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Perturbations of the 1 : 1 : 1 resonance with tetrahedral symmetry: a three degree of freedom analogue of the two degree of freedom Hénon–Heiles Hamiltonian

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Abstract

We study a class of three degree of freedom (3-DOF) Hamiltonian systems that share certain characteristics with the 2-DOF Hénon–Heiles Hamiltonian. Our systems represent a 1 : 1 : 1 resonant three-oscillator whose principal nonlinear perturbation is the cubic potential term xyz with tetrahedral symmetry. After normalizing and reducing the 1 : 1 : 1 oscillator symmetry, we show that near the limit of linearization all our systems can be described as a one-parametric family. Such reduced systems have been suggested earlier by Hecht (1960 *J. Mol. Spectrosc.* **5** 355) and later by Patterson (1985 *J. Chem. Phys.* **83** 4618) to model triply degenerate vibrations of tetrahedral molecules. We describe relative equilibria (RE) of these systems, classify all qualitatively different family members, and discuss bifurcations of RE involved in the transitions from one region of regular parameter values to the other.

Mathematics Subject Classification: 37J15

1. Introduction

Consider a two degree of freedom (2-DOF) Hamiltonian system with phase space $T\boldsymbol{R}^2$, standard symplectic form $dx \wedge dp_x + dy \wedge dp_y$ and Hamiltonian function $H : T\boldsymbol{R}^2 \to \boldsymbol{R}$

$$H(x, y, p_x, p_y) = \tfrac{1}{2}(x^2 + p_x^2) + \tfrac{1}{2}(y^2 + p_y^2) + \epsilon(x^2 y - \tfrac{1}{3}y^3), \qquad (1)$$

where we introduced ϵ to scale the cubic perturbation potential of the 1 : 1 harmonic oscillator. This Hamiltonian was derived by Hénon and Heiles from their original model 3-DOF Hamiltonian in [1]. The specific form of its potential was chosen 'because: (i) it is analytically simple; ... (ii) at the same time, it is sufficiently complicated to give trajectories far from trivial' [1]. Equation (1) is called the 2-DOF Hénon–Heiles Hamiltonian.

As Hénon and Heiles found numerically in [1], an additional integral of motion did not exist in the 2-DOF system with Hamiltonian (1). Their study resulted in the first illustration of

Hamiltonian chaos, and their system has since been studied extensively both numerically and analytically (see [2] for a detailed list of references). It has served not only as a model for the dynamics near the centre of a galaxy but also in molecular physics where it has been used to describe doubly degenerate vibrations of molecules whose equilibrium configuration has one or several threefold symmetry axes [3], such as H_3^+ [4], P_4 or CH_4, and SF_6.

The Hamiltonian (1) is symmetric under the action of the finite group $D_3 \times \mathcal{T}$. Here D_3 is the dihedral symmetry group of the equilateral triangle, and $\mathcal{T} = \{1, T\}$ is the time reversal or momentum reversal group generated by $T : (x, y, z, p_x, p_y, p_z) \to (x, y, z, -p_x, -p_y, -p_z)$. Although the finite symmetry of (1) was not particularly emphasized in [1], it has one very important consequence, namely the *a priori* existence at low energy of eight families of periodic orbits called nonlinear normal modes [5–7]. In fact, this is a property of any $D_3 \times \mathcal{T}$ symmetric 1 : 1 resonant 2-oscillator.

1.1. Generalized Hénon–Heiles Hamiltonian

In this work, we propose and study the following three-dimensional generalization of the Hamiltonian (1):

$$H_\epsilon(x, y, z, p_x, p_y, p_z) = H_0(x, y, z, p_x, p_y, p_z) + V_\epsilon(x, y, z), \tag{2a}$$

where x, y, z are Cartesian coordinates in $\boldsymbol{R}^3$, and p_x, p_y, p_z are the corresponding conjugate momenta. We assume that (x, y, z) transform according to the vector representation of the orthogonal group $O(3)$ of transformations of $\boldsymbol{R}^3$. The zero-order Hamiltonian in (2*a*)

$$H_0(x, y, z, p_x, p_y, p_z) = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \tfrac{1}{2}(x^2 + y^2 + z^2) \tag{2b}$$

represents three harmonic oscillators with equal frequencies, that is, the 1 : 1 : 1 resonant (isotropic) harmonic oscillator. The perturbation potential

$$V_\epsilon(x, y, z) = \epsilon K_3\, xyz + \epsilon^2 K_0(x^2 + y^2 + z^2)^2 + \epsilon^2 K_4(x^4 + y^4 + z^4) \tag{2c}$$

is a polynomial in (x, y, z) of degree 4. The dimensionless smallness parameter ϵ characterizes the magnitude of the perturbation, while parameters K_0, K_3 and K_4 give the relative strength of each perturbation term. We assume that these parameters are of the order of 1. Note that we use K_3 in order to keep track of the contribution of the cubic potential term; in principle, this parameter can be absorbed into ϵ.

Comparison of the family (2) to the two-dimensional Hénon–Heiles Hamiltonian (1).

- (i) Like the two-dimensional Hénon–Heiles system, the system with Hamiltonian (2) is probably not integrable and hence is a genuinely three-dimensional system in the sense that the only exact first integrals are smooth functions of energy.
- (ii) Both (1) and (2) have no connected compact Lie group of symmetries. Both are invariant under the action of a discrete group which includes rotations by $2\pi/3$ and the time reversal group $\mathcal{T}$ described above. Our Hamiltonian (2) is invariant under the action of the group $T_d \times \mathcal{T}$, where $T_d \subset O(3)$ is the tetrahedral group of transformations of $\boldsymbol{R}^3$. This symmetry group has four conjugate subgroups $C_{3v} \times \mathcal{T}$ which are isomorphic to the symmetry group of (1).
- (iii) Both (1) and (2) have principal cubic perturbation terms of the simplest possible analytic form which realize completely the respective point group symmetries: D_3 and T_d. The polynomial V_ϵ in (2*c*) is the most general T_d-invariant polynomial in (x, y, z) of degree 4.

A proof of the nonintegrability of the system with Hamiltonian (2) is beyond the scope of this work and is not really necessary in our context. However, direct computations for high degree $T_d \times \mathcal{T}$-invariant polynomials $F(x, y, z, p_x, p_y, p_z)$ show that $\{H_\epsilon, F\}$ does not vanish. Furthermore, numerical integration also reveals chaotic dynamics.

The symmetry properties of H_ϵ can be verified easily once the action of $O(3) \supset T_d$ is defined on the phase space $T^*\boldsymbol{R}^3$ using the diagonal extension to (p_x, p_y, p_z) of the standard action of $O(3) \supset T_d$ on $\boldsymbol{R}^3$ with coordinates (x, y, z) (see [8–10] and appendix A). The action and structure of $T_d \times \mathcal{T}$ is described in detail in [11].

The ring of polynomial invariants of the T_d action on $\boldsymbol{R}^3$ with coordinates (x, y, z) is generated freely by three principal invariants

$$\mu_2 = x^2 + y^2 + z^2, \qquad \mu_3 = xyz, \qquad \text{and} \qquad \mu_4 = x^4 + y^4 + z^4.$$

This can be confirmed by a direct computation of the corresponding Molien generating function $g(\lambda) = 1/[(1-\lambda^2)(1-\lambda^3)(1-\lambda^4)]$, with λ representing any of the variables (x, y, z) (see, e.g. [12, 13]). It follows that the perturbation V_ϵ in (2c) includes all principal invariants. It is the most general T_d-invariant polynomial in (x, y, z) of degree 4. However, including only the cubic term μ_3 is sufficient to bring the symmetry of (2c) down to T_d.

