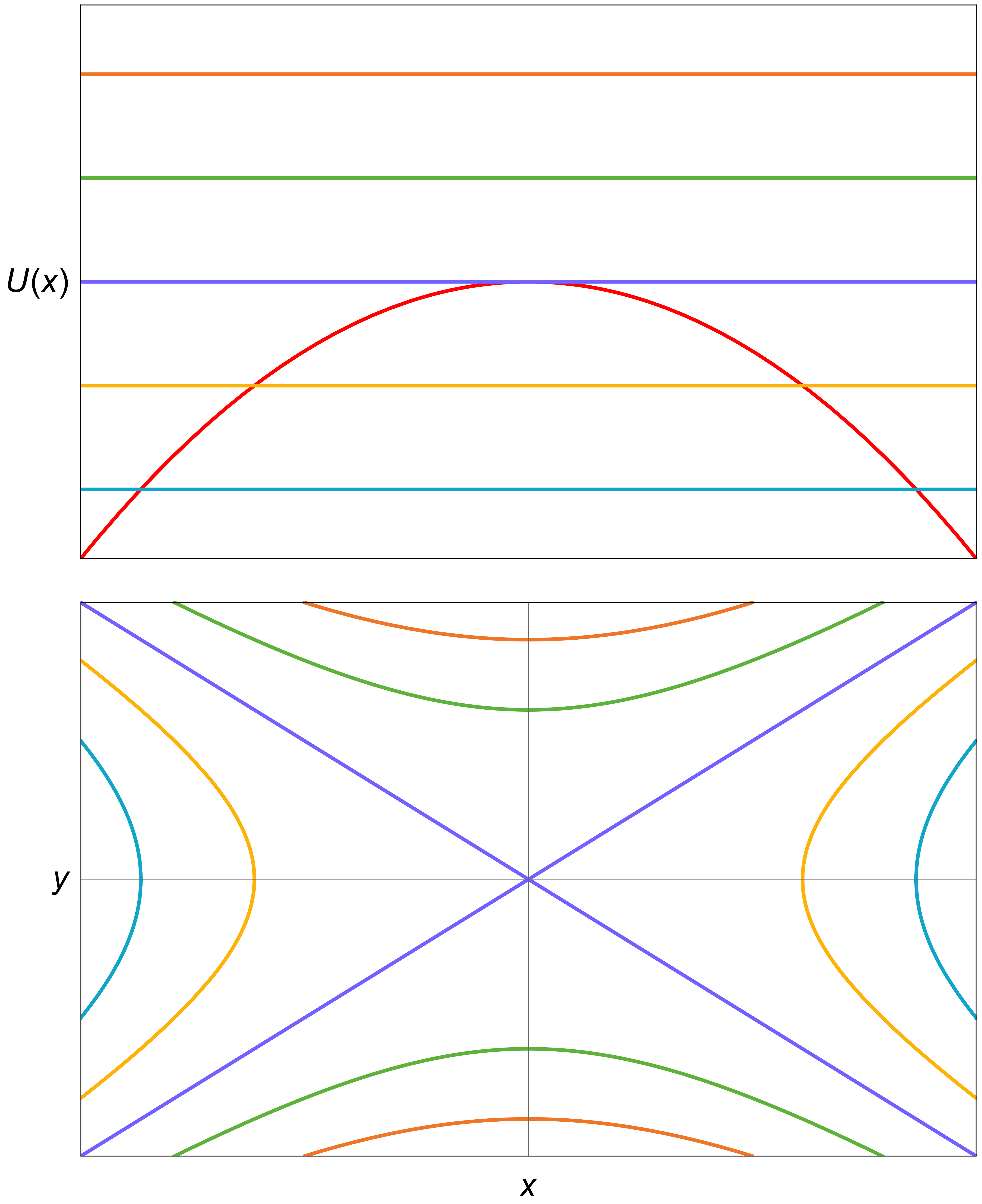
$$E(x, y) = \frac{1}{2}y^2 - \frac{1}{2}b^2x^2.$$

The level sets are hyperbolas when $h \neq 0$. For $h = 0$ we have the straight lines $y = \pm bx$.

