Lecture 22: Lyapunov's Method

MATH 303 ODE and Dynamical Systems

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Isolated equilibria

Definition. An equilibrium (x_e, y_e) of a planar system is an **isolated equilibrium** if there is an open disk $D = B_\delta(x_e, y_e)$ of radius $\delta > 0$ centered at (x_e, y_e) which does not contain any other equilibria.

Positive / Negative (semi-)definite

Definition. Let D be an open disk centered at $(0,0)$ and consider a function $W(x, y)$ which is continuous in D and satisfies $W(0,0) = 0$.

- If $W(x, y) > 0$ for all $(x, y) \in D_* = D \backslash \{(0,0)\}$ then $W(x, y)$ is **positive definite** in D.

- If $W(x, y) \geq 0$ for all $(x, y) \in D$ then $W(x, y)$ is **positive semidefinite** in D.

- If $W(x, y) < 0$ for all $(x, y) \in D_*$ then $W(x, y)$ is **negative definite** in D.

- If $W(x, y) \leq 0$ for all $(x, y) \in D$ then $W(x, y)$ is **negative semidefinite** in D.

Remark. Clearly $W(x, y)$ is positive (semi-)definite in D if and only if $-W(x, y)$ is negative (semi-)definite in D.

Derivative

Consider a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and a real-valued function $V(\mathbf{x})$. Let $\mathbf{x}(t) = (x_1(t), x_2(t))$ be a solution curve of the given planar system. Then

$$\frac{d}{dt}[V(\mathbf{x}(t))] = \frac{d}{dt}[V(x_1(t), x_2(t))] = \frac{\partial V}{\partial x_1}(\mathbf{x}(t))\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}(\mathbf{x}(t))\frac{dx_2}{dt}$$

$$= \frac{\partial V}{\partial x_1}(\mathbf{x}(t))f_1(\mathbf{x}(t)) + \frac{\partial V}{\partial x_2}(\mathbf{x}(t))f_2(\mathbf{x}(t))$$

Given a function $V(\mathbf{x})$ and a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ define a new function $\dot{V}(\mathbf{x})$ by

$$\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial x_1}(\mathbf{x})f_1(\mathbf{x}) + \frac{\partial V}{\partial x_2}(\mathbf{x})f_2(\mathbf{x}).$$

With this notation we have

$$\frac{d}{dt}[V(\mathbf{x}(t))] = \dot{V}(\mathbf{x}(t)).$$

We call $\dot{V}$ the **derivative of V along the flow of $\mathbf{f}$**. Moreover, notice that

$$\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial x_1}(\mathbf{x})f_1(\mathbf{x}) + \frac{\partial V}{\partial x_2}(\mathbf{x})f_2(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \nabla V(\mathbf{x}),$$

or $\dot{V} = \mathbf{f} \cdot \nabla V$. The relation above shows that $\dot{V}$ is the directional derivative of V along the vector field $\mathbf{f}$.

Lyapunov's Stability Theorem

Theorem. Consider a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ for which $(0,0)$ is an isolated equilibrium.

(a) If there is a function $V(\mathbf{x})$ which is positive definite in an open disk D centered at $(0,0)$ while $\dot{V}(\mathbf{x})$ is negative definite in D then $(0,0)$ is asymptotically stable.

(b) If there is a function $V(\mathbf{x})$ which is positive definite in an open disk D centered at $(0,0)$ while $\dot{V}(\mathbf{x})$ is negative semidefinite in D then $(0,0)$ is stable.

Remark. A function V satisfying the conditions of (either part of) this theorem is called a **Lyapunov function**.

Example 1

Consider the system $x' = y - xy^2 - x^3,\ y' = -x - x^2y - y^3$. Then $(0,0)$ is an isolated equilibrium.

Let $V(x, y) = x^2 + y^2$. Then $V(0,0) = 0$, and V is continuous and positive definite on $\mathbb{R}^2$.