Remark 1. The most obvious physical realization of a system with Hamiltonian (2) is an atom (a spherical particle) trapped in a tetrahedral potential well. Hamiltonian (2) can also be used to model triply degenerate vibrational modes of the tetrahedral molecules A_4 or AB_4 [11]. In this latter case, we should include the term representing the squared length of the vibrational angular momentum $[(x, y, z) \times (p_x, p_y, p_z)]^2$. We will see later that omission of such a term in (2) results in no loss of generality because Hamiltonian (2) with fourth degree terms in (2c) already has enough parameters to represent qualitatively the above molecular systems.

The natural starting point of the analysis of the family of systems with Hamiltonian (2) is near the linearization limit $\epsilon \to 0$. According to a theorem of Weinstein [14], a perturbed non-resonant k-oscillator near this limit has k families of short periodic orbits called nonlinear normal modes. In the presence of resonances, the oscillator can have more than k such families. The number and the properties of the modes depend primarily on the resonance and the symmetry of the perturbing nonlinear terms. As in the case of the two-dimensional Hénon–Heiles system [5,6], the nonlinear normal modes of our three-dimensional system with Hamiltonian (2) are special periodic solutions characterized by a nontrivial isotropy group G, or stabilizer, which is a subgroup of the total symmetry group of the system $T_d \times \mathcal{T}$. The description of the subgroups G is given in appendix A and in more detail in [10]. Montaldi *et al* [6] studied special short period solutions for a system, which, near $\epsilon \to 0$, is equivalent to ours. Using their results we immediately obtain.

Theorem 1. *The system with Hamiltonian (2) has at least* 27 *nonlinear normal modes. These modes can be classified according to their stabilizers* $G \subset T_d \times \mathcal{T}$ *as follows.*

Conjugacy class of stabilizers	*Shorthand notation*	*Number of modes*
$D_{2d} \times \mathcal{T}$	A_4	3
$C_{3v} \times \mathcal{T}$	A_3	4
$C_{2v} \times \mathcal{T}$	A_2	6
$S_4 \wedge \mathcal{T}_2$	B_4	6
$C_3 \wedge \mathcal{T}_s$	B_3	8

For a description of these stabilizers see appendix A.1.

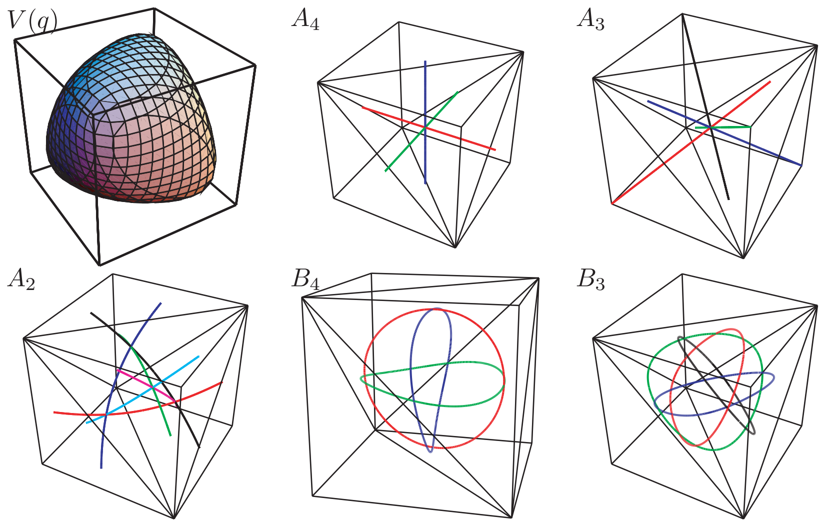

Figure 1. Qualitative representation of the equipotential surface of H (top left). Configuration space representation of the periodic orbits of the system with Hamiltonian H_ϵ (2) that correspond to the critical points of $T_d \times \mathcal{T}$. These periodic orbits have been computed for appropriate values of the parameters ϵ, K_3, K_4, K_0 in (2).

Remark 2. We can also use the approach of [6, 7] as well as of the earlier work on the two-dimensional Hénon–Heiles system [5] (see also [15]) to reconstruct qualitatively the nonlinear normal modes in theorem 1 using their isotropy groups. Figure 1 shows the projection of these periodic orbits in the configuration space $\boldsymbol{R}^3$. A detailed discussion is relegated to appendix A.3 (see also [11]).

In this work, we determine the actual number of nonlinear normal modes for generic members of the family (2) and we find situations where the Hamiltonian has more than the 27 relative equilibria (RE) given by theorem 1. Computing the value of H_ϵ (energy) for these modes and characterizing their linear stability we will give a basic qualitative description of the whole parametric family (2).

Remark 3. In [1], Hénon and Heiles derived the 2-DOF Hamiltonian (1) from a 3-DOF axisymmetric Hamiltonian. This 3-DOF Hamiltonian has been studied in [16–18] where it is called the three-dimensional Hénon–Heiles Hamiltonian. The reduction of the axial symmetry was done in [19] where it is shown that the reduced system is in 1 : 2 resonance.

1.2. Dynamical symmetry; relative equilibria

An alternative proof of theorem 1 was suggested in [15] based on the earlier work by Zhilinskií [20]. In order to follow this latter approach we recall a number of known facts which we formulate as lemmas.

Lemma 1. *For all $n > 0$ the n-level set of the Hamiltonian H_0 in (2b) is a 5-sphere $\boldsymbol{S}^5_n : \{\xi \in T^*\boldsymbol{R}^3,\ H_0(\xi) = h_0 = n > 0\} \subset T^*\boldsymbol{R}^3\backslash\{0\}$. All orbits of the flow $\varphi^t_0 : \boldsymbol{S}^1 \times \boldsymbol{S}^5_n \mapsto \boldsymbol{S}^5_n$ of the Hamiltonian vector field X_{H_0} are periodic with period 2π. This flow defines a symmetry group $\boldsymbol{S}^1$ whose action on $T^*\boldsymbol{R}^3\backslash\{0\}$ and on $\boldsymbol{S}^5_n$ is free. The orbit space $\boldsymbol{S}^5_n/\boldsymbol{S}^1$ is a complex projective 2-space $\boldsymbol{CP}^2$.*

Proof. Identify the phase space $T^*\boldsymbol{R}^3$ with a complex 3-space $\boldsymbol{C}^3$ with coordinates

$$w_1 = x + \mathrm{i}p_x, \qquad w_2 = y + \mathrm{i}p_y, \qquad \text{and} \qquad w_3 = z + \mathrm{i}p_z. \tag{3}$$

In these coordinates, the equation

$$H_0 = \tfrac{1}{2}(w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3) = h_0 = n > 0$$

defines a 5-sphere $\boldsymbol{S}^5_n \subset \boldsymbol{C}^3\backslash\{0\}$ of radius $\sqrt{2n}$. The flow

$$\varphi_0^t : (w, \bar{w}) \to (e^{it}w, e^{-it}\bar{w}) \tag{4}$$

is, obviously, diagonal, and all orbits are circles $\boldsymbol{S}^1_n$ of radius $\sqrt{2n}$. The quotient space $\boldsymbol{S}^5_n/\boldsymbol{S}^1_n$ is obtained by identifying points in each $\boldsymbol{S}^1_n \subset \boldsymbol{S}^5_n$ orbit. We arrive at one of the standard definitions of the complex projective space. See an appropriate textbook, for example [21]. □

We denote by $\boldsymbol{CP}^2(n)$ the orbit space of H_0 for the level set $H_0^{-1}(n)$, $n > 0$. A convenient way to parametrize $\boldsymbol{CP}^2(n)$ globally is by using polynomial invariants of the flow φ_0. (Some other ways to parametrize $\boldsymbol{CP}^2$ are discussed in [22].)

