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A method for accurate computation of the rotation and the twist numbers of invariant circles

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Abstract

A method is proposed for accurate evaluation of the rotation and the twist numbers of invariant circles in two degrees of freedom Hamiltonian systems or two-dimensional symplectic maps. The method uses the recurrence of orbits to overcome the problems usually arising because of the multivalued character of the angles (due to modulo 2π) that have to be added in order to evaluate the above numbers. Furthermore, best convergent demoninators Q_n of these numbers can be estimated and we show that under a proper treatment of the sequences of Q_n iterations the accuracy is of the order of $1/Q_n^4$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The computation of the rotation number (frequency) of invariant circles plays a prominent role in the study of the organized motions in chaotic systems. For the standard map, the breakup hierarchy of the invariant circles is conjectured to be connected with the number theoretical properties of their rotation number [1,2]. Also the non-monotonicity of the frequency map has been used as an indicator of the non-existence of invariant circles in a region of the phase space [3].

For these applications, one needs to compute the rotation number of the invariant circles with high precision. With the direct method for the computation of the rotation number (Section 2), i.e. as the mean value of the rotation angles, the accuracy is of order $1/N$, where N is the number of iterations of the map. A method based on the continued fraction approximations, described in [4], can improve this precision.

When we are interested in the direct computation of the rotation number of invariant circles around a fixed point, a problem is that angles are not determined uniquely, but only up to an integer multiple of 2π. One has to be careful

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about this ambiguity because it can affect the determination of the rotation number and can lead to incorrect results. The same problem also appears in the computation of the twist number [5], i.e. the mean value of the angles between successive vectors tangent to the invariant curve. Usually, in order to determine the rotation or the twist angles, one fixes an interval $[a, a + 2\pi)$ and defines the angles inside this interval. In Section 2, we explain how we can find such an interval if it exists and also explain what we can do when such an interval does not exist. In [6], the notion of the *self-rotation* number was used in order to overcome these problems.

In this paper, we propose a new method that we call improved continued fraction (ICF) method that (i) allows us to overcome the problem of the correct determination of the rotation or of the twist angles, and (ii) improves the accuracy of the computation of the rotation or the twist numbers, respectively, to the order of $1/N^4$. This is achieved by adding the *small angles* introduced in Section 3, that are always inside the interval $(-\pi, \pi)$ and, therefore, there is no ambiguity in their computation. The same order of accuracy can be obtained by Laskar's method [3] which is based on the Fourier analysis of the orbit using a Hanning filter.

In Section 2, we describe the direct method for the computation of the rotation and twist numbers, and explain how we should determine the angles in all possible cases. In Section 3, we define the small angles, describe the ICF method, and give numerical results concerning its accuracy. Finally, in Section 4, we explain the asymptotic properties of the ICF method.

2. Problems in the computation of the rotation and twist numbers

We consider a two-dimensional symplectic map of the plane, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that sends the point $z = (x, y)$ to the point $z' = (x', y') = f(x, y)$. If the map has an elliptic fixed point $z_c = (x_c, y_c)$, then the well-known theorem of Moser [7] ensures that under certain non-resonance conditions the fixed point will be surrounded by invariant curves. These invariant curves are referred in many contexts as librational [8–10], they can, however, be characterized locally as rotational with respect to the fixed point.

Given an initial point $z_0 = (x_0, y_0)$ on such an invariant curve $\mathcal{I}$, successive applications of the map f, produce the orbit

$$\mathcal{O}(z_0) = \{z_0, z_1, \dots, z_n, \dots\}, \tag{1}$$

where $z_n = f^n(z_0)$.

At each point $z \in \mathcal{I}$, let $r(z)$ be the vector $z - z_c$, i.e. the position vector of the point z with respect to the fixed point z_c. We define the rotation angle $w_\theta(z_1, z_2)$ between two points z_1 and z_2 on $\mathcal{I}$, as the angle between the two position vectors $r(z_1)$ and $r(z_2)$ (Fig. 1), inside an appropriate definition interval of the form $[a, a + 2\pi)$. The rotation number of f on $\mathcal{I}$ is given by

$$\nu_\theta = \lim_{N\to\infty} \frac{1}{2\pi} \frac{\sum_{j=0}^{N-1} w_\theta(z_j, z_{j+1})}{N}. \tag{2}$$

At each point $z \in \mathcal{I}$ we can define a unit vector $v(z) \in T_z\mathbb{R}^2 \cong \mathbb{R}^2$ tangent to the invariant curve, where $T_z\mathbb{R}^2$ is the tangent space at z, i.e. the space of all vectors with their base points at z. We define the twist angle $w_\phi(z_1, z_2)$ between two points z_1 and z_2 on $\mathcal{I}$ as the angle between the two tangent vectors $v(z_1)$ and $v(z_2)$ (Fig. 1), inside an appropriate definition interval of the form $[a', a' + 2\pi)$. The twist number of f on $\mathcal{I}$ is given by

$$\nu_\phi = \lim_{N\to\infty} \frac{1}{2\pi} \frac{\sum_{j=0}^{N-1} w_\phi(z_j, z_{j+1})}{N}. \tag{3}$$

