Lecture 26: Bifurcations

MATH 303 ODE and Dynamical Systems

Konstantinos Efstathiou

Systems depending on parameters

We consider **one-dimensional dynamical systems**

$$x' = f(a, x)$$

where $x \in \mathbb{R}$. Here $a \in \mathbb{R}$ is a **parameter**, that is, it is remains fixed as the system evolves.

The main question that we will address is how the dynamics of the system changes as the parameter changes.

Recall that for one-dimensional dynamical systems the dynamics is organized in terms of equilibria. For this reason, we focus on the changes in number and stability of equilibria as the parameter changes.

Implicit function theorem

Theorem. Consider a smooth function $g : \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$ such that $g(x_0, y_0) = 0$. If $g_y(x_0, y_0) \neq 0$ then there is $\delta > 0$ and a unique smooth function $h(x)$ defined on $(x_0 - \delta, x_0 + \delta)$ such that

$$g(x, h(x)) = 0 \text{ and } h(x_0) = y_0$$

for all $x \in (x_0 - \delta, x_0 + \delta)$.

Notation. We denote partial derivatives using subscripts:

$$g_y = \frac{\partial g}{\partial y}, g_x = \frac{\partial g}{\partial x}, g_{xx} = \frac{\partial^2 g}{\partial x^2}, g_{xy} = \frac{\partial^2 g}{\partial x \partial y}.$$

Interpretation

You can think of the expression $g(x, y) = 0$ as a relation between x and y — an equation that must be solved for y in terms of x.

The question that the Implicit Function Theorem answers is when we can solve the equation $g(x, y) = 0$ in terms of y, that is, when we can express y as a function of x. In the theorem, this solution is $y = h(x)$.

Moreover, we assume that for a specific x_0 we know that there is a corresponding y_0 so that $g(x_0, y_0) = 0$ and we want that the function $h(x)$ satisfies $h(x_0) = y_0$.

Then the theorem tells us that if $g_y(x_0, y_0) \neq 0$ then such a function $h(x)$ exists locally (in a neighborhood of x_0).