Lemma 2. *The quadratic invariants of the $\boldsymbol{S}^1$ action (4) are*

$$\nu_1 = \tfrac{1}{2}w_1\bar{w}_1, \qquad \nu_2 = \tfrac{1}{2}w_2\bar{w}_2, \qquad \nu_3 = \tfrac{1}{2}w_3\bar{w}_3, \tag{5a}$$

$$\sigma_1 = \mathrm{Re}(w_2\bar{w}_3), \qquad \sigma_2 = \mathrm{Re}(w_3\bar{w}_1), \qquad \sigma_3 = \mathrm{Re}(w_1\bar{w}_2), \tag{5b}$$

$$\tau_1 = \mathrm{Im}(w_2\bar{w}_3), \qquad \tau_2 = \mathrm{Im}(w_3\bar{w}_1), \qquad \tau_3 = \mathrm{Im}(w_1\bar{w}_2), \tag{5c}$$

Except for the relation

$$\Sigma_0 = \nu_1 + \nu_2 + \nu_3 - n = 0, \tag{6a}$$

which fixes the level set $H_0^{-1}(n)$, these invariants are linearly independent. They satisfy nine algebraic relations $\Sigma_k = 0$, called syzygies of the first order, where

$$\Sigma_1 = 4\nu_2\nu_3 - \sigma_1^2 - \tau_1^2, \quad \Sigma_4 = 2\nu_1\sigma_1 - \sigma_2\sigma_3 + \tau_2\tau_3, \quad \Sigma_7 = 2\nu_1\tau_1 + \sigma_2\tau_3 + \tau_2\sigma_3, \tag{6b}$$

$$\Sigma_2 = 4\nu_3\nu_1 - \sigma_2^2 - \tau_2^2, \quad \Sigma_5 = 2\nu_2\sigma_2 - \sigma_3\sigma_1 + \tau_3\tau_1, \quad \Sigma_8 = 2\nu_2\tau_2 + \sigma_3\tau_1 + \tau_3\sigma_1, \tag{6c}$$

$$\Sigma_3 = 4\nu_1\nu_2 - \sigma_3^2 - \tau_3^2, \quad \Sigma_6 = 2\nu_3\sigma_3 - \sigma_1\sigma_2 + \tau_1\tau_2, \quad \Sigma_9 = 2\nu_3\tau_3 + \sigma_1\tau_2 + \tau_1\sigma_2. \tag{6d}$$

The syzygies (6) are themselves not algebraically independent.

The following two lemmas show why invariants (5) are used extensively in the reduction of the oscillator symmetry (4).

Lemma 3. *We can represent the points on $\boldsymbol{CP}^2(n)$, i.e. the orbits of the $\boldsymbol{S}^1$ action in (4) using $(\nu_1, \nu_2, \nu_3; \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3)$ where the nine parameters satisfy relations (6).*

Lemma 4. *Any $\boldsymbol{S}^1$-invariant smooth function $\boldsymbol{C}^3 \to \boldsymbol{R}$ is a smooth function of basic quadratic invariants (5). In particular, any $\boldsymbol{S}^1$-invariant polynomial can be expressed uniquely in terms of an integrity basis. One possible choice of such a basis is*

$$\boldsymbol{R}[n, \nu_1 - \nu_2, \sigma_1, \sigma_2, \sigma_3] \bullet \{1, \nu_3, \nu_3^2, \tau_1, \tau_2, \tau_3\}.$$

Here, $\boldsymbol{R}[\ldots]$ is the ring generated freely by the principal invariants listed within the square brackets; auxiliary invariants listed within the curly brackets can enter only linearly, so that the whole ring can be represented as follows:

$$\boldsymbol{R}[n, \nu_1 - \nu_2, \sigma_1, \sigma_2, \sigma_3] \cdot 1 + \boldsymbol{R}[n, \nu_1 - \nu_2, \sigma_1, \sigma_2, \sigma_3] \cdot \nu_3 + \cdots$$

Proof. This lemma follows from a standard Gröbner basis computation and Schwarz's theorem [23]. The structure of the polynomial ring is described by the following Molien generating function

$$g(\lambda) = \frac{1 + 4\lambda + \lambda^2}{(1 - \lambda)^5}, \tag{7}$$

where λ represents any of the generators in (5) (see more in [11]). □

Lemma 5. *Invariants (5) generate a Poisson algebra* u(3) *with* $n = \nu_1 + \nu_2 + \nu_3$ *one of its Casimirs. The ring of invariant polynomials, generated multiplicatively by (5), can, therefore, be equipped with a Poisson structure. This structure is used to define Hamiltonian dynamical systems on* $\boldsymbol{CP}^2$.

Proof. The Poisson bracket $\{\,,\}$ of the two invariants in (5) is $\boldsymbol{S}^1$-invariant. By lemma 4 it can be expressed in terms of (5). Moreover, since (5) are all quadratic in (x, y, z, p_x, p_y, p_z), the brackets are linear in (5). The concrete Poisson structure is found straightforwardly by computing $\{\,,\}$ in the coordinates (x, y, z, p_x, p_y, p_z). The brackets satisfy relations of u(3). Note that if we set n to a specific value greater than 0, then the algebra spanned by the linearly independent invariants (5) is isomorphic to $su(3)$. □

Lemma 6. *Near the limit of linearization* $\epsilon \to 0$ *the perturbed Hamiltonian* H_ϵ *in (2) is approximately invariant with respect to the flow* φ_0 *in lemma 1. Near* $\epsilon \to 0$ *we can normalize* H_ϵ *with respect to* H_0 *and make this approximate dynamical symmetry exact.*

After normalization we obtain a formal series $\tilde{H}_\epsilon$ such that $\{\tilde{H}_\epsilon, H_0\} = 0$. In practice, we truncate $\tilde{H}_\epsilon$ and, therefore, $\tilde{H}_\epsilon$ and H_0 Poisson commute up to a certain order k in ϵ: $\{\tilde{H}_\epsilon, H_0\} = \mathcal{O}(\epsilon^k)$.

Definition 1. *RE are periodic orbits of the normalized system with Hamiltonian* $\tilde{H}_\epsilon$ *in* $T^*\boldsymbol{R}^3$, *which are also group orbits of the* $\boldsymbol{S}^1$ *action in lemma 1.*

RE are also sometimes called short periodic orbits, i.e. periodic orbits with period close to 2π, or basic orbits.

To reduce the now exact $\boldsymbol{S}^1$ symmetry of $\tilde{H}_\epsilon$, we pass from the original phase space $T^*\boldsymbol{R}^3$ to the space $\boldsymbol{CP}^2(n)$ of $\boldsymbol{S}^1_n$ orbits or the reduced space as follows. Since $\{\tilde{H}_\epsilon, H_0\} = 0$, the value of $\tilde{H}_\epsilon$ on each orbit of H_0 is constant. This means that we can properly define a function $\hat{H}_\epsilon$ on the phase space $\boldsymbol{CP}^2(n)$ of H_0 by assigning to each $\boldsymbol{S}^1_n$ orbit of H_0 the value of $\tilde{H}_\epsilon$ on any point of the orbit. We call the Hamiltonian $\hat{H}_\epsilon$ on $\boldsymbol{CP}^2(n)$ the reduced Hamiltonian. Reduction results in the 2-DOF system on $\boldsymbol{CP}^2$ or the reduced system. By lemma 5, this system is a Poisson dynamical system with Hamiltonian $\hat{H}$ expressed (uniquely) in terms of the invariants (5) and the integrity basis in lemma 4.