At the right we have drawn the graph of the potential $U(x)$ together with different values of h (top) and the corresponding level curves of $E(x, y)$ (bottom).

Note again that for each value of h the corresponding level curve is defined only in the region where $U(x)$ is below h.

The level curve for $h = 0$ (blue) does not become vertical at the origin. This happens because $U'(0) = 0$ and therefore the previous argument does not apply.



Equilibria

The equilibria of the system

$$x' = y$$
$$y' = f(x)$$

are the points $(x_e, 0)$ for which $f(x_e) = 0$. Since $U'(x) = -f(x)$ we conclude that at an equilibrium we have $U'(x_e) = 0$.

This means that critical points of the potential $U(x)$ correspond to equilibria of the system. In particular, maxima and minima of $U(x)$ correspond to equilibria.

Linearization

The Jacobian matrix corresponding to $\mathbf{F} = [y\ f(x)]^t$ is given by

$$D\mathbf{F}(x, y) = \begin{bmatrix} 0 & 1 \\ f'(x) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -U''(x) & 0 \end{bmatrix}.$$

Assuming that $f(x)$ and $f'(x)$ are continuous functions in a neighborhood of a point x_e with $f(x_e) = 0$, and that $U''(x_e) \neq 0$ we conclude that the system is almost linear at the equilibrium $(x_e,0)$.

We distinguish two cases. First, if $U''(x_e) > 0$ then the potential has a minimum at x_e.

The eigenvalues of the corresponding linear system are $\pm i\sqrt{U''(x_e)}$ and therefore the equilibrium for the linear system is a center. The linearization theorem cannot be used in this case.

The Taylor series of the potential $U(x)$ up to quadratic terms is

$$U(x) \approxeq U(x_e) + \frac{1}{2}U''(x_e)(x - x_e)^2.$$

This quadratic expression is essentially the expression $U(x) = \frac{1}{2}a^2x^2$ that we met in an earlier example.

Even though the linearization theorem does not allow us to determine the stability of the equilibrium, because $E(x, y)$ is a conserved quantity and near the equilibrium it is approximately

$$E(x, y) \approxeq U(x_e) + \frac{1}{2}y^2 + \frac{1}{2}U''(x_e)(x - x_e)^2$$

we conclude that the level curves near $(x_e, 0)$ are approximate ellipses.

Second, if $U''(x_e) < 0$ then the potential has a maximum at x_e.

The eigenvalues of the corresponding linear system are $\pm\sqrt{-U''(x_e)}$ and therefore the equilibrium for the linear system is a saddle. The linearization theorem then ensures that the equilibrium $(x_e, 0)$ is also a saddle for the full system.

We can work similarly as for the case $U''(x_e) > 0$ to write

$$U(x) \approxeq U(x_e) + \frac{1}{2}U''(x_e)(x - x_e)^2.$$

Since $U''(x_e) < 0$ this quadratic expression is essentially the same as the expression $U(x) = -\frac{1}{2}b^2x^2$ that we also met in an earlier example.