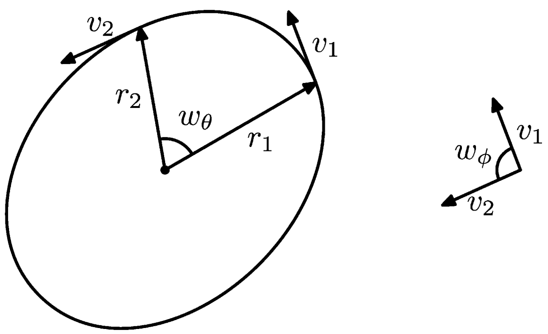

Fig. 1. Illustration of the rotation (w_θ) and the twist (w_ϕ) angles and the respective vectors for a librational invariant curve.

Note that in order to compute the twist angles, we need to know the tangent vector $v(z)$ at each point z of the invariant curve. Since this is not always an easy task, we can consider some other vector $\tilde{v} \in T_z\mathbb{R}^2$. After some iterations of the map the image of the vector $\tilde{v}$ will become almost tangent to the invariant curve [5] and, therefore, we can use the images of $\tilde{v}$ as a good approximation of the correct tangent vector.

The direct method for the computation of the rotation number involves the computation of the quantity

$$\mu_N(z_0) = \frac{1}{2\pi}\frac{\sum_{j=0}^{N-1} w_\theta(z_j, z_{j+1})}{N} \tag{4}$$

for some finite N. This is an approximation of the exact rotation number, which was defined in Eq. (2) as $\nu_\theta = \lim_{N\to\infty}\mu_N(z_0)$. The twist number can also be computed using the analog of Eq. (4), the only difference being that we must first iterate the map several times in order for the vector $\tilde{v}$ to become tangent to the invariant curve as we had explained previously.

Notice that, the twist and rotation numbers are equal for librational invariant circles, but they are different for higher order islands around the fixed point and for chaotic regions. The equality of the rotation and the twist number can be used as an indicator of the existence of invariant circles around a fixed point [11] and, therefore, it is important to know the values of these numbers with high accuracy.

In Eq. (4), the rotation angles (or the twist angles in the respective case) are defined inside an interval $[a, a+2\pi)$. In order to understand how we can find such an interval, we need a definition of the rotation and the twist angles that *does not* depend on the selection of some interval. For this purpose, we consider a segment O_N of an orbit on the invariant curve, consisting of N iterates of the map. After fixing a definite orientation for traversing the curve, we can sort the iterates of the orbit *along* the curve. To be more precise, suppose that we begin at a point z of O_N and following the previously fixed orientation we proceed along the curve until we find another iterate z' of O_N. Then the two first points of the sorted orbit are z and z'. Continuing from z' towards the same direction we will find another iterate z'' of the orbit. This will be the third point of the ordered orbit, and we continue the procedure above until we return to z. Points that are in adjacent places in the sorted orbit will be called adjacent points.

By making N large enough, we can ensure that the rotation angle between two adjacent points of the orbit will be small and, therefore, inside the interval $(-\pi, \pi)$. This means that there is no ambiguity in the computation of the rotation angle between adjacent points. Now we can define the rotation angle between *any two* points z_1 and z_2 on the invariant curve as

$$w_\theta(z_1, z_2) = w_\theta(z_1, z^{(1)}) + w_\theta(z^{(1)}, z^{(2)}) + \cdots + w_\theta(z^{(M-1)}, z^{(M)}) + w_\theta(z^{(M)}, z_2), \tag{5}$$

where $\{z^{(j)}\}_{j=1,M}$ is the sequence of points of the sorted orbit O_N, that we find between z_1 and z_2 when traversing the invariant curve from z_1 to z_2 following the previously fixed orientation. A similar definition can be used for the twist angles. It would be perhaps more natural to give the above definition using an integral along the invariant curve instead of the sum in Eq. (5), but Eq. (5) also proposes a practical way of using our definition.

In order to determine the proper definition interval for the angles w_θ and w_ϕ, we will use the notion of the angular dynamical spectra [5,11,12], that we denote by S_θ for the rotation angles and S_ϕ for the twist angles. This is the distribution function of the angles, namely

$$S_\theta(w) = \lim_{\substack{N\to\infty \\ \delta w \to 0}} \frac{1}{\delta w} \frac{\delta N(w, w+\delta w)}{N},$$

where $\delta N(w, w+\delta w)$ is the number of angles $w_\theta(z_n, z_{n+1})$ in the small interval $(w, w+\delta w)$ after N iterations of the map f. Assuming that $S_\theta(w)$ is integrable, we can express the rotation number as the integral

$$\nu_\theta = \frac{1}{2\pi} \int_{\mathbb{R}} w S_\theta(w)\,\mathrm{d}w, \tag{6}$$

and a similar relation holds for the twist number. It is easy to see that the angular dynamical spectra on invariant circles do not have any gaps and do not extend to infinity.