Lemma 7. *After reduction of the* $\boldsymbol{S}^1$ *symmetry, RE correspond to the equilibria of the reduced system with Hamiltonian* $\hat{H}_\epsilon$ *on* $\boldsymbol{CP}^2$. *Linear Hamiltonian stability and the isotropy group of the RE and of the corresponding stationary point of* $\hat{H}_\epsilon$ *are the same.*

The normal form of (2) is a formal power series whose orders are 'tracked' by the degrees of the smallness parameter ϵ. Since this series diverges for typical values of parameters in (2), it is truncated at the order of interest, which is in our case the principal order ϵ^2. At this order $\hat{H}_\epsilon$ is a Morse function on $\boldsymbol{CP}^2$ except for five isolated values of the appropriate parameter introduced in section 2. We analyse only the non-exceptional (or ϵ^2-generic) cases.

As is well-known (see, e.g. appendix 7 of [24]), the system described by such truncated normal form $\tilde{H}_\epsilon$ and the original system can be profoundly different. At the same time, it is possible to use $\tilde{H}_\epsilon$ to analyse the short time average behaviour of the original system, and in particular its short periodic orbits. The adequateness (validity) of the truncated normal form approximation for the study of orbits of a given short period is clearly limited by the value of ϵ which should be sufficiently small. (Note that decreasing ϵ is equivalent to decreasing the energy and approaching the equilibrium $x = y = z = 0$ of (2) where $H = 0$.) This makes $\tilde{H}_\epsilon$ particularly suited for the analysis of the nonlinear normal modes which exist and can be studied anywhere close to the limit $\epsilon \to 0$.

Lemma 8. *For small enough ϵ, RE of the normalized system with Hamiltonian $\hat{H}_\epsilon$ correspond to the nonlinear normal modes of the original system with Hamiltonian (2).*

The correspondence between the RE of the normalized system and the nonlinear normal modes is used in many applications. In particular, this was discussed in detail by Duistermaat in [25] who uncovered the relation of normalization near the equilibrium and Lyapunov–Schmidt reduction.

We conclude that the study of the nonlinear normal modes of the system with Hamiltonian (2) becomes the study of the equilibria of the reduced system, i.e. of the stationary points of the appropriately truncated reduced Hamiltonian $\hat{H}_\epsilon$.

1.3. Symmetry and topology

In order to describe qualitatively the systems with Hamiltonian (2) in terms of their nonlinear normal modes, we should find the equilibria of Hamiltonians $\hat{H}_\epsilon$ on $\boldsymbol{CP}^2$ (and hence the RE of the normalized system) and characterize them in terms of their energy and linear stability type. When searching for the stationary points of $\hat{H}_\epsilon$ we should account for the action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$ and the topology of this space.

Consider the action of a compact or finite group G on a manifold M. The isotropy group (or stabilizer) of $m \in M$ is the subgroup G_m of elements of G that leave m fixed. A point $m \in M$ is called a fixed point of the G action when $G_m = G$, that is, when it is fixed by all the elements of G. The G-orbit of m is the set $G{\cdot}m = \{g \cdot m : g \in G\}$. We are primarily interested in points $m_c \in M$ such that there is a punctured neighbourhood of m_c in which there are no points m with isotropy group G_m that belongs to the same conjugacy class in G as G_{m_c}. We call such points m_c and the orbit $G{\cdot}m_c$ critical (for more details see, e.g. [13]). The importance of the critical points is due to the following theorem of Louis Michel [26].

Theorem. *Critical points of the action of a group G on a smooth manifold M are stationary points of every smooth, G-invariant function H on M.*

Consequently, analysis of the critical points of the $T_d \times \mathcal{T}$ group action on the reduced space $\boldsymbol{CP}^2$ provides a number of RE of the normalized system and by lemma 8, nonlinear normal modes of the original system with Hamiltonian (2). A concrete study of this action results in the following conclusion (Zhilinskií [20], see also [10, 11, 15] and appendix A).

Theorem 2. *The action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$ induced by the action of $T_d \times \mathcal{T}$ on $T^*\boldsymbol{R}^3 \sim \boldsymbol{C}^3$ has 27 critical (i.e. isolated fixed) points grouped into five critical orbits with the conjugacy classes of stabilizers given in theorem 1. Table 1 presents the position of these points on $\boldsymbol{CP}^2(n)$ characterized by the values of the invariants (5).*

Remark 4. Table A3 of appendix A lists isotropy groups of the 27 points in theorem 2. Isotropy groups of the points in the same critical orbit of the $T_d \times \mathcal{T}$ action are conjugate in $T_d \times \mathcal{T}$.

Table 1. Critical points of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2(n)$. Coordinates are given by $(\nu_1, \nu_2, \nu_3; \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3)$ with $\alpha = \frac{1}{2}$, $\beta = \sqrt{3}/2$, $\bar{\alpha} = -\frac{1}{2}$, $\bar{\beta} = -\sqrt{3}/2$, and $\bar{1} = -1$.

Point	Coordinates on $\boldsymbol{CP}^2(n)$	Point	Coordinates on $\boldsymbol{CP}^2(n)$
A_4^x	$n(1, 0, 0; 0, 0, 0; 0, 0, 0)$	B_4^x	$n(0, \alpha, \alpha; 0, 0, 0; 1, 0, 0)$
A_4^y	$n(0, 1, 0; 0, 0, 0; 0, 0, 0)$	$B_4^{\bar{x}}$	$n(0, \alpha, \alpha; 0, 0, 0; \bar{1}, 0, 0)$
A_4^z	$n(0, 0, 1; 0, 0, 0; 0, 0, 0)$	B_4^y	$n(\alpha, 0, \alpha; 0, 0, 0; 0, 1, 0)$
A_3^a	$\frac{2n}{3}(\alpha, \alpha, \alpha; 1, 1, 1; 0, 0, 0)$	$B_4^{\bar{y}}$	$n(\alpha, 0, \alpha; 0, 0, 0; 0, \bar{1}, 0)$
A_3^b	$\frac{2n}{3}(\alpha, \alpha, \alpha; \bar{1}, \bar{1}, 1; 0, 0, 0)$	B_4^z	$n(\alpha, \alpha, 0; 0, 0, 0; 0, 0, 1)$
A_3^c	$\frac{2n}{3}(\alpha, \alpha, \alpha; 1, \bar{1}, \bar{1}; 0, 0, 0)$	$B_4^{\bar{z}}$	$n(\alpha, \alpha, 0; 0, 0, 0; 0, 0, \bar{1})$
A_3^d	$\frac{2n}{3}(\alpha, \alpha, \alpha; \bar{1}, 1, \bar{1}; 0, 0, 0)$	B_3^a	$\frac{2n}{3}(\alpha, \alpha, \alpha; \bar{\alpha}, \bar{\alpha}, \bar{\alpha}; \beta, \beta, \beta)$
A_2^x	$n(0, \alpha, \alpha; 1, 0, 0; 0, 0, 0)$	$B_3^{\bar{a}}$	$\frac{2n}{3}(\alpha, \alpha, \alpha; \bar{\alpha}, \bar{\alpha}, \bar{\alpha}; \bar{\beta}, \bar{\beta}, \bar{\beta})$
$A_2^{\bar{x}}$	$n(0, \alpha, \alpha; \bar{1}, 0, 0; 0, 0, 0)$	B_3^b	$\frac{2n}{3}(\alpha, \alpha, \alpha; \alpha, \alpha, \bar{\alpha}; \beta, \beta, \bar{\beta})$
A_2^y	$n(\alpha, 0, \alpha; 0, 1, 0; 0, 0, 0)$	$B_3^{\bar{b}}$	$\frac{2n}{3}(\alpha, \alpha, \alpha; \alpha, \alpha, \bar{\alpha}; \bar{\beta}, \bar{\beta}, \beta)$
$A_2^{\bar{y}}$	$n(\alpha, 0, \alpha; 0, \bar{1}, 0; 0, 0, 0)$	B_3^c	$\frac{2n}{3}(\alpha, \alpha, \alpha; \bar{\alpha}, \alpha, \alpha; \bar{\beta}, \beta, \beta)$
A_2^z	$n(\alpha, \alpha, 0; 0, 0, 1; 0, 0, 0)$	$B_3^{\bar{c}}$	$\frac{2n}{3}(\alpha, \alpha, \alpha; \bar{\alpha}, \alpha, \alpha; \beta, \bar{\beta}, \bar{\beta})$
$A_2^{\bar{z}}$	$n(\alpha, \alpha, 0; 0, 0, \bar{1}; 0, 0, 0)$	B_3^d	$\frac{2n}{3}(\alpha, \alpha, \alpha; \alpha, \bar{\alpha}, \alpha; \beta, \bar{\beta}, \beta)$
		$B_3^{\bar{d}}$	$\frac{2n}{3}(\alpha, \alpha, \alpha; \alpha, \bar{\alpha}, \alpha; \bar{\beta}, \beta, \bar{\beta})$