Moreover,

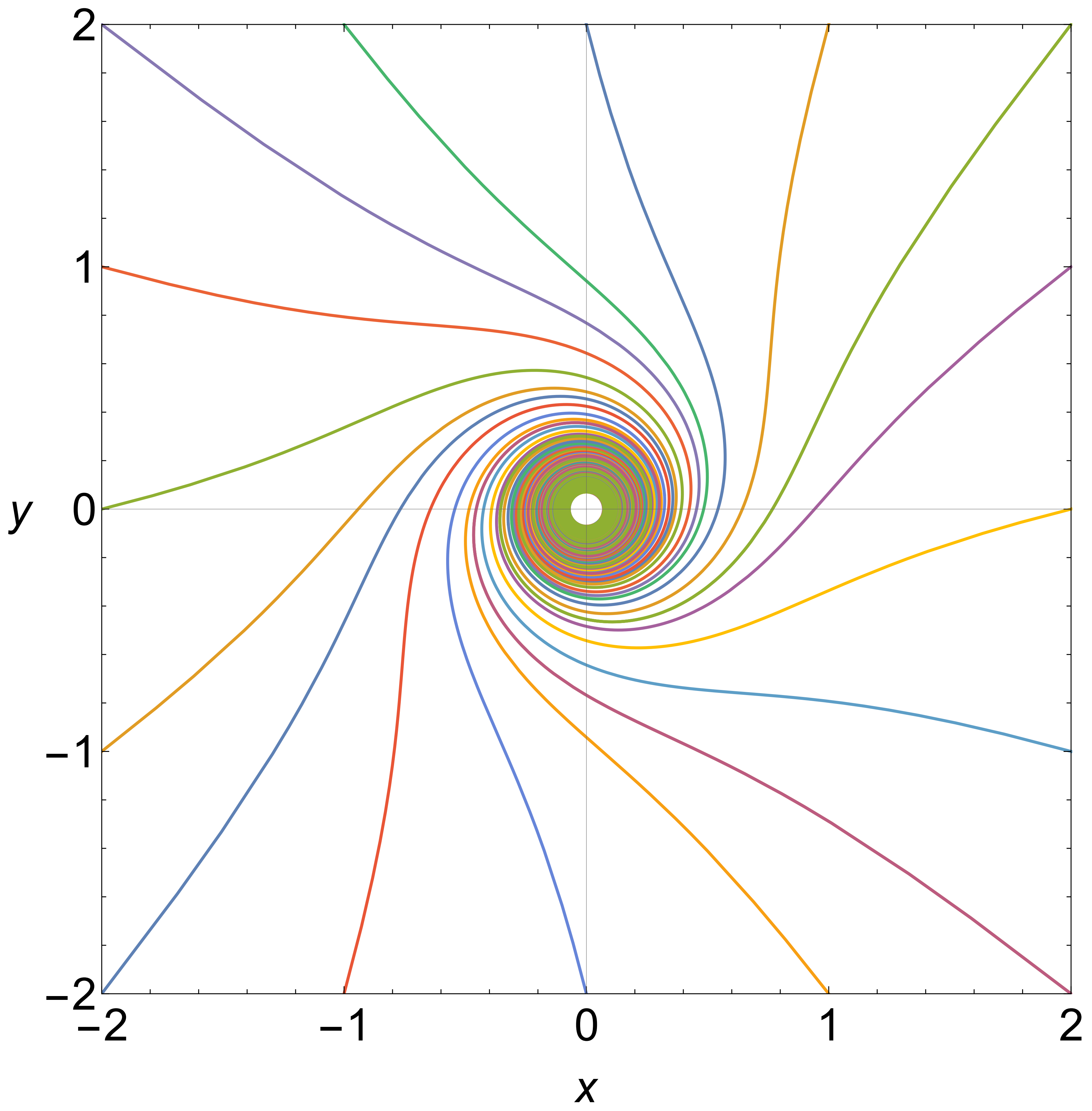
$$\dot{V} = \frac{\partial V}{\partial x} \cdot (y - xy^2 - x^3) + \frac{\partial V}{\partial y} \cdot (-x - x^2y - y^3)$$

$$= 2x(y - xy^2 - x^3) + 2y(-x - x^2y - y^3) = -2(x^4 + y^4)$$

We have $\dot{V}(0,0) = 0$ and $\dot{V}$ is continuous and negative definite on $\mathbb{R}^2$. We conclude that $(0,0)$ is asymptotically stable.