Proposition 1. *The support of the spectrum of the angles is compact and connected.*

In order to see this, consider the mapping δ that to each point of the circle associates the corresponding rotation angle, i.e. $\delta(z) = w_\theta(z, f(z))$. This mapping is continuous. Notice that the support of the spectrum is equal to the image of the circle under δ. Since the invariant circle is compact and connected so is its image under δ. For our purposes, this limits considerably the possible forms of the spectra.

As we already mentioned, we define w_θ inside an interval of the form $[a, a+2\pi)$. If a belongs in the support of the spectrum, then by defining angles inside $[a, a+2\pi)$, a part of the spectrum will appear near a, another part of the spectrum will appear near $a+2\pi$ and there will be a gap between these two parts. The same happens when $a+2\pi$ is inside the support of the spectrum. Since Proposition 1 gaps are not admissible, the particular choice would be evidently wrong. Therefore, we must define the angles inside an interval of the form $[a, a+2\pi)$ such that it contains completely the whole spectrum. It is easy to construct an algorithm that can take care of this automatically.

One case where the above method is not applicable is when the spectrum has width greater than 2π, thus it is not contained completely in any interval of the form $[a, a+2\pi)$. A case like this can be identified numerically by computing the angles inside some interval of the form $[a, a+2\pi)$. If the width of the spectrum is larger than 2π then this interval will be completely covered by the spectrum. Notice, that methods based on the Fourier analysis of the orbit may fail in this case, due to the problem of selecting the correct peak of the Fourier spectrum. A solution in this case is to sort the iterates of the orbit along the invariant curve and then apply Eq. (5) in order to compute the angles.

3. Small angles and the ICF method

As we have mentioned in Section 2, one computes the rotation number using Eq. (4) for some finite but large N. If we know a best convergent P_n/Q_n [13] of the rotation number ν_θ we can write

$$\mu_{Q_n}(z_0) = \frac{1}{2\pi} \frac{2\pi P_n + u_{Q_n}(z_0)}{Q_n} = \frac{P_n}{Q_n} + \frac{1}{2\pi} \frac{u_{Q_n}(z_0)}{Q_n}, \tag{7}$$

where we have expressed the numerator of (4) as a sum of P_n circles and a *small angle* $u_{Q_n}(z_0)$. Obviously, the small angle is the rotation angle between the vectors $r(z_0)$ and $r(z_{Q_n})$.

The numerators and denominators of the best convergents will be called *best numerators* and *best denominators*, respectively. We denote by $[a_0, a_1, a_2, \dots]$ the continued fraction expansion of ν_θ [13]. The best numerators of ν_θ satisfy the recursive relation

$$P_n = a_n P_{n-1} + P_{n-2} \tag{8}$$

with $P_{-2} = 0$ and $P_{-1} = 1$. The recursive relation

$$Q_n = a_n Q_{n-1} + Q_{n-2}, \tag{9}$$

holds for the best denominators with $Q_{-2} = 1$ and $Q_{-1} = 0$.

It is well known that if the dynamics on the invariant curve are diffeomorphically conjugate to a rigid rotation, then u_{Q_n} is of the same order as $|Q_n\nu_\theta - P_n|$. From the theory of best convergents [13], we know that $|Q_n\nu_\theta - P_n| < 1/Q_n$. This means that u_{Q_n} is of the order of $1/Q_n$, thus for a large enough Q_n we can ensure that u_{Q_n} is inside the interval $[-\pi, \pi)$.

Eq. (7) is the basis of the ICF method, therefore the method relies heavily on the correct determination of the best convergent P_n/Q_n of the rotation number ν_θ. In order to determine Q_n we are using the fact that the dynamics on a circle with an irrational rotation number ν_θ, are always topologically conjugate to a rigid rotation by $2\pi\nu_\theta$ [14]. This means that if an orbit begins at a point z_0 of an invariant curve $\mathcal{I}$, the points of the orbit $\mathcal{O}(z_0)$ that will come closer to z_0 are the iterates $f^{Q_n}(z_0)$. Thus by recording the points of the orbit that minimize the distance with z_0 we can compute the best denominators Q_n. According to [4], this method was originally proposed by Hénon.

Once we have determined Q_n, we also need to determine P_n. In this case, we want to find how many times the orbit winds around the invariant curve in Q_n iterations. The determination of P_n can be simplified by the fact that if we determine two consecutive best convergents P_{n-1}/Q_{n-1} and P_n/Q_n of the rotation number then the rest of the best convergents can be determined if we just determine the best denominators $Q_{n+1}, Q_{n+2}, \dots$ Specifically, if we compute Q_{n+1} then by Eq. (9), we get $a_{n+1} = (Q_{n+1} - Q_{n-1})/Q_n$ and then P_{n+1} can be computed from Eq. (8).

An important benefit of selecting best denominators of ν_θ as the number of iterations to use, is that the difference $|\mu_{Q_N}(z_0) - \nu_\theta|$, i.e. the error of the approximation, is of the same order as $|\nu_\theta - P_n/Q_n| < 1/Q_n^2$. This means that without any significant extra computational effort we have a method for the computation of the rotation number that is accurate up to a quantity of order $1/Q_n^2$, which is much better than $1/N$ which is the guaranteed accuracy if we select at random a number of iterations N.

