Lecture 23: Limit Cycles and Periodic Solutions

MATH 303 ODE and Dynamical Systems

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Example

Consider the planar system

$$x' = x - y - x(x^2 + y^2)$$
$$y' = x + y - y(x^2 + y^2)$$

To understand the dynamics we will use polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

We have

$$rr' = xx' + yy'$$
$$r^2\theta' = xy' - x'y$$

We have

$$
\begin{aligned}
rr' &= x(x - y - x(x^2 + y^2)) + y(x + y - y(x^2 + y^2)) \\
&= x^2 - x^2(x^2 + y^2) + y^2 - y^2(x^2 + y^2) \\
&= (x^2 + y^2) - (x^2 + y^2)^2 = r^2 - r^4
\end{aligned}
$$

Therefore,

$$r' = r - r^3 = r(1 - r^2).$$

Moreover,

$$r^2\theta' = x(x + y - y(x^2 + y^2)) - y(x - y - x(x^2 + y^2)) = x^2 + y^2 = r^2$$

Therefore,

$$\theta' = 1.$$

From the equation $\theta' = 1$ we conclude that we have a rotation with constant angular velocity around the origin, combined with motion in the radial direction.

To understand the radial motion we look at the equation (one-dimensional autonomous system) $r' = r(1 - r^2)$.

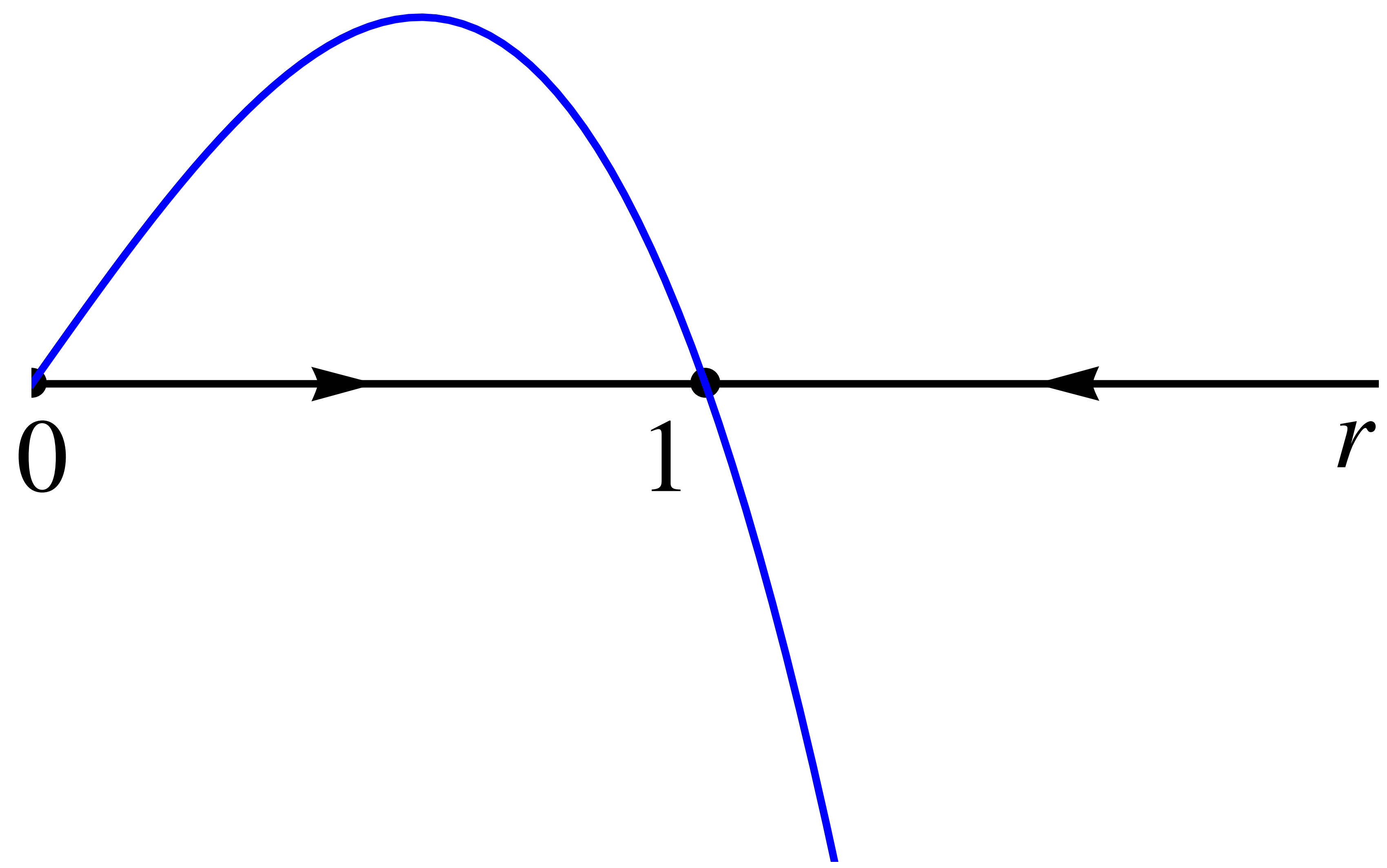
The equilibria for the radial dynamics are $r = 0$ and $r = 1$ (since $r \geq 0$, there is no equilibrium $r = -1$).