Remark. The phase portrait of the system is shown at the right.

Remark. The linearization of this system is $x' = y,\ y' = -x$ which corresponds to a center. Therefore, the linearization theorem cannot be used in this case to determine stability.



Example 2

Consider the system $x' = -2y^3,\ y' = x - 3y^3$. Then $(0,0)$ is an isolated equilibrium.

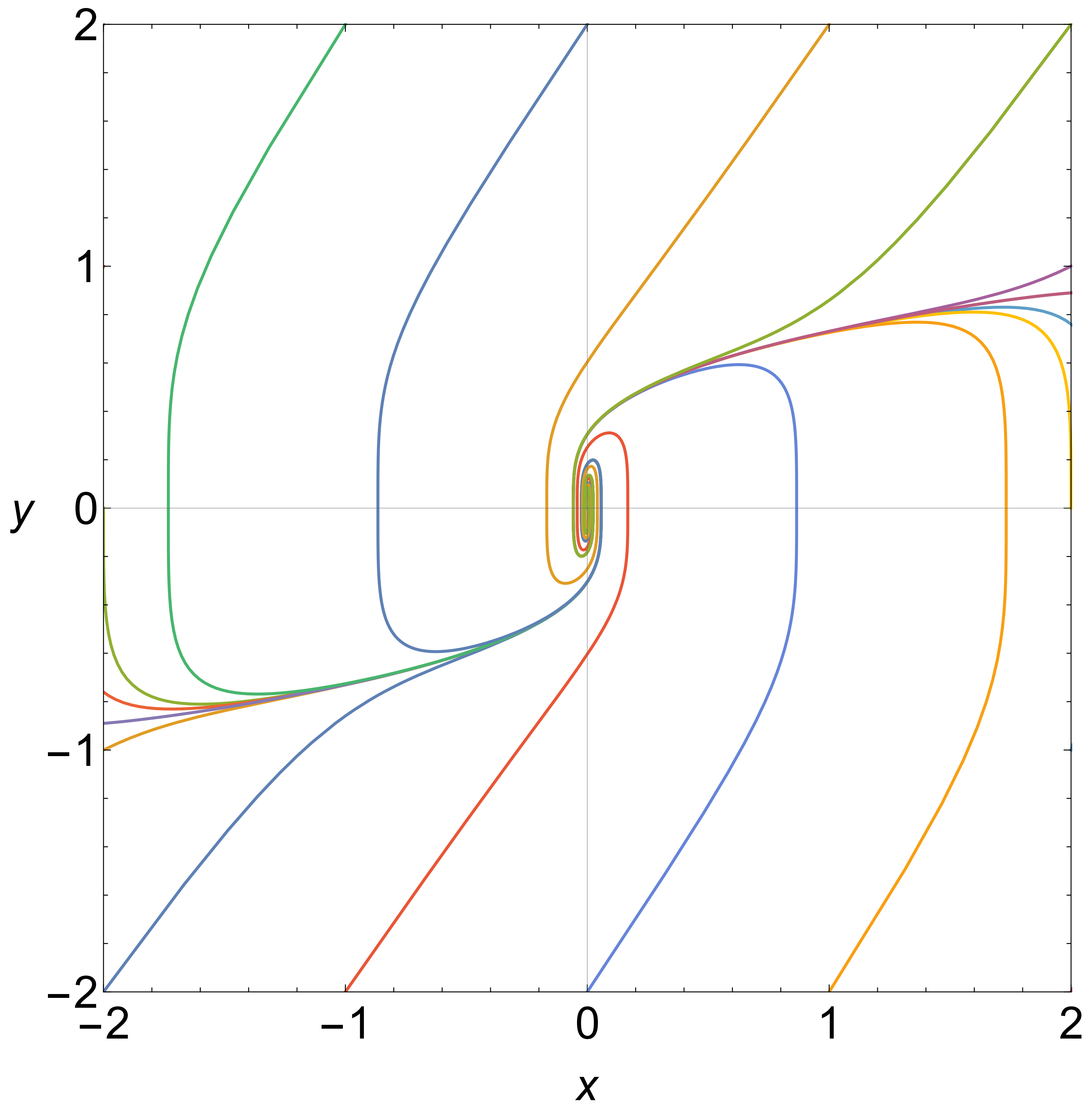
Consider the function $V(x, y) = x^2 + y^4$. Then $V(x, y)$ is positive definite and continuous on $\mathbb{R}^2$ with $V(0,0) = 0$. We have

