Lecture 27: Bifurcations in Planar Systems

MATH 303 ODE and Dynamical Systems

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Planar systems depending on a parameter

We consider planar systems of the form

$$x' = f(a, x, y),$$
$$y' = g(a, x, y),$$

where $a \in \mathbb{R}$ is a parameter.

Persistence of equilibria

If (x_0, y_0) is an equilibrium for a_0 then the corresponding version of the implicit function theorem tells us that the equilibrium persists for small changes of a provided that

$$\det \begin{bmatrix} f_x(a_0, x_0, y_0) & f_y(a_0, x_0, y_0) \\ g_x(a_0, x_0, y_0) & g_y(a_0, x_0, y_0) \end{bmatrix} \neq 0.$$

Bifurcations in planar systems

There are two important types of bifurcations in planar systems: the fold bifurcation which is a simple generalization of the one-dimensional fold bifurcation, and the Hopf bifurcation which is related to the appearance of limit cycles.

Fold (saddle-node) bifurcation

Fold (saddle-node) bifurcation in planar systems

We first discuss the fold bifurcation through two examples.

First, consider the planar system

$$x' = a - x^2,$$
$$y' = -y\,.$$

We know that the one-dimensional system $x' = a - x^2$ undergoes a fold bifurcation at $a = 0$. For $a < 0$ there are no equilibria in the x-direction while for $a > 0$ there are two equilibria $x = \pm\sqrt{a}$ where $\sqrt{a}$ is stable and $-\sqrt{a}$ is unstable.

Therefore, the planar system has the equilibria $(\sqrt{a}, 0)$ and $(-\sqrt{a}, 0)$ when $a > 0$ and it has no equilibria when $a < 0$.

It is easy to determine the stability of these equilibria.

$(\sqrt{a}, 0)$ is stable in the x-direction and also stable in the y-direction (since $y' = -y$). Therefore it is a **stable node**.

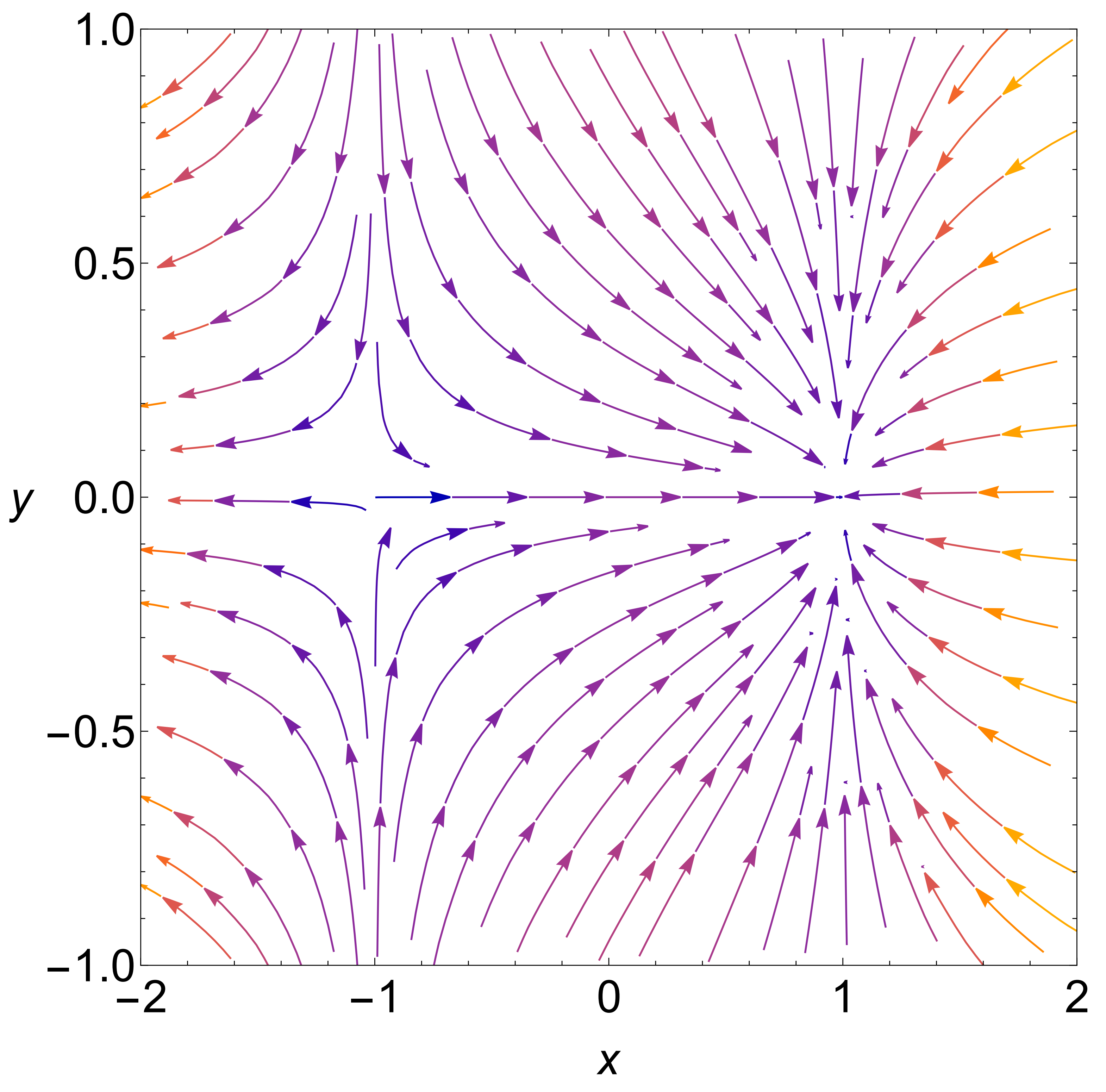
$(-\sqrt{a}, 0)$ is unstable in the x-direction and stable in the y-direction. Therefore it is a **saddle**.