These points are equivalent: the dynamics in their neighbourhood is identically the same, so it is usually sufficient to study one representative of each critical orbit. We denote different points of the same critical orbit by a superscript; we drop this index when referring to the entire orbit or when our results apply identically to all points in the orbit.

We can now prove theorem 1. We rely on lemma 6 in order to establish the correspondence of the nonlinear normal modes of the initial system with Hamiltonian (2) and the RE of the normalized system near the limit $\epsilon \to 0$. We then use the theorem of Michel and theorem 2.

Consider a smooth Hamiltonian function $\mathcal{H} : \boldsymbol{CP}^2 \to \boldsymbol{R}$ whose stationary points are nondegenerate. We call $\mathcal{H}$ a Morse type Hamiltonian. The isotropy group of a stationary point c of $\mathcal{H}$ also restricts the possible types of linear Hamiltonian stability and the Morse index of c. Recall that linear stability is given by the eigenvalues of the 4×4 Hamiltonian matrix, which describes the linearized equations of motion near $c \in \boldsymbol{CP}^2$, while the Morse index of c gives the number of negative eigenvalues of the Hessian matrix of the Hamiltonian at c. Depending on the eigenvalues of this matrix we will distinguish six linear stability types EE, HH, EH, CH, 2E, 2H, described in appendix B.2.

Theorem 3. *Equilibria of a $T_d \times \mathcal{T}$-invariant Morse type Hamiltonian function on $\boldsymbol{CP}^2$, which are the critical points listed in theorem 2, have the following linear Hamiltonian stability types and Morse indices.*

Critical	*Orbit*		*Stability*	*Index*	*Critical*	*Orbit*		*Stability*	*Index*
$D_{2d} \times \mathcal{T}$	A_4	*3*	*2E*	*0, 4*	$S_4 \wedge \mathcal{T}_2$	B_4	*6*	*EE*	*0, 2, 4*
			2H	*2*				*EH*	*1, 3*
$C_{3v} \times \mathcal{T}$	A_3	*4*	*2E*	*0, 4*	$C_3 \wedge \mathcal{T}_s$	B_3	*8*	*EE, CH*	*0, 2, 4*
			2H	*2*					
$C_{2v} \times \mathcal{T}$	A_2	*6*	*EE*	*0, 2, 4*					
			EH	*1, 3*					
			HH	*2*					

Proof. See [10]. □

Theorem 3 is a local statement which concerns an open neighbourhood $D_c \subset \boldsymbol{CP}^2$ of each critical point c of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2$. Thus far we have no information about the set of stationary points characterized in theorems 2 and 3. This information can only be obtained from the global (topological) analysis. Note that we have already used the topology of $\boldsymbol{CP}^2$ in order to find the action of $T_d \times \mathcal{T}$ (initially defined on $\boldsymbol{R}^3_{x,y,z}$) on this space, and of the isotropy group G_c of c on D_c. In appendix B we summarize how Morse theory [27, 28] is applied in order to check the consistency of any set of stationary points using the Morse inequalities (four inequalities and one equality) imposed by the topology of $\boldsymbol{CP}^2$ on the number and types of stationary points of Morse functions $\mathcal{H}$. In particular, we can determine if it is possible for a $T_d \times \mathcal{T}$-invariant Morse function $\mathcal{H}$ to have stationary points solely at the critical points of the $T_d \times \mathcal{T}$ symmetry given in theorem 2.

Definition 2. *A simplest (or perfect) G-invariant Morse function on a manifold M is one that has the minimal possible number of nondegenerate stationary points.*

Note that there is no guarantee that all the stationary points of a perfect function lie on a critical orbit of the G-action. In our case, we have the following lemma.

Lemma 9. *The simplest $T_d \times \mathcal{T}$-invariant Morse Hamiltonian $\mathcal{H}$ on $\boldsymbol{CP}^2$ has 27 equilibria which are critical points of the $T_d \times \mathcal{T}$ action in theorems 2 and 3. In this case the six A_2 and six B_4 stationary points of $\mathcal{H}$ are of odd Morse index and have stability EH.*

Proof. According to theorem 3, points A_3, A_4, and B_3 are of even Morse index. The 27 points can have the right Morse indices whose alternating sum gives the Euler characteristic of $\boldsymbol{CP}^2$ only if A_2 and B_4 are of odd Morse index. □

Remark 5. Linear stability types of the nonlinear normal modes in theorem 1 correspond to the stability types in theorem 3. In the simplest possible case, the system with Hamiltonian (2) near the limit of linearization has exactly 27 families of short periodic orbits.

Remark 6. Morse theory provides necessary conditions that must be satisfied by the stationary points of any Morse function on $\boldsymbol{CP}^2$. At the same time, when a set of known stationary points of a Morse function $\mathcal{H}$ satisfies all these conditions, $\mathcal{H}$ can still have other stationary points.

Remark 7. A more complete consistency check of a known system of stationary points of a $T_d \times \mathcal{T}$-invariant Morse function $\mathcal{H}$ on $\boldsymbol{CP}^2$ requires that the Morse inequalities hold not only for $\boldsymbol{CP}^2$ but also for all G-invariant subspaces of $\boldsymbol{CP}^2$ with G a subgroup of $T_d \times \mathcal{T}$.

1.4. Overview

We give a brief overview of this paper. In section 2, we continue a general study of $T_d \times \mathcal{T}$ symmetric systems with Hamiltonian (2) and show how these systems can be described as a one-parameter family. In section 3, we normalize the Hamiltonian (2) to the second (principal) order in ϵ (degree 4) and then reduce it. In section 4, we determine the local properties (linear stability type and Morse index) of the equilibria of the reduced Hamiltonian $\hat{H}_\epsilon$, which are critical points of the $T_d \times \mathcal{T}$ action. In section 5, we describe other stationary points of $\hat{H}_\epsilon$, which do not lie on a critical orbit of the $T_d \times \mathcal{T}$ action. This concludes the concrete study of the family of systems with Hamiltonian (2) near the limit $\epsilon \to 0$. Finally, in section 6, we make some comments on the bifurcations of the RE of this family. This paper contains two appendices where we detail the action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$ and describe how we find the linear stability types and the Morse indices of the critical points on $\boldsymbol{CP}^2$.

2. One-parameter classification

Theorem 1 has already highlighted the important consequences of the presence of the additional finite symmetry $T_d \times \mathcal{T}$. The next lemma shows that this symmetry causes important modifications of the standard $\boldsymbol{S}^1$-invariant polynomial basis in lemma 4.