The equilibrium $r = 0$ corresponds to $x = y = 0$ and thus corresponds to the origin which we can directly see from the original equations that is indeed an equilibrium.

The equilibrium $r = 1$ is an equilibrium for the radial dynamics, meaning that r is constant, but we must remember that θ changes and therefore the solution curve traces a circle of radius $r = 1$ in time 2π.

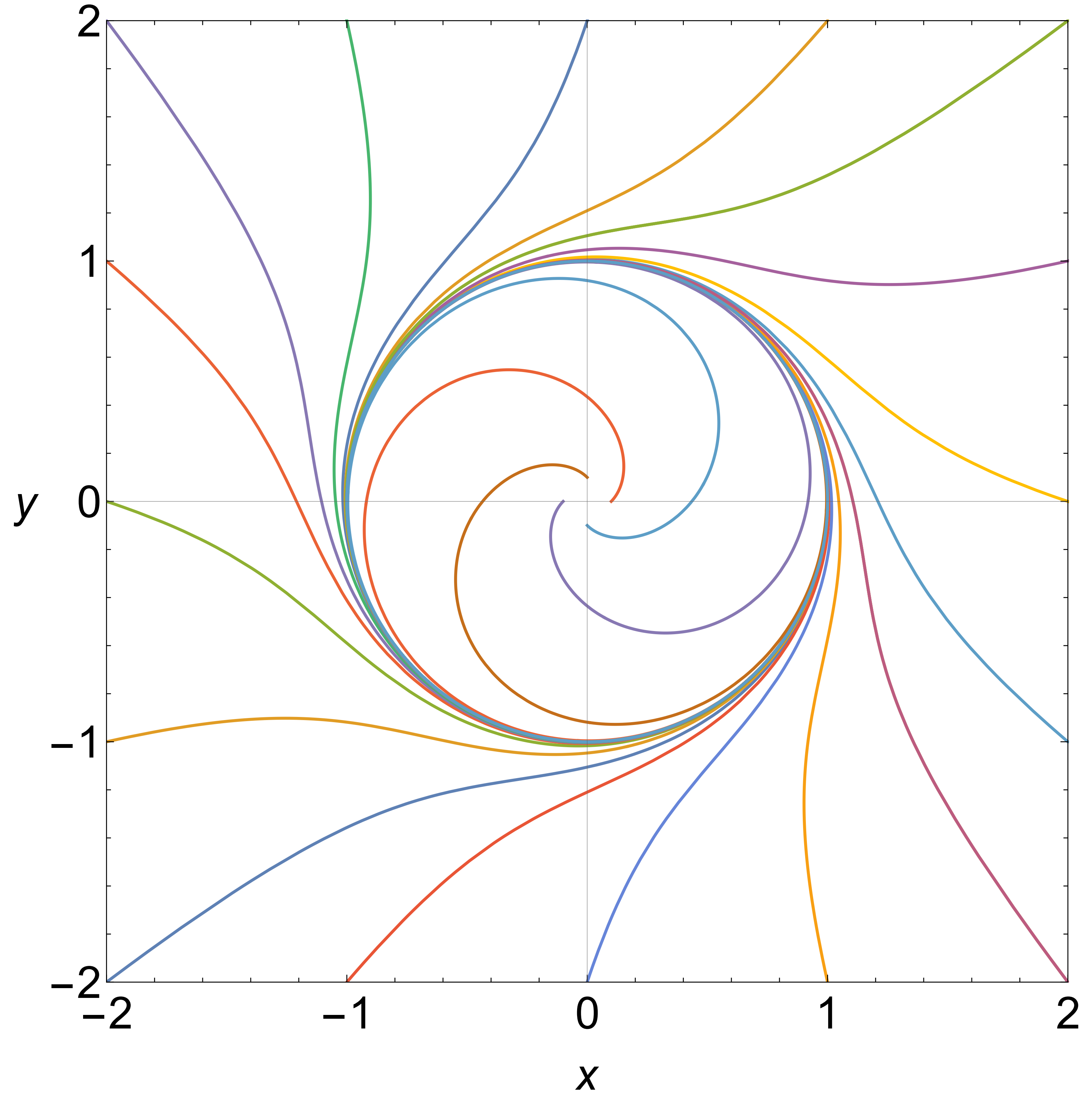
This means that $r = 1$ corresponds to a **periodic orbit** of the original system.