This means that when a becomes positive we have the appearance of two equilibria, one of which is a node while the other one is a saddle. For this reason, this planar fold bifurcation is also called a **saddle-node bifurcation**.

The stability of the equilibria can also be determined through linearization. The Jacobian matrix at (x, y) is $\begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}$.

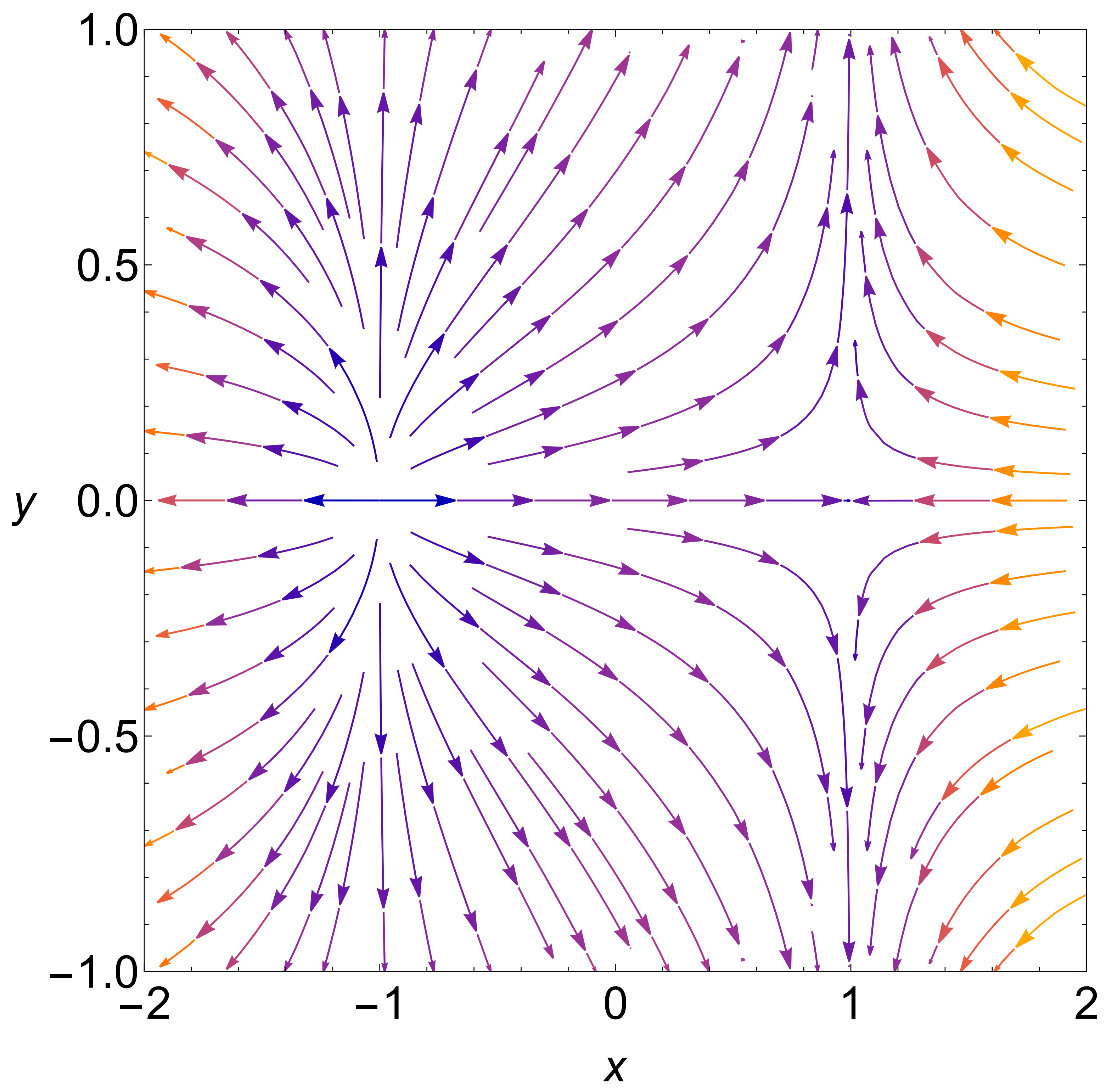
At the equilibrium $(\sqrt{a}, 0)$ the Jacobian matrix becomes $\begin{bmatrix} -2\sqrt{a} & 0 \\ 0 & -1 \end{bmatrix}$. Therefore, there are two negative eigenvalues and the equilibrium is a stable node.