$$\dot{V} = \frac{\partial V}{\partial x} \cdot (-2y^3) + \frac{\partial V}{\partial y} \cdot (x - 3y^3) = -4xy^3 + 4y^3(x - 3y^3) = -12y^6 \le 0.$$

We also have $\dot{V}(0,0) = 0$ and $\dot{V}$ is continuous on $\mathbb{R}^2$, therefore $\dot{V}$ is negative semidefinite on $\mathbb{R}^2$. Therefore, according to the theorem, the origin is stable.

Actually, the origin is asymptotically stable but the theorem cannot ensure this.

The phase portrait of the system is shown at the right.



Example 3

We consider the pendulum with damping:

$$x' = y, \quad y' = -by - \sin x,$$

where $b > 0$. Recall that when $b = 0$ the energy

$$E(x, y) = \frac{1}{2}y^2 - \cos x$$

is a conserved quantity. In this case we have

$$\dot{E} = y\frac{dy}{dt} + \sin x\frac{dx}{dt} = -by^2 - y\sin x + y\sin x = -by^2 \leq 0.$$

We want to use $E(x, y)$ to show that the isolated equilibrium $(0,0)$ is stable.

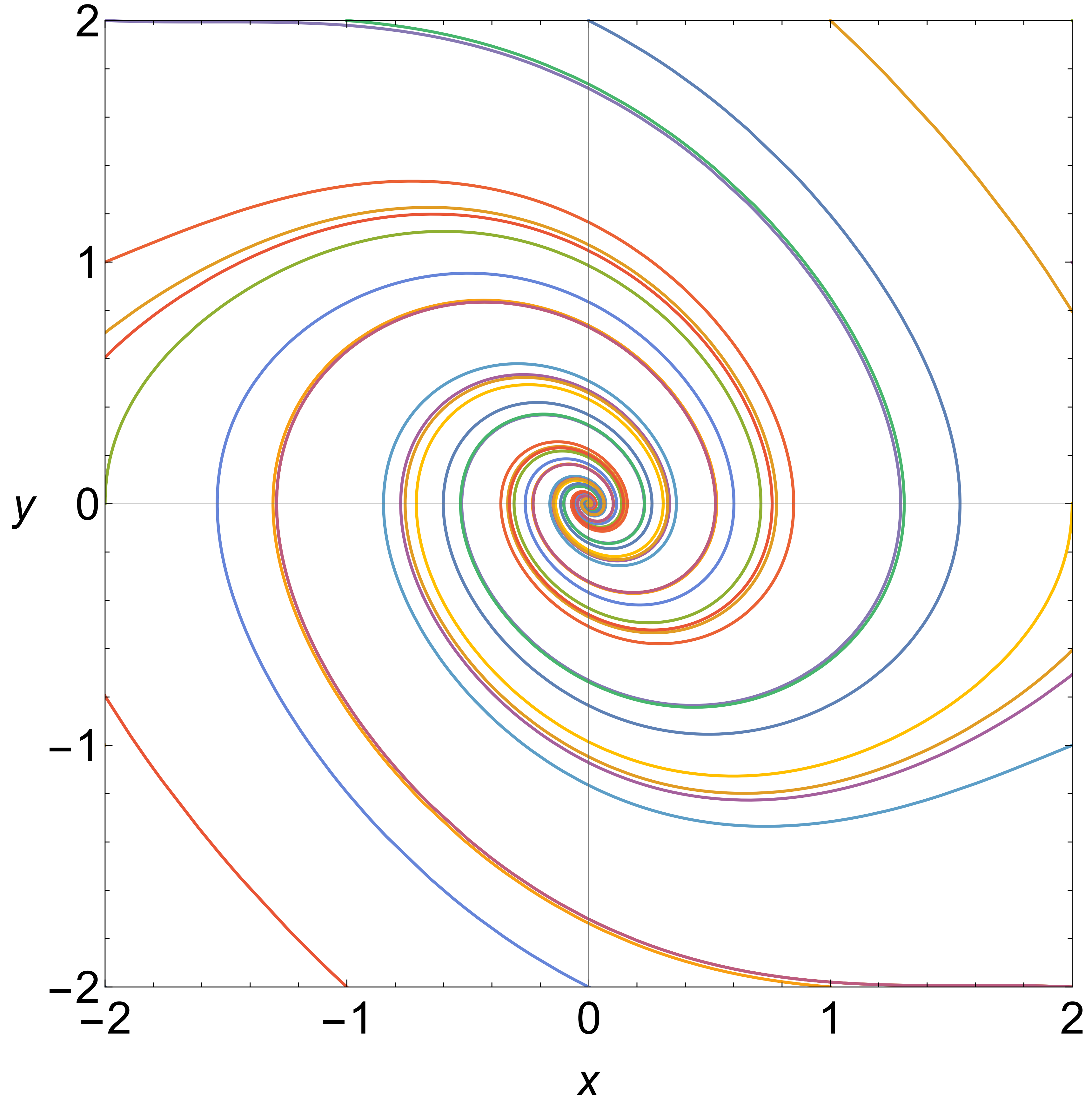
Since we have $E(0,0) = -1$ we choose

$$V(x, y) = E(x, y) + 1 = \frac{1}{2}y^2 + (1 - \cos x).$$

Then $V(0,0) = 0$ and $V(x, y)$ is continuous on $\mathbb{R}^2$. Moreover, $V(x, y)$ is positive definite on $(-\pi, \pi) \times \mathbb{R}$ and certainly there is a disk D centered at $(0,0)$ and contained in this set, for example, the open disk with radius π. Therefore, $V(x, y)$ is positive definite in D.

Moreover, $\dot{V}(x, y) = -by^2$ is continuous on $\mathbb{R}^2$, has $\dot{V}(0,0) = 0$, and $\dot{V}(x, y) = -by^2 \leq 0$ for all $(x, y) \in D$. Therefore, $\dot{V}$ is negative semidefinite on D. Then Lyapunov's Stability Theorem allows us to conclude that $(0,0)$ is (at least) stable.