This can be seen clearly in Fig. 2a–c where we have plotted the logarithm of the error of the approximation $|\mu_N - \nu_\theta|$ as a function of $\log N$ for three invariant curves $\mathcal{I}_\mathrm{A}, \mathcal{I}_\mathrm{B}, \mathcal{I}_\mathrm{C}$ of the standard map

$$(x, y) \to (x', y') = \left(x + y + \frac{K}{2\pi}\sin(2\pi x), y + \frac{K}{2\pi}\sin(2\pi x)\right) \bmod 1,$$

where $x, y \in [0, 1)$. These invariant curves are rotational and the rotation angle $\Delta\theta$ is equal to $2\pi(x' - x) = 2\pi y - K\sin(2\pi x)$.

The invariant curve $\mathcal{I}_\mathrm{A}$ has initial conditions

$$x_0 = 0, \qquad y_0 = 0.6604939698315303$$

for $K = 0.9$. The rotation number of $\mathcal{I}_\mathrm{A}$ is equal (up to numerical error) to the golden number

$$\varphi = [0, (1)^\infty] = \tfrac{1}{2}(\sqrt{5} - 1)$$

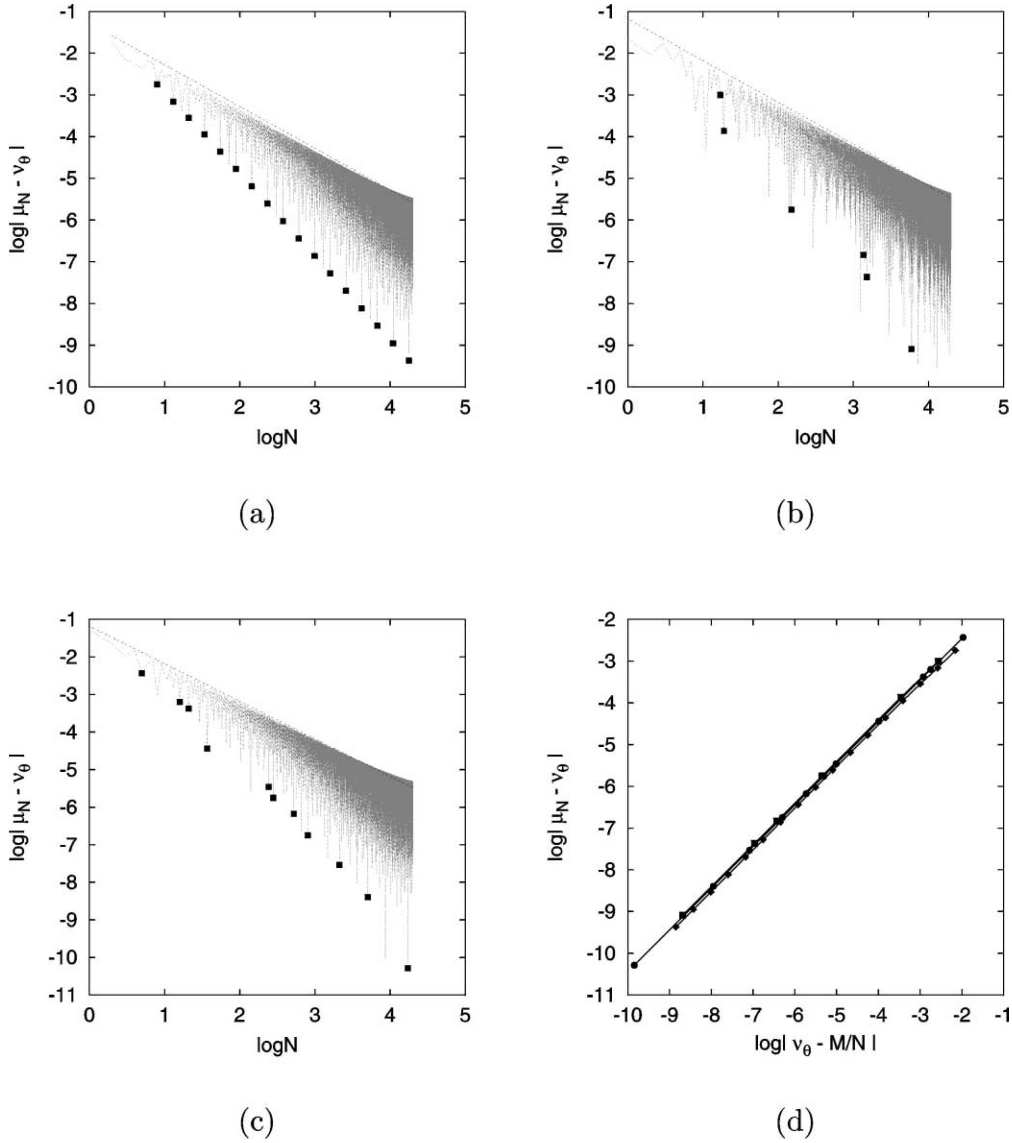

Fig. 2. The error of the approximation $|\mu_N - \nu_\theta|$. (a–c) Log–log diagrams of $|\mu_N - \nu_\theta|$ vs. N for the invariant curves $\mathcal{I}_\mathrm{A}$, $\mathcal{I}_\mathrm{B}$ and $\mathcal{I}_\mathrm{C}$, respectively. The straight lines have slope -1. (d) Log–log diagram of $|\mu_N - \nu_\theta|$ vs. $|\nu_\theta - M/N|$ for values of N that are best denominators of the corresponding rotation numbers for the three invariant curves. The slope of the three lines is equal to 1.