At the equilibrium $(-\sqrt{a}, 0)$ the Jacobian matrix becomes $\begin{bmatrix} 2\sqrt{a} & 0 \\ 0 & -1 \end{bmatrix}$. Therefore, there is one positive and one negative eigenvalue and the equilibrium is a saddle.



We now see a second example. Consider the planar system

$$x' = a - x^2,$$
$$y' = y\,.$$



Nothing changed in the x-direction. However, the equilibrium $y = 0$ in the y-direction is now unstable.

Therefore, $(\sqrt{a}, 0)$ is stable in the x-direction but unstable in the y-direction. Therefore it is a **saddle**.

$(-\sqrt{a}, 0)$ is unstable in both the x-direction and the y-direction. Therefore it is an **unstable node**.

This means that again when a becomes positive we have the appearance of two equilibria, one of which is a node while the other one is a saddle.

Remarks

Remark. Note that in both of the previous examples at $(a, x, y) = (0,0,0)$ the Jacobian matrix becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

The determinant of this matrix is 0 and therefore according to the "planar" implicit function theorem the equilibria may not persist.

Remark. For a matrix A with eigenvalues λ_1, λ_2 we have $\det A = \lambda_1 \lambda_2$. If A is the Jacobian matrix of the system at (a_0, x_0, y_0) then to have a change in the number of equilibria we need $\det A = 0$ and therefore at least one of the eigenvalues should become 0.