Let's now analyze further the radial dynamics. The corresponding phase line is shown at the top right. We observe that $r = 0$ is an unstable equilibrium, while $r = 1$ is asymptotically stable.



This implies that $(x, y) = (0,0)$ is an **unstable spiral** for the system (this can also be easily obtained from the linear analysis), while $r = 1$ corresponds to an **asymptotically stable periodic orbit**.

The phase portrait of the system is shown at the bottom right.



Limit cycles

Definition. If $\mathbf{x}(t)$ is a (non-constant) T-periodic solution of a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ the closed curve $L = \{\mathbf{x}(t) : 0 \leq t \leq T\}$ is called a **limit cycle** if there is at least one other solution $\mathbf{x}_1(t)$ such that

$$\lim_{t\to\infty} d(\mathbf{x}_1(t), L) = 0 \text{ or } \lim_{t\to-\infty} d(\mathbf{x}_1(t), L) = 0.$$

Here, $d(\mathbf{x}, L) = \min\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in L\}$ is the distance between the point $\mathbf{x}$ and the set L.

Remark. In the previous example, the periodic solution $r = 1$ is a limit cycle.

Remark. In the general case, limit cycles are closed curves but they are not circles.

Example

Consider the planar system

$$x' = x - y - xf(r)$$
$$y' = x + y - yf(r)$$

where $f(r) = 3r - r^2 - 1$. Then we have

$$rr' = xx' + yy' = x(x - y - xf(r)) + y(x + y - yf(r)) = r^2(1 - f(r))$$

Therefore,

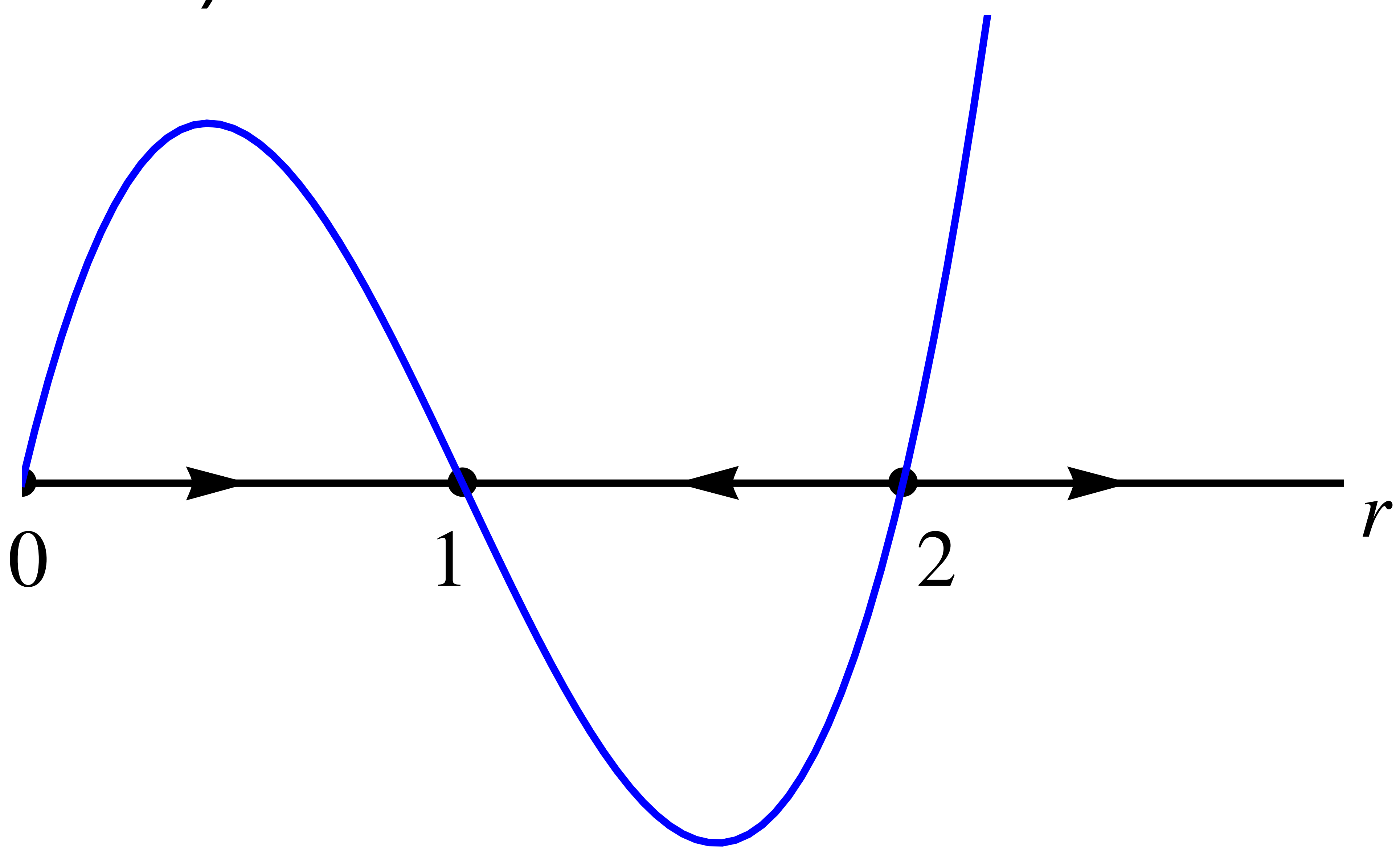
$$r' = r(1 - f(r)) = r(r^2 - 3r + 2).$$

Moreover,

$$r^2\theta' = xy' - x'y = x(x + y - yf(r)) - y(x - y - xf(r)) = r^2$$

Therefore, $\theta' = 1$.