Actually, again the equilibrium is not just stable but asymptotically stable. The phase portrait of the system near the origin is shown at the right.



Lyapunov's Instability Theorem

Theorem. Consider a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ for which $(0,0)$ is an isolated equilibrium. Assume that there is a function $V(\mathbf{x})$ and an open disk D centered at $(0,0)$ such that $V(\mathbf{x})$ is continuous in D, $V(0,0) = 0$, and $\dot{V}(\mathbf{x})$ is positive definite in D.

If for every open disk B centered at $(0,0)$ there exists $\mathbf{a} \in B$ such that $V(\mathbf{a}) > 0$ then the origin is unstable.

Example 4

Consider the planar system $x' = -y^3,\ y' = -x^3$. The only equilibrium is $(0,0)$ which clearly is isolated.

Consider $V(x, y) = -xy$ which is continuous on $\mathbb{R}^2$ and satisfies $V(0,0) = 0$. We have

$$\dot{V} = \frac{\partial V}{\partial x} \cdot (-y^3) + \frac{\partial V}{\partial y} \cdot (-x^3) = y^4 + x^4.$$

We have $\dot{V} > 0$ for all $(x, y) \neq (0,0)$. Therefore, $\dot{V}$ is positive definite on $\mathbb{R}^2$.

Consider now any open disk B centered at $(0,0)$. Denote by r the radius of B.

Then consider the point $(x, y) = (r/2, -r/2) \in B$. We have

$$V(\frac{1}{2}r, -\frac{1}{2}r) = \frac{1}{4}r^2 > 0.$$

This shows that the given function V satisfies the conditions of Lyapunov's second theorem and therefore the equilibrium is unstable.

The phase portrait of the system near the origin is shown at the right.