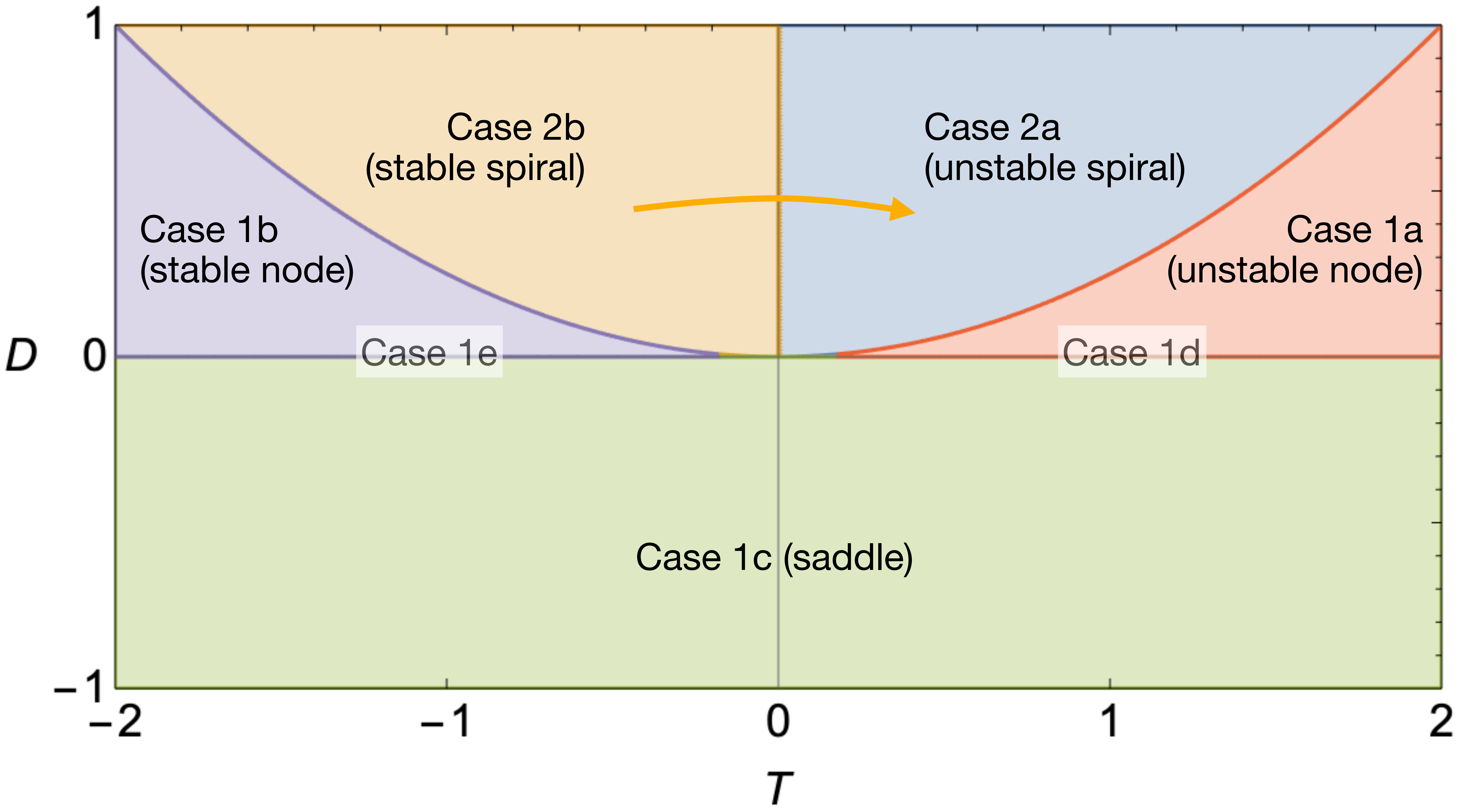
Hopf bifurcation

Another remark

In one-dimensional systems $x' = f(a, x)$ an equilibrium changes stability when f_x changes sign. For this to occur we must have $f_x(a_0, x_0) = 0$.

However, this is exactly the situation where we cannot apply the Implicit Function Theorem to ensure the persistence of the equilibria. Therefore, typically in one-dimensional systems the change of stability of an equilibrium is accompanied by a change of the number of equilibria.

In planar systems, if we denote by A the Jacobian matrix at the equilibrium, we can have a change of stability from stable spiral to unstable spiral, when $\mathrm{tr}(A) = 0$ and $\det(A) > 0$. In this case, since $\det(A) \neq 0$ we conclude that the equilibrium persists while its stability changes.