Lemma 10. *Consider the most general $(T_d \times \mathcal{T}) \times S^1$-invariant polynomial $P_k(w, \bar{w})$ of degree k in variables $(w, \bar{w})$ (3). Then $P_k = 0$ if k is odd, and*

$$P_2 = c'n = c'(\nu_1 + \nu_2 + \nu_3) \tag{8a}$$

$$P_4 = cn^2 + a(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + b(\tau_1^2 + \tau_2^2 + \tau_3^2) \tag{8b}$$

where a, b, c, and c' are arbitrary constants.

Proof. The action of $O(3) \times \mathcal{T}$ and its subgroup $T_d \times \mathcal{T}$ on polynomials (5) can be found by direct computation (see appendix A and [10, 11]). In particular, we can verify that all $\boldsymbol{S}^1$-invariants in (5) are invariant with respect to spatial inversion, and that

$$n, \qquad (\tau_1, \tau_2, \tau_3), \qquad \text{and} \qquad \left(\sigma_1, \sigma_2, \sigma_3, \frac{3\nu_3 - n}{\sqrt{3}}, n_1 - n_2\right)$$

transform according to the irreducible representations of $O(3)$ of indices 0_g, 1_g, and 2_g, respectively. In other words, n and (τ_1, τ_2, τ_3) transform as a scalar and an axial 3-vector, respectively. We can also easily verify that for the generator T of $\mathcal{T}$

$$T : (\nu, \sigma, \tau) \to (\nu, \sigma, -\tau).$$

Knowing the action of $T_d \times \mathcal{T}$, we can further symmetrize the basis of lemma 4. In particular, we obtain the generating function

$$g(\lambda) = \frac{1 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6 + \lambda^9}{(1 - \lambda)(1 - \lambda^2)^2(1 - \lambda^3)(1 - \lambda^4)} \tag{9}$$

which describes the symmetrized integrity basis (see [11]). The symmetrized invariants have generators of high degree when expressed in terms of the generators in (5): of the five denominator factors in (9), which describe principal invariants, $(1 - \lambda)$ corresponds to n, while $(1 - \lambda^2)^2$ represents two invariants of degree two in generators (5). Direct computation shows that τ^2 and σ^2 are $T_d \times \mathcal{T}$-invariant and can be chosen as these two invariants. The function (9) indicates that there are no other principal or auxiliary invariants of this degree. □

Remark 8. The 3-vector $\tau = (\tau_1, \tau_2, \tau_3)$ is the angular momentum vector. Both n and $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$ are totally symmetric with respect to the larger group $O(3) \times \mathcal{T}$. The only term in (8) which represents a $T_d \times \mathcal{T}$ symmetry is σ^2.

Definition 3.

(i) Two reduced Hamiltonians $\hat{H}_\epsilon$ and $\hat{H}'_\epsilon$ will be called ϵ^2-equivalent if $a(n)\hat{H}_\epsilon = b(n)\hat{H}'_\epsilon + c(n) + \mathcal{O}(\epsilon^2)$, where $a(n) \neq 0$, $b(n) \neq 0$ and $c(n)$ are functions of the parameter n.

(ii) Two normalized Hamiltonians $\tilde{H}_\epsilon$ and $\tilde{H}'_\epsilon$ are ϵ^2-equivalent if $a(H_0)\hat{H}_\epsilon = b(H_0)\hat{H}'_\epsilon + c(H_0) + \mathcal{O}(\epsilon^2)$, where $a \neq 0$, $b \neq 0$ and c are functions of H_0 in (2b).

(iii) Two systems with Hamiltonians H_ϵ and H'_ϵ of the form (2) will also be called ϵ^2-equivalent if their respective normalized Hamiltonians are ϵ^2-equivalent.

We now come to a central result which provides the basis for our work.

Theorem 4. *At the level of ϵ^2-equivalence, all systems with Hamiltonian H_ϵ in (2) with generic values of K_3, K_0, and K_4, can be characterized using one parameter.*

Proof. Lemma 6 establishes the correspondence of the initial system with Hamiltonian H_ϵ and the normalized system with Hamiltonian $\tilde{H}_\epsilon$. According to lemma 10, $\tilde{H}_\epsilon$ has the form $P_2 + \epsilon^2 P_4 + \cdots$. Since $\{H_0, \tilde{H}_\epsilon\} = 0$, H_0 is replaced by its value n when we define the reduced Hamiltonian $\hat{H}_\epsilon$. Then, up to the constant $c'n + \epsilon^2 cn^2$ and the overall scaling factor of ϵ^2,

$$\hat{H} = K_s(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + K_t(\tau_1^2 + \tau_2^2 + \tau_3^2) + \cdots. \tag{10}$$

It remains to verify that the family of systems (2) is generic in the sense that for practically all members of this family $K_s^2 + K_t^2 \neq 0$. (For the exceptional members we would have to normalize to higher orders.) This will be done in section 3 after computing explicitly the normal form $\tilde{H}_\epsilon$ of (2).

In the generic situation when $R = \sqrt{K_s^2 + K_t^2} > 0$ we can define a one-parameter family by setting $K_s = R\sin\theta$ and $K_t = R\cos\theta$ and rescaling the reduced Hamiltonian $\hat{H}$ in (10) by R. Then,

$$\hat{H} = \sin\theta\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \cos\theta\,(\tau_1^2 + \tau_2^2 + \tau_3^2) + \cdots, \tag{11}$$

where, in general, θ can take any value in $[0, 2\pi)$. Systems with the same value of θ but different values of R have qualitatively the same dynamics but different timescales. Specifically, for smaller R the dynamics is slower. □

Remark 9. Lemma 10 and theorem 4 apply, in fact, to a larger family of systems with an extended Hamiltonian $H_\epsilon + W_\epsilon(x, y, z, p_x, p_y, p_z)$, where H_ϵ is defined in (2), and W_ϵ is a general $T_d \times \mathcal{T}$-invariant ϵ-series perturbation of degree 3 or higher in all the dynamical variables (x, y, z, p_x, p_y, p_z).

The Hamiltonian (10) is equivalent to the model quantum Hamiltonian proposed by Hecht [29] in order to describe the structure of polyads formed by triply degenerate vibrational modes of tetrahedral and octahedral molecules. Hecht's Hamiltonian was analysed later by Patterson [30]. Both authors exploited the one-parameter property in their analysis. The classical RE system and quantum–classical comparison for this Hamiltonian are presented in section 11.1 of [11]. To continue classifying reduced systems with Hamiltonian (11) and respective original systems with Hamiltonian (2) we use the equivalence relation, which takes into account only the families of RE and respective nonlinear normal modes.

Definition 4. *Consider two different systems with Hamiltonians H_ϵ and H'_ϵ in (2).*

(i) Nonlinear normal modes of these two systems are equivalent if these modes have the same stability and the same isotropy group.
(ii) The sets of modes of the two systems are equivalent if we can establish a 1–1 correspondence between all modes in each respective set using the equivalence in (i).
(iii) Two systems belong to the same class of mode-equivalent systems if their sets of nonlinear normal modes are equivalent.

Definition 5. *Consider two reduced systems with Hamiltonians $\hat{H}_\epsilon$ and $\hat{H}'_\epsilon$ in (11).*

(i) Equilibria of these two systems are equivalent if they have the same stability and the same isotropy group.
(ii) The sets of equilibria are equivalent if we can establish a 1–1 correspondence between the equivalent equilibria in the sets.

(iii) Two reduced systems are equivalent if their sets of equilibria are equivalent.