The invariant curve $\mathcal{I}_\mathrm{B}$ has initial condition

$$x_0 = 0, \qquad y_0 = 0.54783921738192783921$$

for $K = 0.5$ and rotation number

$$\nu_{\theta\mathrm{B}} = [0, 1, 1, 8, 1, 7, 9, 1, 3, 13, 1, 1, 169, 1, 18, 1, \dots].$$

Finally, the invariant curve $\mathcal{I}_C$ has initial conditions

$$x_0 = 0, \qquad y_0 = 0.86356172536726678432$$

for $K = 0.5$ and rotation number

$$\nu_{\theta C} = [0, 1, 4, 3, 1, 1, 6, 1, 1, 1, 2, 2, 3, 22, 1, 8, 17, 1, 2, \ldots].$$

All the computations were performed using multiprecision numbers from the LiDIA C++ library [15] with 50 accurate digits.

One important feature of these figures is that the error, in all three cases, is bounded above by a quantity proportional to $1/N$ (the dashed lines in Fig. 2a–c). The constant of proportionality is different for the three invariant curves. The most striking feature of the graphs, though, are the data points that we have plotted with squares. These correspond to values of N that are best denominators of the corresponding rotation numbers. We see that the error for these values of N is consistently smaller than for most other values of N and, in fact, it falls much faster than $1/N$.

In Fig. 2d, we have plotted the logarithm of the error of the approximation as a function of $\log|\nu_\theta - M/N|$, only for those values of N that are best denominators of the corresponding rotation numbers. We can see that the error is proportional to $|\nu_\theta - M/N|$ in all three cases.

The ICF method is completed by going one step further. Specifically, we can improve the accuracy of the computation of the rotation number from $1/Q_n^2$ to $1/Q_n^4$. This can be achieved by defining a small angle $u_{Q_n}(z_j)$ *for each point* z_j of the orbit, where Q_n is a best denominator of the rotation number. It is clear that the *average of the small angles* over these Q_n points will be much closer to the exact value than the value of the small angle at only one point. This is happening because the values of u_{Q_n} appear alternately above and below their mean value and, therefore, they cancel each other in order to give a much more accurate result. Therefore, instead of computing $\mu_{Q_n}(z_0)$ from Eq. (7) as an approximation of the rotation number, we can substitute the small angle $u_{Q_n}(z_0)$ that appears in Eq. (7) with the mean value

$$\langle u_{Q_n} \rangle_{Q_n}(z_0) = \frac{1}{2\pi} \frac{\sum_{j=0}^{Q_n-1} u_{Q_n}(z_j)}{Q_n} \tag{10}$$

of the small angles over the first Q_n points of the orbit. Notice here, that we need $2Q_n$ points of the orbit in order to compute $\langle u_{Q_n} \rangle_{Q_n}(z_0)$. This way we define a new approximation $\nu_{Q_n,Q_n}(z_0)$ to the rotation number that is more accurate than $\mu_{Q_n}(z_0)$. Explicitly, the new approximation is given by the relation

$$\nu_{Q_n,Q_n}(z_0) = \frac{P_n}{Q_n} + \frac{\langle u_{Q_n} \rangle_{Q_n}(z_0)}{Q_n}, \tag{11}$$

and as we will see the accuracy of the method, i.e. $|\nu_{Q_n,Q_n} - \nu_\theta|$ is of order $(\nu_\theta - P_n/Q_n)^2$ (or $1/Q_n^4$) for large Q_n.

In Fig. 3, we have plotted $\log|\nu_{Q_n,Q_n} - \nu_\theta|$ as a function of $\log|Q_n|$ for the three invariant curves $\mathcal{I}_A$, $\mathcal{I}_B$, $\mathcal{I}_C$. We can see that the error is of the order $1/Q_n^4$ for large Q_n. Note that by computing ν_{Q_n,Q_n}, instead of μ_{Q_n}, we increased the accuracy of the determination of ν_θ from $1/Q_n^2$ to $1/Q_n^4$ *by only doubling the number of iterations*.

We must notice that if the rotation number ν_θ is close to a rational, then the best denominators of ν_θ are distributed sparsely among the integers, in the sense that for a given positive integer N we expect to find fewer best denominators of ν_θ that are smaller than N than if the rotation number is "very irrational". This means that for rotation numbers close to rationals it will be harder to find an appropriate best denominator from which we will be able to compute the rotation number with very high accuracy.