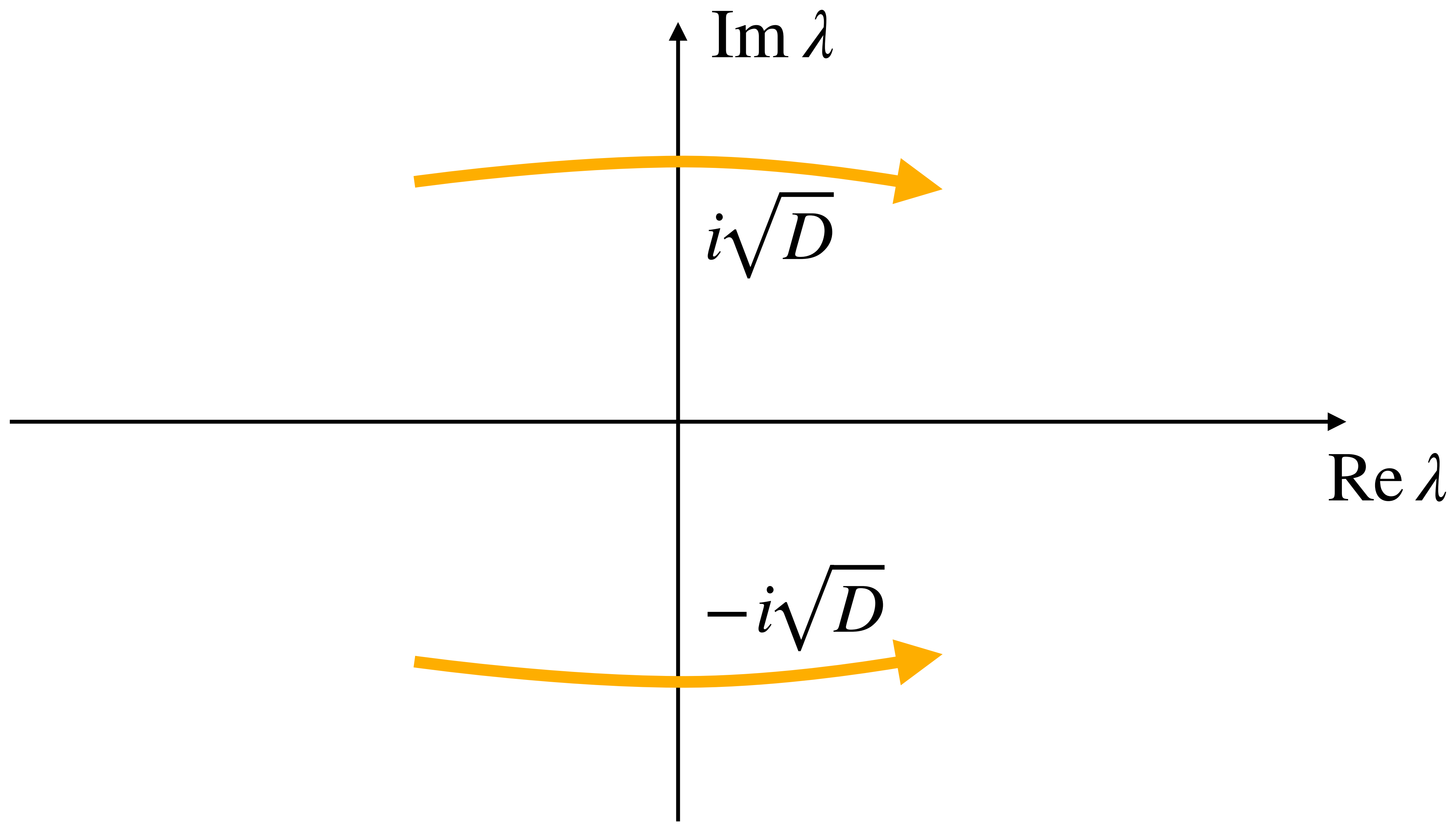


Case 2b
(stable spiral)
Case 2a
(unstable spiral)
Case 1b
(stable node)
Case 1a
(unstable node)
Case 1e
Case 1d
Case 1c (saddle)
D
T
1
0
−1
−2
−1
0
1
2

Recall that in the region of interest $T^2 - 4D < 0$ and $D > 0$ the eigenvalues are given by

$$\lambda = \frac{T}{2} \pm i\frac{\sqrt{4D - T^2}}{2}.$$

As T increases and crosses the axis $T = 0$ with $D > 0$ the real part of λ also increases and changes from negative to positive. For $T = 0$ we have $\lambda = \pm i\sqrt{D}$.



Example

Consider the planar system

$$x' = ax - y + kx(x^2 + y^2),$$
$$y' = x + ay + ky(x^2 + y^2)\,.$$

Here $a \in \mathbb{R}$ is a parameter that changes continuously while k takes the values ± 1.

It is easy to check that this system has the unique equilibrium $(0,0)$.

The linearization of this system at the origin is

$$\begin{aligned} x' &= ax - y, \\ y' &= x + ay\,. \end{aligned}$$

The corresponding matrix is $A = \begin{bmatrix} a & -1 \\ 1 & a \end{bmatrix}$. We compute that

$$T = \operatorname{tr} A = 2a \quad \text{and} \quad D = \det A = a^2 + 1.$$

The corresponding eigenvalues are

$$\lambda = \frac{T}{2} \pm i \frac{\sqrt{4D - T^2}}{2} = a \pm i\,.$$

For $a < 0$ the origin is a stable spiral while for $a > 0$ it becomes an unstable spiral. At $a = 0$ the eigenvalues $\pm i$ are on the imaginary axis (and $T = 0,\ D = 1$).

To understand the full dynamics of the system we use polar coordinates. Recall that we use the equations $rr' = xx' + yy'$ and $r^2\theta' = xy' - x'y$.