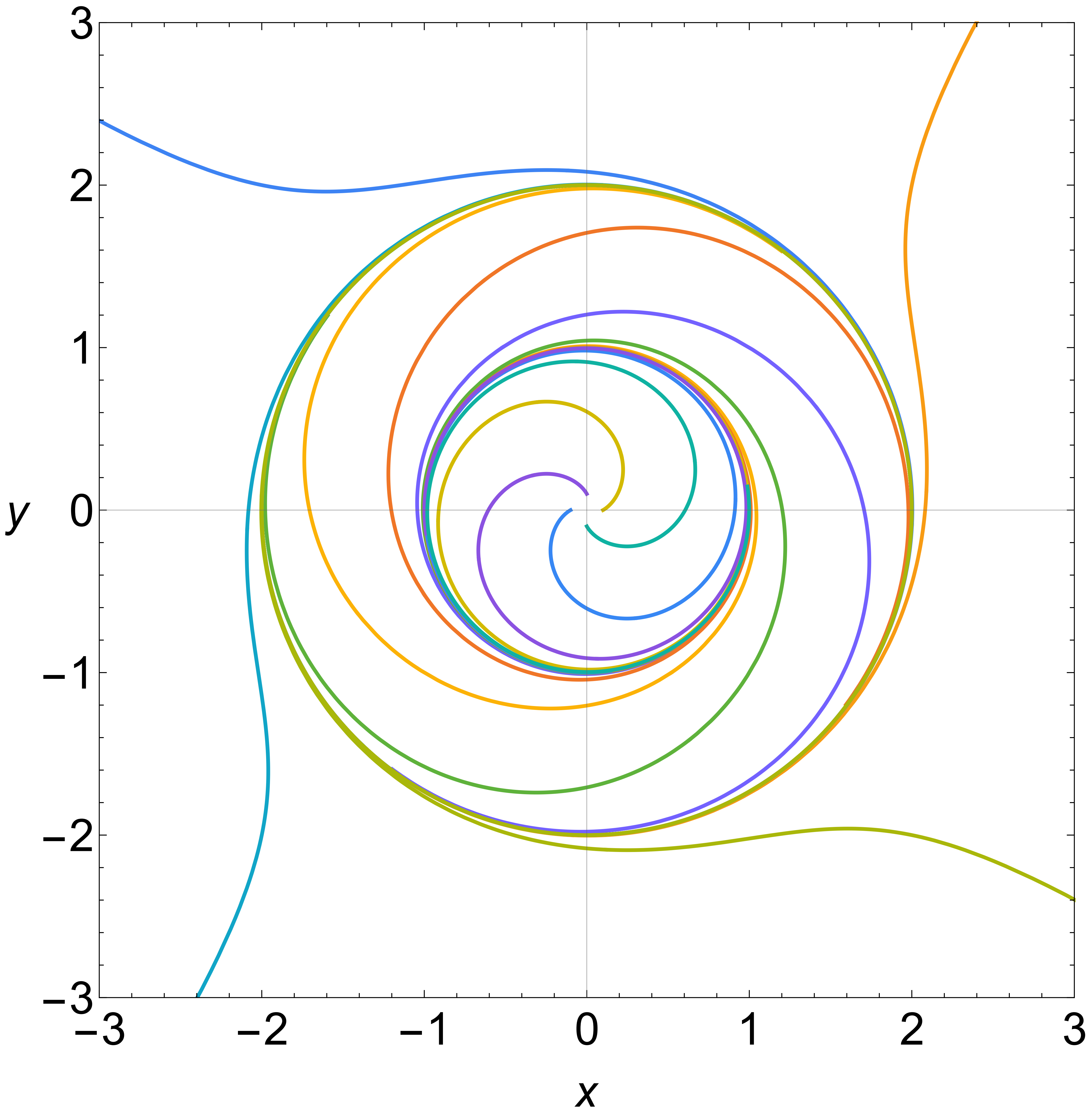
For the dynamics in the radial direction we have equilibria $r = 0,\ r = 1,\ r = 2$. Here, $r = 0$ corresponds to a fixed point of the system, while $r = 1$ and $r = 2$ correspond to periodic orbits. For their stability we check the phase line for the radial dynamics (shown below).



Therefore, we see that the origin is an unstable spiral, the periodic orbit $r = 1$ is asymptotically stable, and the periodic orbit $r = 2$ is unstable.

The phase portrait for the system is shown at the right.

To draw the flow arrows (not shown here) recall that all rotations are counterclockwise since $\theta' = 1$.



Limit cycles enclose equilibria

Theorem. If L is a limit cycle of the planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ then it must enclose at least one equilibrium of the system. If L encloses exactly one equilibrium then the equilibrium cannot be a saddle point.

Bendixson's negative criterion

Theorem. Consider a planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ defined in a simply connected domain $D \subseteq \mathbb{R}^2$. Assume that f_1, f_2 have continuous partial derivatives in D, and that $\operatorname{div} \mathbf{f} = \nabla \cdot \mathbf{f} = \partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not change sign in D. Then the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ does not have any (non-constant) periodic orbits in D.