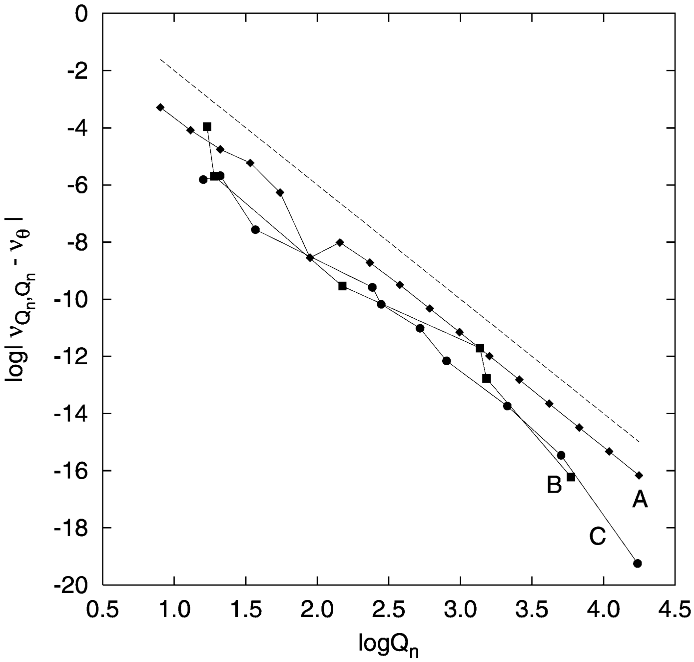

Fig. 3. Log–log diagram of the error of the approximation $|\nu_{Q_n,Q_n} - \nu_\theta|$ vs. Q_n, where P_n/Q_n are best convergents of the corresponding rotation numbers. The straight line has slope −4.

4. Explanation of the asymptotic behavior of the ICF method

In order to understand the asymptotic behavior of the method, we express $u_{Q_n}(z)$ as $2\pi(Q_n\nu_\theta - P_n)(1 + v_{Q_n}(z))$ and substituting in Eq. (7) we get

$$\mu_{Q_n}(z) = \nu_\theta + \left(\nu_\theta - \frac{P_n}{Q_n}\right) v_{Q_n}(z). \tag{12}$$

Since the rotational circles of the standard map are graphs over x (i.e. there is a function J such that $y = J(x)$) we can describe points on a circle using only x instead of $z = (x, y)$ and we can define the restriction $F_{\mathcal{I}}$ of the standard map on a circle $\mathcal{I}$ as

$$F_{\mathcal{I}}(x) = \pi_x F_{\mathrm{sm}}(x, J(x)),$$

where π_x is the projection on x.

We consider a lift F of $F_{\mathcal{I}}$, i.e. a continuous function $F\colon \mathbb{R} \to \mathbb{R}$ such that

$$F(x) \bmod 1 = F_{\mathcal{I}}(x \bmod 1),$$

and for which $F(x + 1) = F(x) + 1$ [16]. We will also assume that F is a C^∞ diffeomorphism with diophantine rotation number and, therefore, there is a C^∞ diffeomorphism H such that $FH = HF_{\nu_\theta}$, where F_{ν_θ} is the rigid rotation $F_{\nu_\theta}\colon \mathbb{R} \to \mathbb{R}\colon x \to x + \nu_\theta$ [17].

We will use the following theorem that we prove in Appendix A.

Theorem 2. *Consider a C^∞ function G: $\mathbb{R} \to \mathbb{R}$, such that $G(x+1) = G(x)$, and a C^∞ diffeomorphism F: $\mathbb{R} \to \mathbb{R}$, such that $F(x+1) = F(x) + 1$ with diophantine rotation number ν_θ. If Q_n is a best denominator of ν_θ then*

$$\langle G\rangle_{Q_n}^F(x) - \langle G\rangle^F = g_{Q_n}(x)\frac{Q_n\nu_\theta - P_n}{Q_n},$$

where g_{Q_n}: $\mathbb{R} \to \mathbb{R}$ is a C^∞ function with $\langle g\rangle_{Q_n} = 0$ and $g_{Q_n}(x+1) = g_{Q_n}(x)$. Also the family g_{Q_n} converges pointwise to a C^∞ function.

Here, where $\langle G\rangle_N^F(x)$ is the finite time average of G on the orbit of F with initial point x

$$\langle G\rangle_N^F(x) = \frac{1}{N}\sum_{j=0}^{N-1} G(F^j(x)), \tag{13}$$

and

$$\langle G\rangle^F = \lim_{N\to\infty}\langle G\rangle_N^F(x). \tag{14}$$

Remark. We have implicitly assumed that the time average of G exists and is independent of x. In Appendix A, we prove that this is true under the same conditions as Theorem 2.

In order to apply the last theorem, we define $\Delta(x) = F(x) - x$ and we notice that $\nu_\theta = \langle\Delta\rangle^F$ and $\mu_{Q_n}(x) = \langle\Delta\rangle_{Q_n}^F(x)$. Then applying Theorem 2 with $G = \Delta$ we have that

$$\mu_{Q_n}(x) - \nu_\theta = \langle\Delta\rangle_{Q_n}^F(x) - \nu_\theta = v_{Q_n}(x)\frac{Q_n\nu_\theta - P_n}{Q_n}.$$

According to Theorem 2, the sequence $v_{Q_n}(x)$ converges pointwise to a C^∞ function v_∞. In this particular case, it is easy to see (following the argument given in the proof of Theorem 2 in Appendix A) that

$$v_\infty(x) = H'(H^{-1}(x)) - 1,$$

where $H'(x) = \mathrm{d}H(x)/\mathrm{d}x$.