Consider now two reduced systems with Hamiltonians $\hat{H}_\epsilon$ and $\hat{H}'_\epsilon$ which correspond to the systems in definition 4. By lemma 8 we obtain the following lemma.

Lemma 11. *Two systems belong to the same class of mode-equivalent systems in definition 4 if the corresponding reduced systems are equivalent in the sense of definition 5.*

Proof. Relate the equilibria of the reduced system, the RE of the normalized system and the nonlinear normal modes of the original system. □

Definition 6. *The system with Hamiltonian H_ϵ in (2) is called ϵ^2-generic if it is mode-equivalent to the respective normalized system with Hamiltonian $\tilde{H}_\epsilon$ truncated at the principal order ϵ^2.*

In this work, we will only consider ϵ^2-generic systems with Hamiltonian (2). In order to characterize all such systems we will study all possible sets of stationary points $\xi \in \boldsymbol{CP}^2(n)$ of the Hamiltonian (11). Symmetry properties of ξ can be obtained from the study of the action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2(n)$. Stability of ξ is given by four eigenvalues $(\pm\lambda_1, \pm\lambda_2)$ of the Hamiltonian matrix of the locally linearized Hamiltonian $\hat{H}|_\xi$. We also use the eigenvalues of the Hessian matrix to compute the Morse index of ξ.

Remark 10. It is sufficient to study (11) for $\theta \in [0, \pi)$ because $\hat{H}(\theta + \pi) = -\hat{H}(\theta)$. Both Hamiltonian and Hessian matrices change signs when $\theta \to \theta + \pi$. This does not affect stability (since both λ and $-\lambda$ are eigenvalues). On the other hand, if the Morse index for θ is d then for $\theta + \pi$ it becomes $4 - d$.

We would like to point out that our classification has both qualitative and quantitative aspects. We will find several different classes of ϵ^2 generic systems with Hamiltonian (2). At the same time we represent all systems in the same class as a continuous one-parameter family and describe quantitatively the evolution of their RE as the parameter varies.

Remark 11. It is instructive to compare our analysis to a similar single parameter classification of the two-dimensional Hénon–Heiles systems with Hamiltonian (1) which is summarized in appendix C. The principal difference between our case and the two-dimensional case is that systems with different θ in (11) have the same principal action n and do not differ seriously in energy. Figuratively, our scheme is 'isoenergetic', while any possible scheme for the two-dimensional systems has an energy-dependent parametrization.

3. Normalization and reduction

Normalization is the first step of the concrete study of systems with Hamiltonian H_ϵ in (2). This well-known procedure can be performed using the Lie series method [31–33]. Up to the second-order terms ϵ^2 we obtain the normalized Hamiltonian

$$\tilde{H}_\epsilon(w, \bar{w}) = \tilde{H}_0(w, \bar{w}) + \epsilon^2 \tilde{H}_2(w, \bar{w}) , \tag{12a}$$

where variables $(w, \bar{w})$ are given in (3), $\tilde{H}_0(w, \bar{w}) = H_0(w, \bar{w}) = \frac{1}{2} \sum w_i \bar{w}_i$, and

$$\begin{aligned}\tilde{H}_2(w, \bar{w}) = {} & -\tfrac{3}{2}(K_0 + K_4)(w_1^2\bar{w}_1^2 + w_2^2\bar{w}_2^2 + w_3^2\bar{w}_3^2) \\ & + (-2K_0 + \tfrac{1}{6}K_3^2)(w_1\bar{w}_1w_2\bar{w}_2 + w_2\bar{w}_2w_3\bar{w}_3 + w_3\bar{w}_3w_1\bar{w}_1) \\ & + (-\tfrac{1}{2}K_0 + \tfrac{1}{8}K_3^2)(w_1^2\bar{w}_2^2 + \bar{w}_1^2w_2^2 + w_2^2\bar{w}_3^2 + \bar{w}_2^2w_3^2 + w_3^2\bar{w}_1^2 + \bar{w}_3^2w_1^2).\end{aligned} \tag{12b}$$

Reduction of the Hamiltonian (12) gives

$$\hat{H}_\epsilon = \hat{H}_0 + \epsilon^2 \hat{H}_2, \tag{13a}$$

where

$$\hat{H}_0 = \nu_1 + \nu_2 + \nu_3 = n \tag{13b}$$

$$\hat{H}_2 = \tfrac{3}{2}(K_4 + K_0)n^2 + K_s(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + K_t(\tau_1^2 + \tau_2^2 + \tau_3^2) \tag{13c}$$

with

$$K_s = \frac{-5K_3^2 - 36K_4}{48}, \qquad K_t = \frac{K_3^2 - 36K_4 - 24K_0}{48}. \tag{13d}$$

Dropping the constant terms in $\hat{H}_\epsilon$ and rescaling by ϵ^2 we arrive at the Hamiltonian $\hat{H}$ in (10). Furthermore, since K_0, K_3, and K_4 can take arbitrary values (of order 1) we can rewrite $\hat{H}$ in the one-parameter form $\hat{H}(\theta)$ in (11), where $0 \leqslant \theta < \pi$.

Remark 12. There are two values of θ at which the reduced system with Hamiltonian $\hat{H}(\theta)$ has large Lie group of symmetries and is Liouville integrable, see below.

Value of θ	First integrals	Symmetry
0	$n,\ \tau_1^2 + \tau_2^2 + \tau_3^2,\ \tau_3$	$O(3)$
$\pi/4$	$\nu_1,\ \nu_2,\ \nu_3$	$SU(3)$

Remark 13. Including the terms μ_4 and μ_2^2 in (2) results in the most general $T_d \times \mathcal{T}$-invariant fourth-order reduced system. With $K_4 = K_0 = 0$ and $K_3 \neq 0$ in (2) the reduced Hamiltonian is

$$\hat{H} = -5K_3^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + K_3^2(\tau_1^2 + \tau_2^2 + \tau_3^2),$$

which represents only one member of the family (11) with $\theta = \pi - \tan^{-1}(5) \approx 0.563\pi$.

4. RE corresponding to critical points

We study equilibria of the reduced system with Hamiltonian (11) which are the critical points of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2(n)$.

Lemma 12. *The nonlinear normal modes in theorem 1 correspond to the RE of the normalized system with Hamiltonian (2). On the phase space $\boldsymbol{CP}^2$ of the corresponding reduced system, these RE are critical points of the action of the symmetry group $T_d \times \mathcal{T}$ given in theorem 2. The principal terms in the energy-action characteristics for these modes are given below.*

Conjugacy class of stabilizers	*Shorthand notation*	*Number of modes*	*Energy $\hat{H}(n,\theta)$*
$D_{2d} \times \mathcal{T}$	A_4	3	0
$C_{3v} \times \mathcal{T}$	A_3	4	$\frac{4}{3}n^2 \sin\theta$
$C_{2v} \times \mathcal{T}$	A_2	6	$n^2 \sin\theta$
$S_4 \wedge \mathcal{T}_2$	B_4	6	$n^2 \cos\theta$
$C_3 \wedge \mathcal{T}_s$	B_3	8	$\frac{1}{3}n^2(\sin\theta + 3\cos\theta)$

Note that n is equal to the action $I = \oint p\,\mathrm{d}q$ computed along the respective periodic orbit, and θ is defined in the proof of theorem 4. For the members of the family of systems with

Hamiltonian (2), the absolute maximum and minimum energy, $E_{\max}(n)$ and $E_{\min}(n)$, for a given fixed action n can be estimated as follows: $E_{\min} = E_{A_4}$ and $E_{\min} = E_{B_4}$ in the regions $\theta \in [0, \frac{1}{2}\pi]$ and $\theta \in [\frac{1}{2}\pi, \pi]$, respectively, while $E_{\max} = E_{B_3}$ and $E_{\max} = E_{A_3}$ in the regions $\theta \in [0, \frac{1}{4}\pi]$ and $\theta \in [\frac{1}{4}\pi, \pi]$, respectively.