Proof. Suppose that there is a non-constant T-periodic solution $\mathbf{x}(t)$ in D, let $P = \{\mathbf{x}(t) : 0 \leq t \leq T\}$ be the closed curve traced by the periodic solution, and denote by U the domain enclosed by P. U is simply connected. Then by **Green's Theorem** we have

$$\oint_P f_1 dx_2 - f_2 dx_1 = \int_U \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2.$$

Since $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not change sign we can consider, for example, that it is strictly positive. Then

$$\oint_P f_1 dx_2 - f_2 dx_1 = \int_U \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 > 0.$$

However, we have

$$\oint_P f_1 dx_2 - f_2 dx_1 = \int_0^T \left(f_1 \frac{dx_2}{dt} - f_2 \frac{dx_1}{dt} \right) dt = \int_0^T \left(f_1 f_2 - f_2 f_1 \right) dt = 0.$$

This contradictions leads to the conclusion that no such periodic orbit may exist.

Example

Consider the damped pendulum

$$\begin{aligned} x' &= y \\ y' &= -by - \sin x \end{aligned}$$

We have

$$\nabla \cdot \mathbf{f} = \frac{\partial y}{\partial x} + \frac{\partial(-by - \sin x)}{\partial y} = -b < 0.$$

Moreover, all functions and their partial derivatives are continuous on $\mathbb{R}^2$ and $\mathbb{R}^2$ is simply connected. Therefore, all the conditions of Bendixson's negative criterion are satisfied and we conclude that the system has no periodic orbit.

Poincaré — Bendixson Theorem

Theorem. Consider the planar system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ in a closed bounded region R. Assume that f_1, f_2 have continuous partial derivatives in R and that the system has no equilibria in R. Then any solution $\mathbf{x}(t)$ of the planar system for which $\mathbf{x}(t) \in R$ for all $t \geq 0$ is either periodic or it approaches a limit cycle in R.

Example

Consider the planar system

$$x' = y$$
$$y' = 4y - x^3 - 4x^2y - y^3$$

The only equilibrium of the system is $(0,0)$.

We want to prove that the system also has a (non-constant) periodic solution. For this, we try to find a closed region R for which we can prove that if a solution starts in R then it stays in R for all $t \geq 0$. If we can do this then the Poincaré-Bendixson theorem ensures that there must be a periodic solution in R.

To find such region R we consider the function $V(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4$.

Then we find

$$\dot{V}(x,y) = x^3x' + yy' = x^3y + y(4y - x^3 - 4x^2y - y^3) = -y^2(4x^2 + y^2 - 4)$$

We have $\dot{V} \le 0$ when $4x^2 + y^2 \ge 4$ and $\dot{V} \ge 0$ when $4x^2 + y^2 \le 4$. Therefore, ellipse $4x^2 + y^2 = 4$ separates the plane into two regions. In the inner region the value of V increases along solution curves and in the outer region the value of V decreases along solution curves.