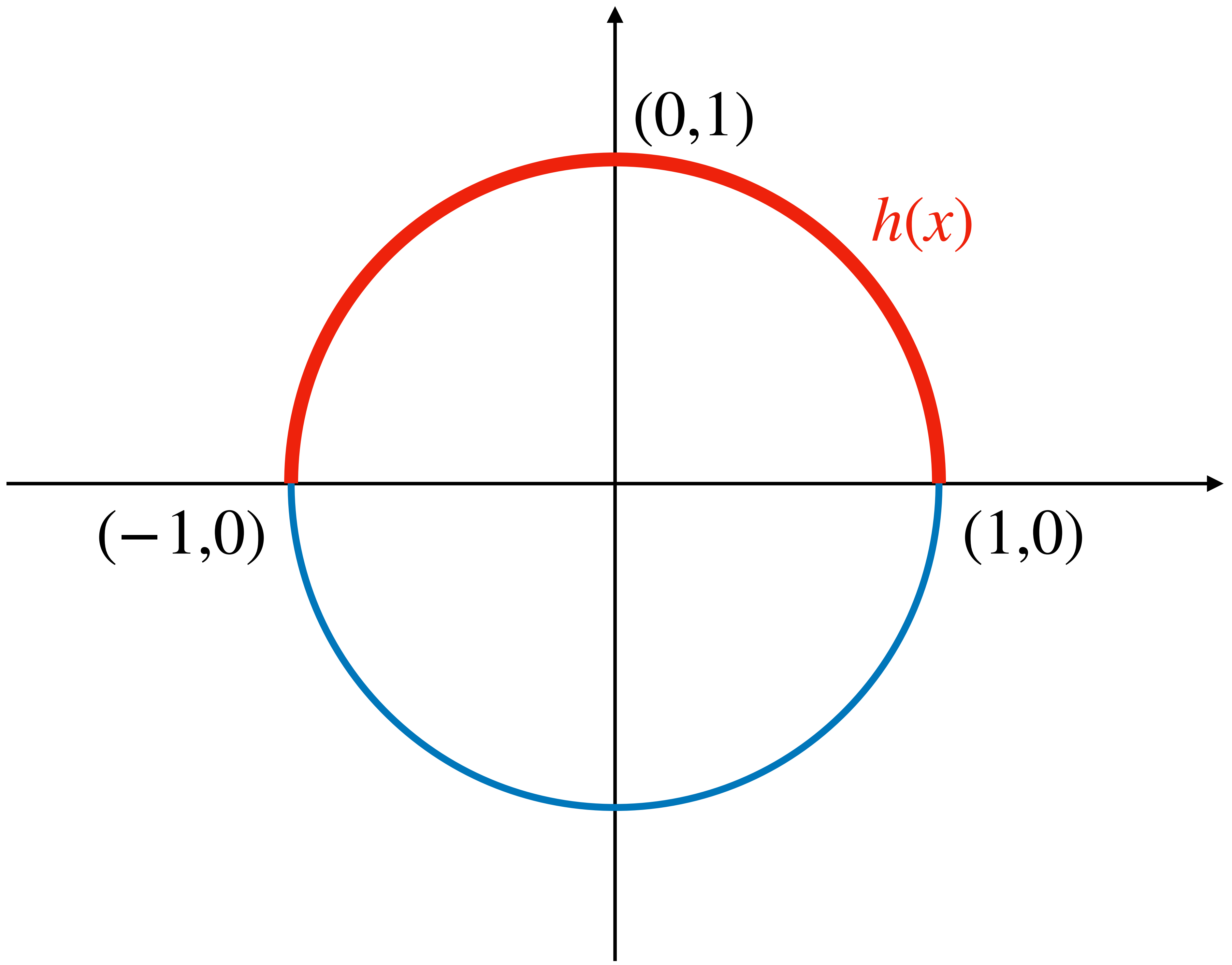
Example

Let $g(x, y) = x^2 + y^2 - 1$.

We can check that $g(0,1) = 0$. Moreover, $g_y = 2y$ and, therefore, $g_y(0,1) = 2 \neq 0$.

Therefore, the implicit function theorem tells us that there is an interval $(-\delta, \delta)$ and a function $h(x)$ defined in this interval such that

$$h(0) = 1 \text{ and } x^2 + h(x)^2 - 1 = 0.$$



In this example, we **can** compute $h(x)$. Solving the last equation we find

$h(x) = \pm\sqrt{1-x^2}$ and since $h(0) = 1$ we finally get $h(x) = \sqrt{1-x^2}$.

Notice that the function $h(x) = \sqrt{1-x^2}$ is smooth for $x \in (-1,1)$.

At $x = \pm 1$ we have $y = 0$ and therefore $g_y(\pm 1,0) = 0$. This shows that we cannot apply the implicit function theorem at the points $(\pm 1,0)$ and therefore we cannot extend the solution $h(x)$ outside the interval $(-1,1)$.

Note that if we consider the point $(1,0)$ then we can solve for x in terms of y. This works because the implicit function theorem requires that $g_x(x_0, y_0)$ to be able to write $x = j(y)$ with $x_0 = j(y_0)$. In our case, $g_x = 2x$ and $g_x(1,0) = 2 \neq 0$. Here $j(y) = \sqrt{1-y^2}$.

Remark

A last remark. In this example we were able to solve in terms of y or x. The strength of the theorem is that it tells us that a unique solution exists even if we cannot solve for x or y.

Moreover, the theorem can be applied in general settings where we may not even know the function $g(x, y)$. Later today we will do exactly this in the proof of the fold bifurcation theorem.

Persistence of equilibria

Theorem. Consider the system $x' = f(a, x)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and assume that x_0 is an equilibrium of $x' = f(a_0, x)$, that is, $f(a_0, x_0) = 0$, and $f_x(a_0, x_0) \neq 0$. Then there is $\delta > 0$ and a unique smooth function $g(a)$ defined in $(a_0 - \delta, a_0 + \delta)$ such that

$$f(a, g(a)) = 0 \ \text{ and } \ g(a_0) = x_0,$$

for all $a \in (a_0 - \delta, a_0 + \delta)$, that is, for each a the point $g(a)$ is an equilibrium of $x' = f(a, x)$.