Averaging Eq. (12) over the first Q_n points of the orbit we get

$$\nu_{Q_n,Q_n}(x) - \nu_\theta = \left(\nu_\theta - \frac{P_n}{Q_n}\right)\langle v_{Q_n}\rangle_{Q_n}(x), \tag{15}$$

where $\langle v_{Q_n}\rangle_{Q_n}^F(x) = \sum_{j=0}^{Q_n-1} v_{Q_n}(F^j(x))/Q_n$.

The functions v_{Q_n} are C^∞ and we have that $\langle v_{Q_n}\rangle = 0$ and $v_{Q_n}(x+1) = v_{Q_n}(x)$. Therefore, we can apply again Theorem 2, this time with $G = v_{Q_n}$ and we get

$$\langle v_{Q_n}\rangle_{Q_n}(x) = c_{Q_n}(x)\frac{Q_n\nu_\theta - P_n}{Q_n}. \tag{16}$$

This behavior is also evident in Fig. 4, where we have plotted the value of $|\langle v_{Q_n}\rangle_{Q_n}|$ as a function of $|\nu_\theta - P_n/Q_n|$.

Substituting the last result in Eq. (15) we find

$$\nu_{Q_n,Q_n}(x) - \nu_\theta = c_{Q_n}(x)\left(\nu_\theta - \frac{P_n}{Q_n}\right)^2. \tag{17}$$

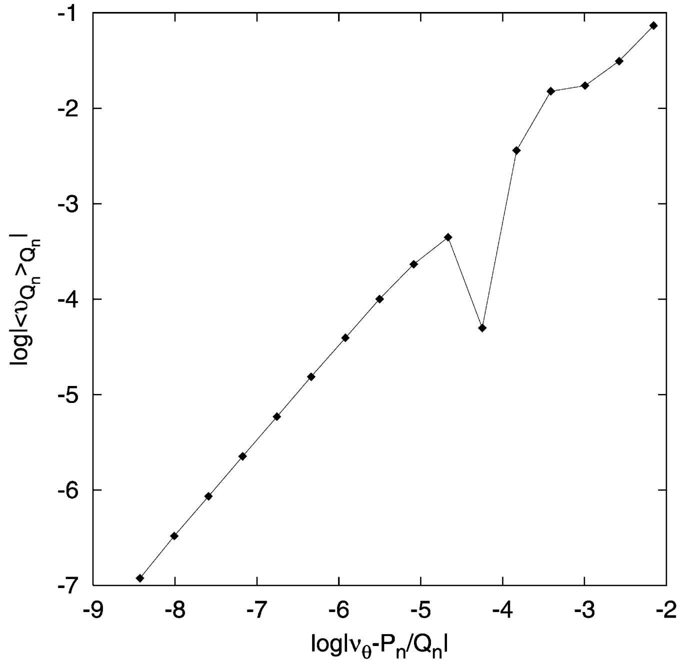

Fig. 4. Log–log diagram of $\langle \upsilon_{Q_n} \rangle_{Q_n}$ vs. $|\nu_\theta - P_n/Q_n|$ for the invariant curve $\mathcal{I}_{\mathrm{A}}$ of the standard map. The slope of the curve is asymptotically equal to 1.

The sequence $c_{Q_n}(x)$ is bounded since it converges pointwise to the C^∞ function $c_\infty(x)$. This means that there is a positive number M such that

$$|c_{Q_n}(x)| < M. \tag{18}$$

We find that

$$|\nu_{Q_n,Q_n}(x) - \nu_\theta| < M\left(\nu_\theta - \frac{P_n}{Q_n}\right)^2 < \frac{M}{Q_n^4}. \tag{19}$$

We conclude that the accuracy of ν_{Q_n,Q_n} is of order $\mathrm{O}(1/Q_n^4)$.

5. Summary and conclusions

We investigated the problem of the definition of the angles used for the correct computation of the rotation and twist numbers. We showed that in some cases one can find an interval of the form $[a, a + 2\pi)$ and define the angles inside this interval. This can be done using the spectrum of the angles. However, there are cases where such an interval does not exist. In this case, one can sort the points on the invariant curve and compute the individual rotation (twist) angles as a sum of rotation (twist) angles between adjacent points of an orbit on the invariant curve.

Another way of bypassing the above problem is by defining and adding the small angles (Section 3). The computation of the mean value of the small angles is more accurate, compared to the computation of the mean value of the rotation angles. The result is that the proposed ICF method has an accuracy of order $|\nu_\theta - P/Q|^2$ if P/Q is a best convergent of ν_θ. Since $|\nu_\theta - P/Q| < 1/Q^2$ the accuracy of the method is better than $1/Q^4$. We have proven that this asymptotic behavior is true for any invariant curve that has diophantine rotation number and on which the dynamics is diffeomorphic to a rigid rotation.