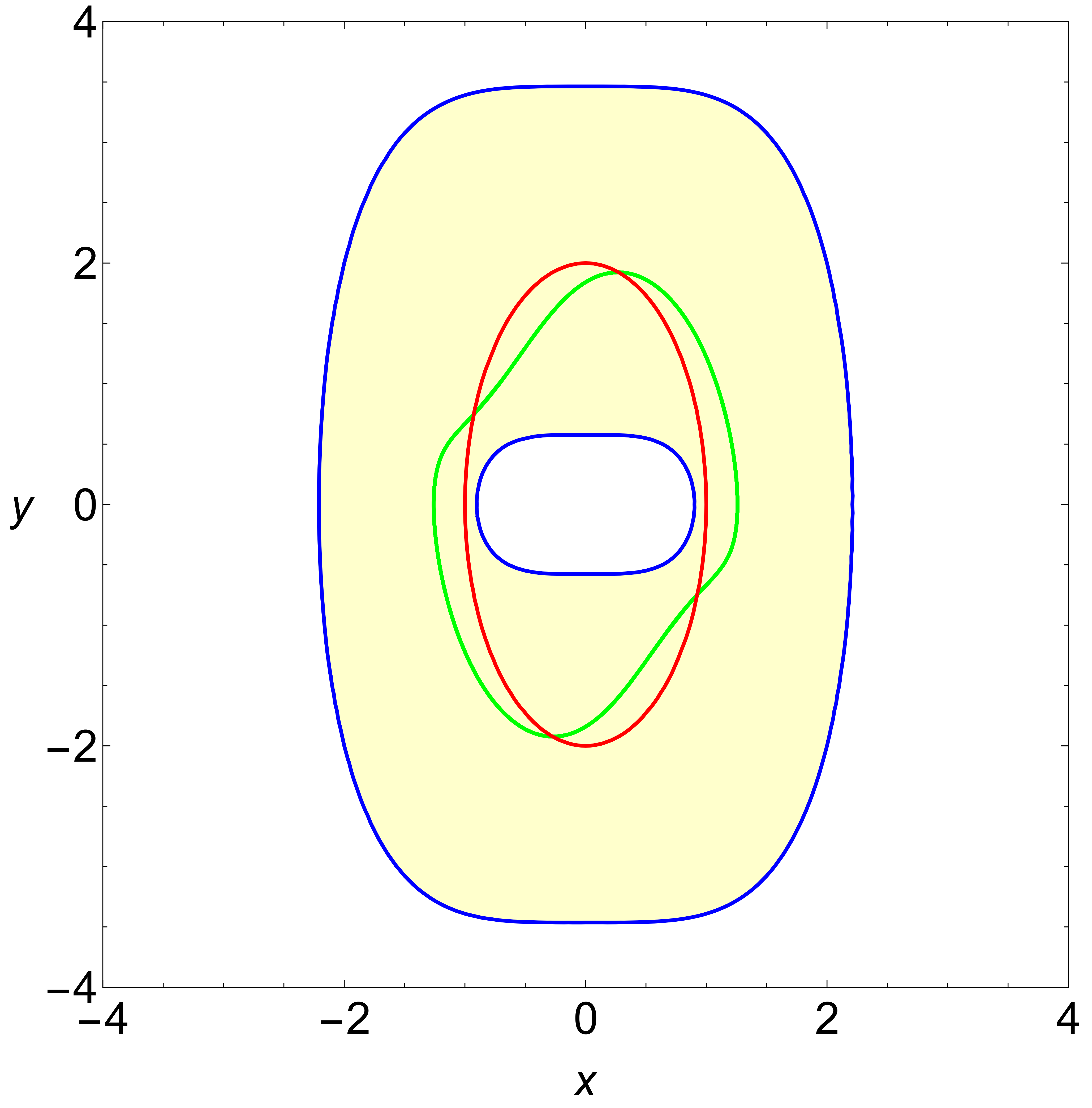
To define the region R we consider the level set $V(x,y) = 1/6$ which can be proved to belong in the inner region, and the level set $V(x,y) = 6$ which can be proved to belong in the outer region, and we define R to be the region between these two level sets.

In the picture at the right, the level sets of $V(x, y)$ for values $1/6$ and 6 are drawn in blue. The curve $\dot{V} = 0$ is drawn in red, and R is shaded light yellow.

Since the level set $V(x, y) = 1/6$ is in the region where $\dot{V} \geq 0$ we conclude that a solution inside R cannot cross the level set going toward the origin since this would need $\dot{V} < 0$.

Similarly, since the level set $V(x, y) = 6$ is in the region where $\dot{V} \leq 0$ we conclude that a solution inside R cannot cross the level set going outward since this would need $\dot{V} > 0$.

Therefore, any solution starting inside R will stay in R for all $t \geq 0$.



Moreover, notice that there are no equilibria of the system in R (the reason we needed to consider also the inner level set $V(x, y) = 1/6$ is that we do not want that the origin is in R).

Since all the conditions of the Poincaré-Bendixson theorem are satisfied we conclude that there must be a periodic orbit in R. The plot in the previous slide shows the limit cycle (drawn in green) in R.