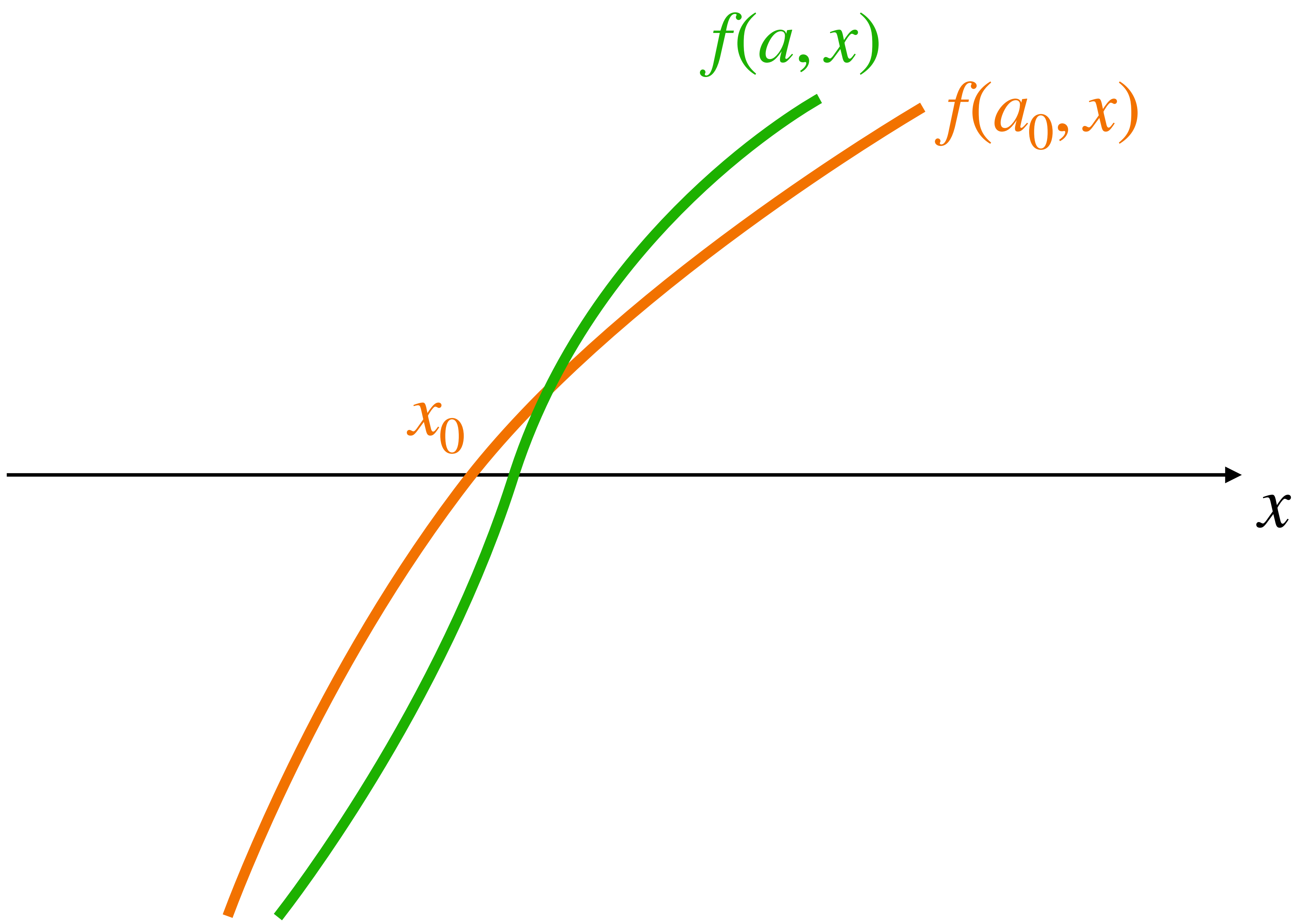
Interpretation

The theorem about persistence tells us that an equilibrium x_0 persists (keeps existing) as the parameter a changes — provided that $f_x(a_0, x_0) \neq 0$.

This means that we can draw the position of the equilibrium as a function of the parameter a on the (a, x)-plane and this will give a curve which is the graph of a function.

Understanding

If $f(a_0, x_0) = 0$ and $f_x(a_0, x_0) \neq 0$ then for small changes in a the equilibrium persists and it does not change stability.



Fold (saddle-node) bifurcation

When the condition $f_x(a_0, x_0) \neq 0$ is not satisfied then we may have a change in the number of equilibria as the parameter a changes.