Although in Sections 3 and 4 we described the application of the ICF method in the case of the computation of the rotation number, we notice that we can also use the ICF method in order to compute the twist number, provided that we have an accurate approximation of the initial tangent vector.

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Appendix A. Proof of Theorem 2

1. Since F is C^∞ and its rotation number is diophantine there is a C^∞ diffeomorphism $H\colon \mathbb{R} \to \mathbb{R}$, such that $FH = HF_{\nu_\theta}$ and $H(x+1) = H(x) + 1$ [17] (we denote the composition of functions by concatenation of their symbols).
2. We seek a periodic function K, such that
$$K(x+\nu_\theta) - K(x) = G(H(x)) - \langle GH\rangle_s, \tag{A.1}$$
where for a periodic function f, we denote the space average by
$$\langle f\rangle_s = \int_0^1 f(x)\,\mathrm{d}x.$$
Eq. (A.1) is the homological equation. It has a C^∞ solution, when the right-hand side of the equation is C^∞ and has zero space average and the number ν_θ is diophantine [17]. These conditions obviously hold, therefore, Eq. (A.1) has a C^∞ solution K, such that $K(x+1) = K(x)$.
3. We have
$$Q_n\langle G\rangle^F_{Q_n}(x) = \sum_{j=0}^{Q_n-1} GF^j(x) = \sum_{j=0}^{Q_n-1} GHF^j_{\nu_\theta}(H^{-1}(x)). \tag{A.2}$$
Solving Eq. (A.1) for $GH(x)$ and substituting into the previous equation we find
$$\begin{aligned} Q_n\langle G\rangle^F_{Q_n}(x) &= \sum_{j=0}^{Q_n-1} (KF^{j+1}_{\nu_\theta}(H^{-1}(x)) - KF^j_{\nu_\theta}(H^{-1}(x)) + \langle GH\rangle_s) \\ &= KF^{Q_n}_{\nu_\theta}(H^{-1}(x)) - K(H^{-1}(x)) + Q_n\langle GH\rangle_s \\ &= K(H^{-1}(x) + Q_n\nu_\theta) - K(H^{-1}(x)) + Q_n\langle GH\rangle_s \\ &= K(H^{-1}(x) + Q_n\nu_\theta - P_n) - K(H^{-1}(x)) + Q_n\langle GH\rangle_s = g_n(x)(Q_n\nu_\theta - P_n) + Q_n\langle GH\rangle_s, \end{aligned}$$
where we have defined
$$g_n(x) = \frac{K(H^{-1}(x) + Q_n\nu_\theta - P_n) - K(H^{-1}(x))}{Q_n\nu_\theta - P_n}. \tag{A.3}$$

Making the above computation with Q_n substituted by N we find that

$$\langle G\rangle_N^F(x) = \frac{K(H^{-1}(x) + N\nu_\theta - M_N) - K(H^{-1}(x))}{N} + \langle GH\rangle_s,$$

where M_N is an integer (that depends on N), such that $N\nu_\theta - M_N \in [0, 1)$. Letting $N \to \infty$, we see that $\lim_{N\to\infty}\langle G\rangle_N^F(x) = \langle GH\rangle_s$ for all $x \in \mathbb{R}$.

4. From Eq. (A.3), we easily deduce that g_{Q_n} is C^∞ and $g_{Q_n}(x+1) = g_{Q_n}(x)$. We also have

$$\begin{aligned}\langle g_{Q_n}\rangle^F &= \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} g_{Q_n}(F^j(x)) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\frac{K(H^{-1}(F^j(x)) + Q_n\nu_\theta - P_n) - K(H^{-1}(F^j(x)))}{Q_n\nu_\theta - P_n}\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\frac{K(F_{\nu_\theta}^j H^{-1}(x) + Q_n\nu_\theta - P_n) - K(F_{\nu_\theta}^j H^{-1}(x))}{Q_n\nu_\theta - P_n}\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\frac{K(H^{-1}(x) + Q_n\nu_\theta - P_n + j\nu_\theta) - K(H^{-1}(x) + j\nu_\theta)}{Q_n\nu_\theta - P_n}\\ &= \frac{\langle K\rangle^{F_{\nu_\theta}} - \langle K\rangle^{F_{\nu_\theta}}}{Q_n\nu_\theta - P_n} = 0,\end{aligned}$$

where in the last equation we used the fact that

$$\langle K\rangle^{F_{\nu_\theta}} = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} K(x + j\nu_\theta)$$

is independent of x provided that ν_θ is irrational.

5. What remains to be proven is that the family $\{g_{Q_n}\}_{n=1,\infty}$ converges pointwise to a C^∞ function. This follows directly from the definition of g_{Q_n} (A.3), the fact that $\lim_{n\to\infty}(Q_n\nu_\theta - P_n) = 0$ and that K is C^∞. Specifically, we have that

$$\lim_{n\to\infty} g_{Q_n}(x) = K'(H^{-1}(x)).$$

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