Example

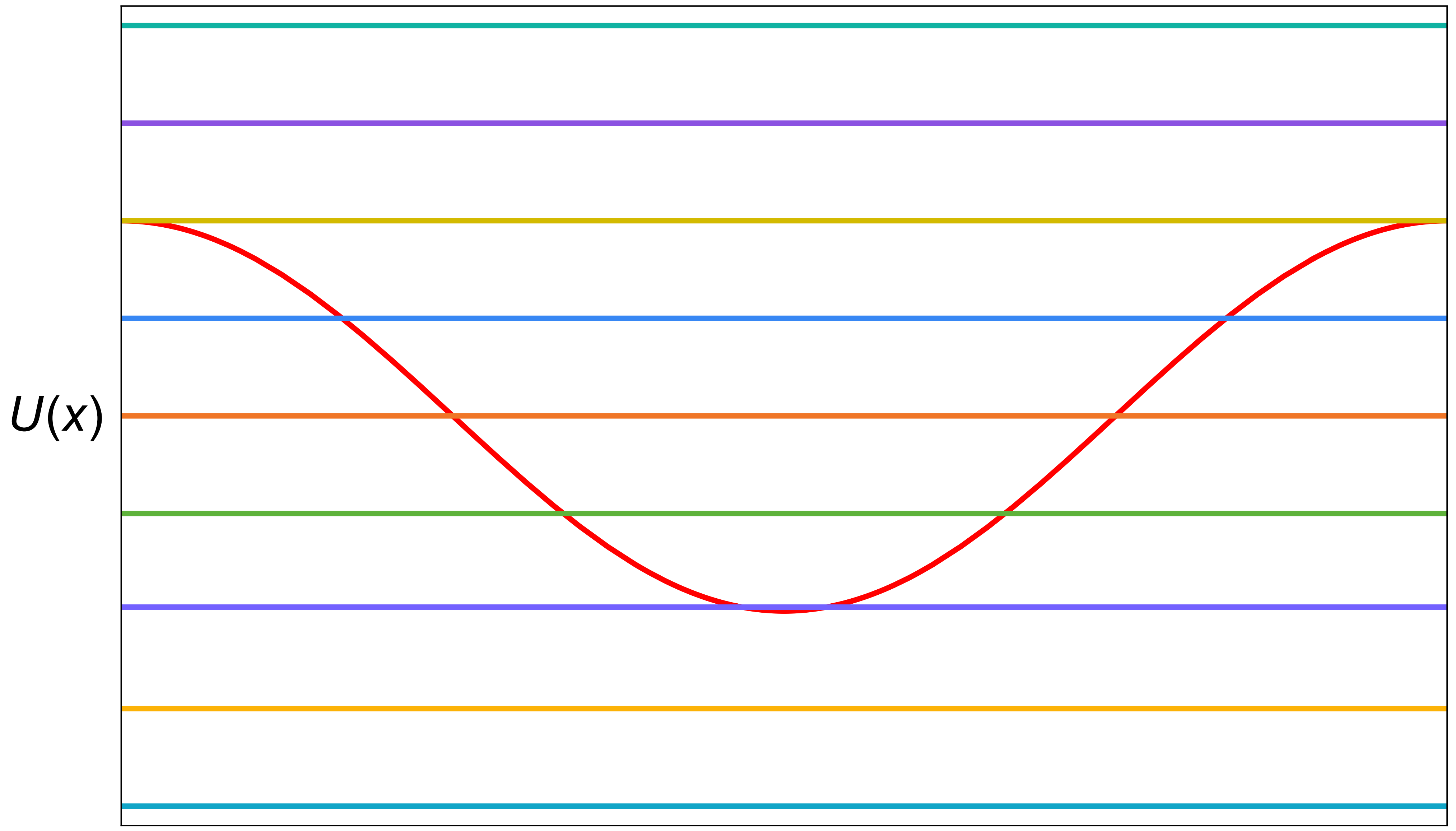
Consider the equations describing the motion of a pendulum:

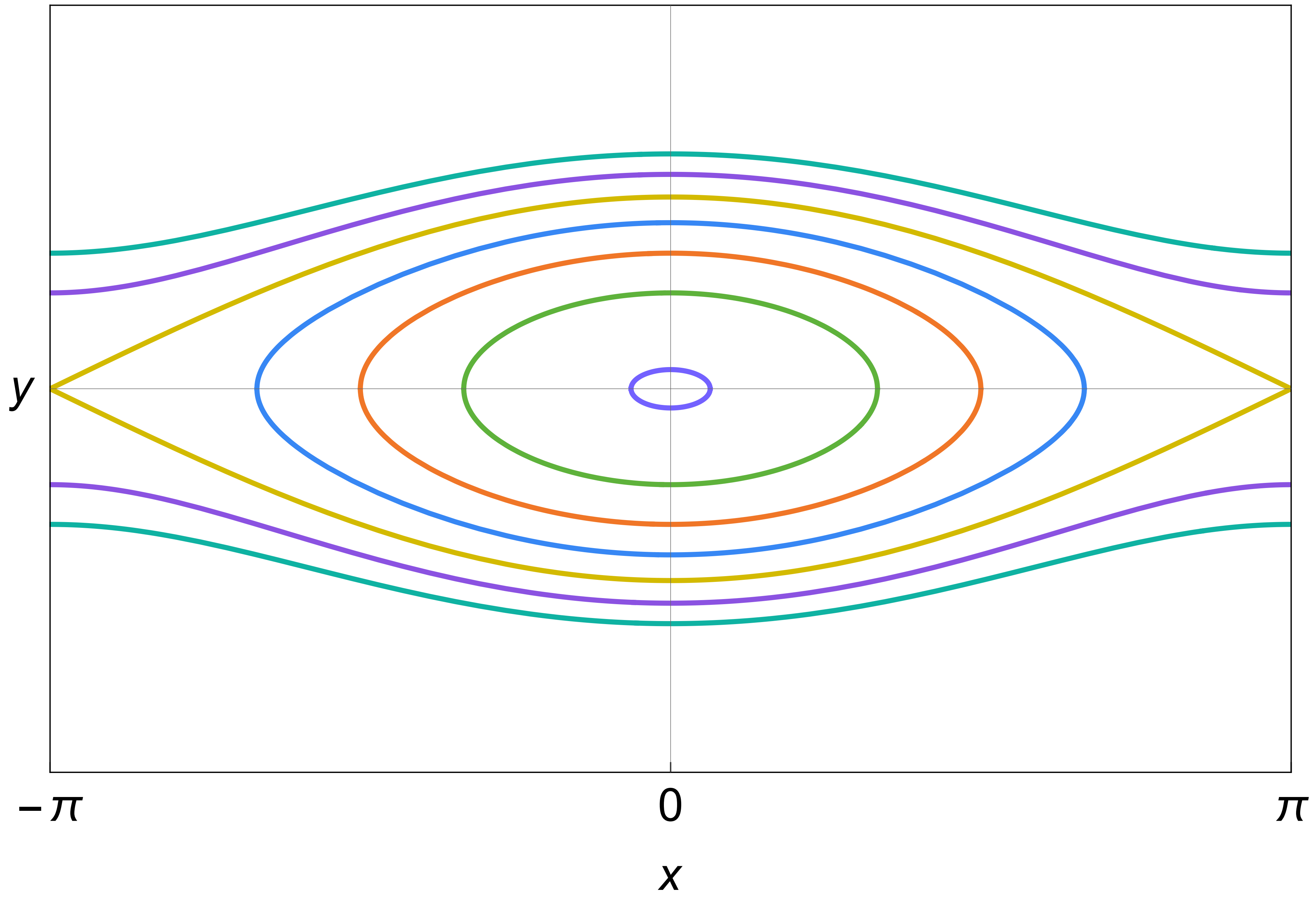
$$x' = y,$$
$$y' = -\sin x\,.$$

In this case, $f(x) = -\sin x$ and therefore we can take $U(x) = -\cos x$. The energy is

$$E(x, y) = \frac{1}{2}y^2 - \cos x.$$

The level curves for different values of h are shown at the right.