Computing the expressions for r' and θ' gives

$$
\begin{aligned}
r' &= r(a + kr^2), \\
\theta' &= 1.
\end{aligned}
$$

In the radial direction we have the equilibrium $r = 0$ which corresponds to the origin in the original planar system.

For $k = -1$, another equilibrium in the radial direction is given by the solutions of $a + kr^2 = a - r^2 = 0$.

For $a < 0$ there are no solutions; for $a \geq 0$ there is a single solution $r = \sqrt{a}$.

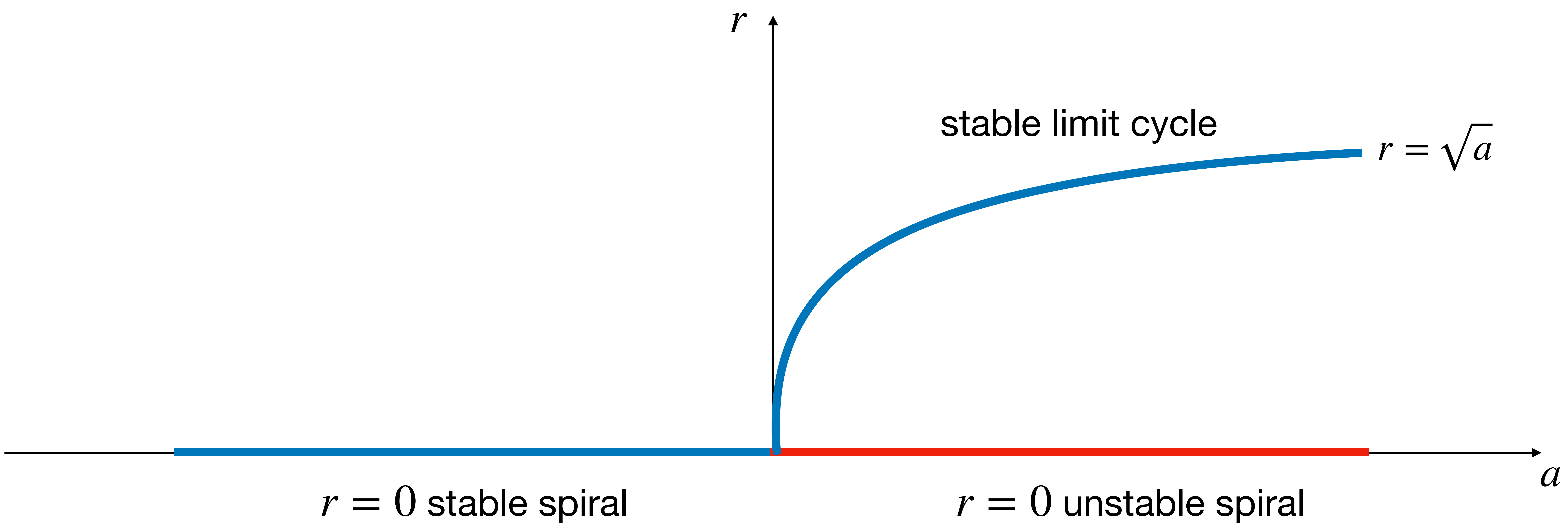
The equilibrium $r = \sqrt{a}$ of the radial dynamics corresponds to a periodic solution (actually, a limit cycle) of the planar system with $r = \sqrt{a}$ and period 2π.

For the stability of the periodic solution we check in more detail the radial dynamics and we find $r = \sqrt{a}$ is asymptotically stable. The latter implies that other solutions approach the limit cycle $r = \sqrt{a}$.

We can summarize this discussion in the bifurcation diagram shown in the next slide.

This type of bifurcation is called a **supercritical Hopf bifurcation**. The name supercritical corresponds to the **limit cycle** being **stable**.

Bifurcation diagram (supercritical)



For $k = 1$ we have the equilibrium $r = 0$ and also the equation $a + r^2 = 0$. The latter has the solution $r = \sqrt{-a}$ for $a \leq 0$ and it has no solutions for $a > 0$.

Note that in terms of stability nothing changes for the origin (since this is determined by the linearization and there is no k in the linearized system).

However, analyzing the radial dynamics we find that $r = \sqrt{-a}$ is unstable and therefore it corresponds to an unstable limit cycle. The bifurcation diagram is shown on the next slide.

This is called a **subcritical Hopf bifurcation**. The name subrcritical corresponds to the **limit cycle** being **unstable**.

Bifurcation diagram (subcritical)