We will first see an example. Consider the system

$$x' = f(a, x) = a - x^2.$$

For the equilibria of the system we need to consider three cases:

$a > 0$: then there are two equilibria $x = \pm\sqrt{a}$;

$a = 0$: then there is an equilibrium $x = 0$;

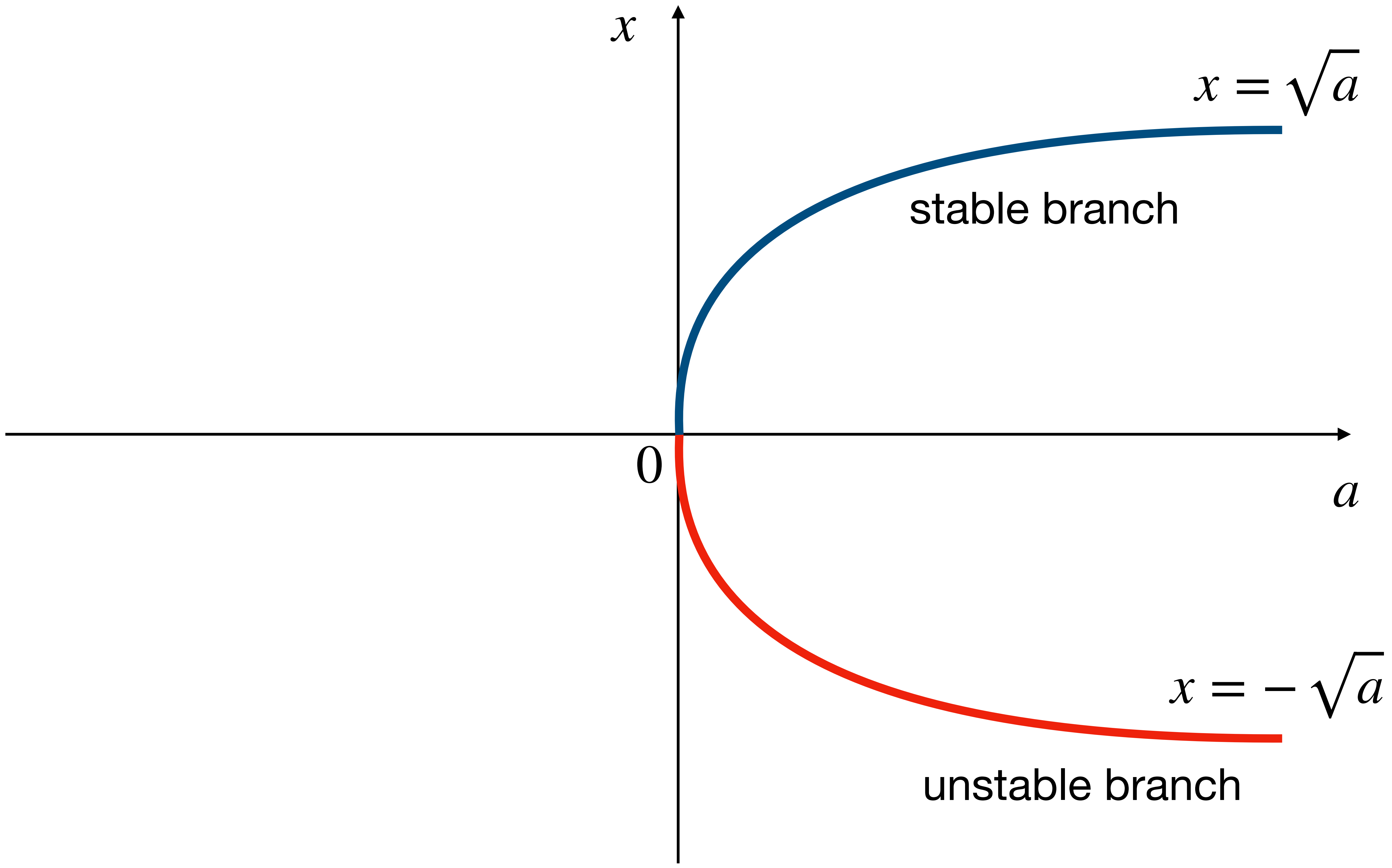
$a < 0$: then there are no equilibria.

For the stability of the equilibria we check that $f_x = -2x$ and therefore

$$f_x(a, \pm\sqrt{a}) = \mp 2\sqrt{a}.$$

This means that the equilibrium $\sqrt{a}$ is **asymptotically stable** since $f_x(a, \sqrt{a}) = -2\sqrt{a} < 0$ while the equilibrium $-\sqrt{a}$ is **unstable** since $f_x(a, -\sqrt{a}) = 2\sqrt{a} > 0.$

We can summarize this discussions in the **bifurcation diagram** on the next slide that shows the positions of the equilibria as functions of a and the stability of the equilibria (when they exist). **Stable equilibria are blue** and **unstable equilibria are red**.