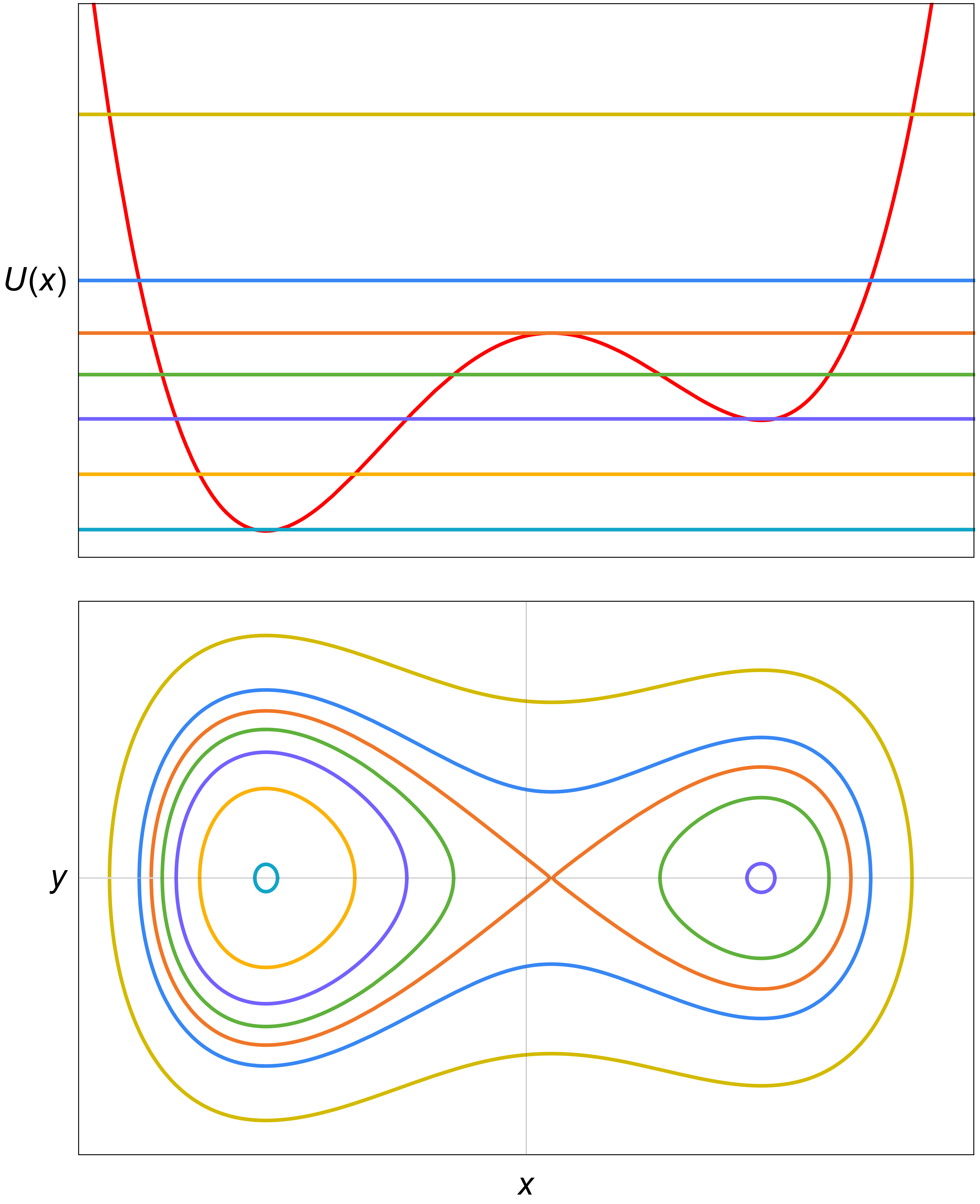


Example

We consider the potential function

$$U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{10}x.$$

The graph of $U(x)$ and some level curves are shown at the right.



Example

We consider the potential function

$$U(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{10}x.$$

The graph of $U(x)$ and some level curves are shown at the right.