In the bifurcation diagram we observe that the curves $\pm\sqrt{a}$ join at $a = 0$ and therefore the theorem on the persistence of equilibria cannot be valid there. Indeed we check that $f_x(0,0) = 0$ and thus it is not surprising that the equilibria do not persist at $a = 0$.

Fold bifurcation theorem

Theorem. Consider the system $x' = f(a, x)$ where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function and assume that:

(i) $f(a_0, x_0) = 0$ (the point x_0 is an equilibrium at a_0);

(ii) $f_x(a_0, x_0) = 0$ (otherwise the equilibrium persists);

(iii) $f_{xx}(a_0, x_0) \neq 0$;

(iv) $f_a(a_0, x_0) \neq 0$ (transversality condition).

[...]

[...] Then the system $x' = f(a, x)$ undergoes a fold bifurcation at a_0 in the sense that there is a smooth curve $a = h(x)$ defined for x near x_0 such that $f(h(x), x) = 0$ and

$$h(x_0) = a_0,\ \ h'(x_0) = 0,\ \ h''(x_0) = -\frac{f_{xx}(a_0, x_0)}{f_a(a_0, x_0)} \neq 0.$$

Remark. The theorem essentially states that near (a_0, x_0) the curve of equilibria looks like a parabola, just as in the example $f(a, x) = a - x^2$.

Proof

The function $f(a, x)$ satisfies $f(a_0, x_0) = 0$ and $f_a(a_0, x_0) \neq 0$.

Therefore, we can use the Implicit Function Theorem to assert that there is a smooth function $a = h(x)$ near x_0 such that $a_0 = h(x_0)$ and $f(h(x), x) = 0$.

We want to show that $h'(x_0) = 0$, $h''(x_0) \neq 0$, that is, that $h(x)$ looks locally like a parabola.

Define $g(x) = f(h(x), x) = 0$. Then

$$g'(x) = f_a(h(x), x)h'(x) + f_x(h(x), x) = 0.$$

Evaluating $f_a(h(x), x)h'(x) + f_x(h(x), x) = 0$ at x_0 we find

$$0 = g'(x_0) = f_a(h(x_0), x_0)h'(x_0) + f_x(h(x_0), x_0)$$
$$= f_a(a_0, x_0)h'(x_0) + f_x(a_0, x_0) = f_a(a_0, x_0)h'(x_0)$$

Since $f_a(a_0, x_0)h'(x_0) = 0$ and $f_a(a_0, x_0) \neq 0$ we conclude that $h'(x_0) = 0$.

Then we take the derivative of the relation

$$g'(x) = f_a(h(x), x)h'(x) + f_x(h(x), x) = 0$$

with respect to x. We find

$$g''(x) = f_{aa}(h(x), x)[h'(x)]^2 + f_{ax}(h(x), x)h'(x) + f_a(h(x), x)h''(x)$$
$$+ f_{xa}(h(x), x)h'(x) + f_{xx}(h(x), x) = 0$$

Evaluating at x_0 and using that $h'(x_0) = 0$ we find

$$0 = g''(x_0) = f_a(a_0, x_0)h''(x_0) + f_{xx}(a_0, x_0).$$

Since $f_a(a_0, x_0) \neq 0$ and $f_{xx}(a_0, x_0) \neq 0$ we can solve for $h''(x_0)$ to find

$$h''(x_0) = -\frac{f_{xx}(a_0, x_0)}{f_a(a_0, x_0)} \neq 0.$$

For the stability of equilibria we note that it is given by the sign of f_x. That is we need to compute $s(x) = f_x(h(x), x)$.

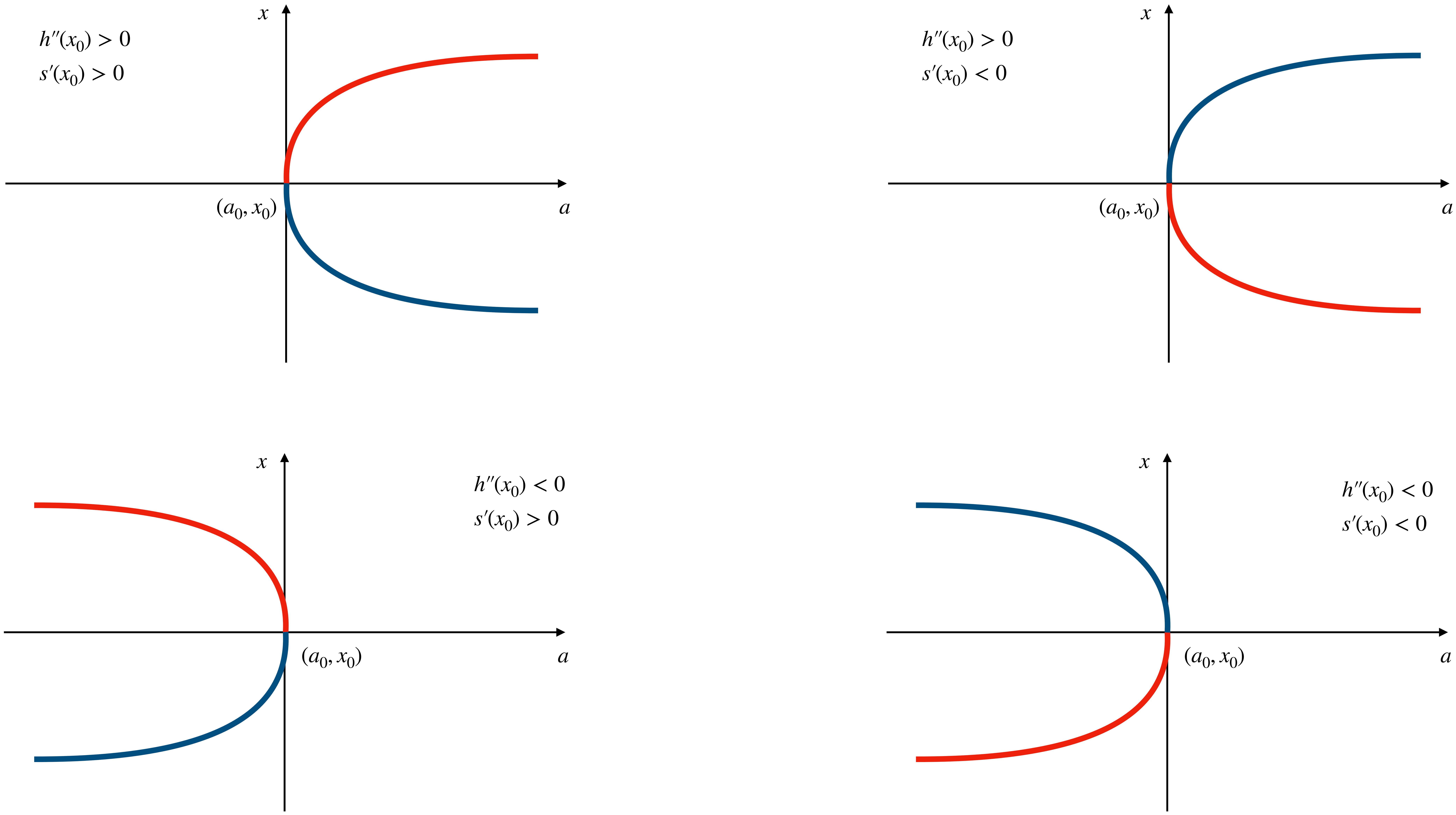
Clearly, $s(x_0) = f_x(a_0, x_0) = 0$ and

$$s'(x) = f_{xa}(h(x), x)h'(x) + f_{xx}(h(x), x).$$

Evaluating at x_0 and using that $h'(x_0) = 0$ we find $s'(x_0) = f_{xx}(a_0, x_0) \neq 0$.

Since $s(x_0) = 0$ and $s'(x_0) \neq 0$ we conclude that $s(x)$ changes sign at x_0.

If $s'(x_0) > 0$ then the equilibrium $x < x_0$ is stable (i.e., $s(x) < 0$) and the equilibrium $x > x_0$ is unstable (i.e., $s(x) > 0$). For $s'(x_0) < 0$ we have the opposite situation.



$h''(x_0) > 0$
$s'(x_0) > 0$
x
a
(a_0, x_0)
$h''(x_0) > 0$
$s'(x_0) < 0$
x
a
(a_0, x_0)
$h''(x_0) < 0$
$s'(x_0) > 0$
x
a
(a_0, x_0)
$h''(x_0) < 0$
$s'(x_0) < 0$
x
a
(a_0, x_0)