Proof. We find the energy for each type of RE by substituting the coordinates in table 1 into (11). We present the result in figure 2 and table 2. We now prove that $\hat{H}/n^2$ takes the values represented by the grey region in figure 2. We do this in a number of steps. First, note that when $0 \leqslant \theta \leqslant \pi/2$ we have $\sin\theta \geqslant 0$ and $\cos\theta \geqslant 0$; therefore $\hat{H} \geqslant 0 = \hat{H}_{A_4}$ in that region.

Let $\nu^2 \equiv \nu_1^2 + \nu_2^2 + \nu_3^2$, $\sigma^2 \equiv \sigma_1^2 + \sigma_2^2 + \sigma_3^2$, and $\tau^2 \equiv \tau_1^2 + \tau_2^2 + \tau_3^2$. From (6*a*) we find that $\nu^2 \geqslant n^2/3$. In the region $\pi/4 \leqslant \theta \leqslant \pi$ we express $\hat{H}$ as $\hat{H} = \sin\theta(\sigma^2+\tau^2)+(\cos\theta-\sin\theta)\tau^2$. Using the syzygies (6) we find that $\sigma^2 + \tau^2 = 2(n^2 - \nu^2)$ and since $\nu^2 \geqslant n^2/3$ we get $\sigma^2 + \tau^2 \leqslant 4n^2/3$. Also, when $\pi/4 \leqslant \theta \leqslant \pi$ we have $\cos\theta - \sin\theta \leqslant 0$. Therefore, $\hat{H} \leqslant \sin\theta(\sigma^2 + \tau^2) \leqslant 4n^2 \sin\theta/3 = \hat{H}_{A_3}$.

In order to complete the argument we need to show that $\tau^2 \leqslant n^2$. By remark 8, any rotation in the original phase space $T^*\boldsymbol{R}^3$ leaves τ^2 unchanged. Note also that the form of the syzygies in (6) remains invariant under such a rotation. Therefore we can rotate coordinate axes so that in the new coordinate system we have $\tau_1' = \tau_2' = 0$ and $\tau_3'^2 = \tau^2$. If $\tau_3' = 0$, then what we want to prove is true. If $\tau_3' \neq 0$, then using the syzygies we find that $\nu_3' = \sigma_1' = \sigma_2' = 0$ and $\tau_3'^2 = 4\nu_1'(n - \nu_1') - \sigma_3'^2$. It follows from the last relation that $\tau_3'^2$ and, therefore, τ^2 is less than or equal to n^2.

We complete the proof. In the region $\pi/2 \leqslant \theta \leqslant \pi$ we have that $\hat{H} = \sin\theta\sigma^2 - |\cos\theta|\tau^2$. Since $\sin\theta \geqslant 0$ we get $\hat{H} \geqslant -|\cos\theta|\tau^2 \geqslant -|\cos\theta|n^2 = \hat{H}_{B_4}$. Finally, in the region $0 \leqslant \theta \leqslant \pi/4$ we have $\hat{H} = \sin\theta(\sigma^2 + \tau^2) + (\cos\theta - \sin\theta)\tau^2 \leqslant (4n^2/3)\sin\theta + n^2(\cos\theta - \sin\theta) = (n^2/3)(\sin\theta + 3\cos\theta) = \hat{H}_{B_3}$. □

We now study linear stability of the RE found in lemma 12 and the Morse index of the corresponding stationary points. Theorem 3 leaves a number of different possibilities that require a concrete study of the Hamiltonian (11).

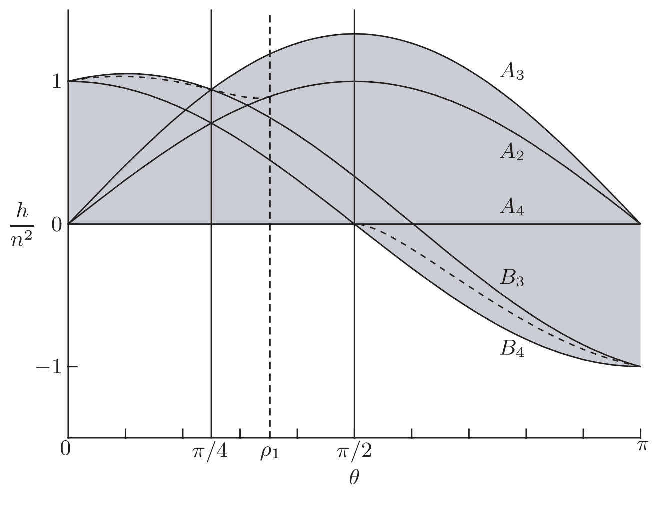

Figure 2. Scaled energy h/n^2 of the stationary points of $\hat{H}$ (11) as a function of the parameter θ. The dashed curve marks the energy of the C_s point (see section 5).

Table 2. Stability (2E, 2H, etc, as explained in appendix B.2) and Morse index $(0, \ldots, 4)$ of the stationary points of $\hat{H}$ in (11). For the C_s points see section 5; $\rho_1 = \cos^{-1}(1/\sqrt{5})$ and $\rho_2 = \cos^{-1}(-1/\sqrt{10})$.

Region		$E_{\min}$	$E_{\max}$	A_4	A_3	A_2	B_4	B_3	C_s
I	$(0, \frac{1}{4}\pi)$	$\hat{H}_{A_4}$	$\hat{H}_{B_3}$	2E 0	2H 2	EH 1	EE 2	EE 4	EH 3
IIa	$(\frac{1}{4}\pi, \rho_1)$	$\hat{H}_{A_4}$	$\hat{H}_{A_3}$	2E 0	2E 4	EE 2	EH 1	CH 2	EH 3
IIb	$(\rho_1, \frac{1}{2}\pi)$			2E 0	2E 4	EH 3	EH 1	CH 2	
IIIa	$(\frac{1}{2}\pi, \rho_2)$	$\hat{H}_{B_4}$	$\hat{H}_{A_3}$	2H 2	2E 4	EH 3	EE 0	CH 2	EH 1
IIIb	(ρ_2, π)			2H 2	2E 4	EH 3	EE 0	EE 2	EH 1

Lemma 13. *The one-parameter family of reduced Hamiltonians (11) and corresponding Hamiltonians (2) can be separated into five qualitatively different subfamilies, which correspond to five open intervals of the values of the parameter θ. Concrete values are listed in table 2. Each subfamily is distinguished by a particular pattern of the linear stability of the RE in lemma 12.*

Proof. According to remark 4, it suffices to study one critical point for each of the five critical orbits of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2(n)$. We begin by finding an appropriate local symplectic chart in the neighbourhood of the critical point, and then compute the quadratic part of the Hamiltonian (11) in this local chart. We define the charts $(\chi_1, \chi_2, \psi_1, \psi_2)$ in terms of the polynomial invariants as described in detail in appendix B. These charts are given in table 3. Note that our local coordinates (χ, ψ) are canonical only up to the constant terms in the Poisson brackets,

$$\{\chi_k, \psi_k\} = 1 + \cdots, \quad \{\chi_1, \psi_2\} = 0 + \cdots, \quad \{\chi_2, \psi_1\} = 0 + \cdots, \qquad k = 1, 2.$$

This is adequate only for the study of the linearized equations of motion. We now express Hamiltonian (11) in each local chart and expand it to the second-order terms. This gives

$$H_{A^4}(\chi, \psi) = 2n\cos\theta(\psi_1^2 + \psi_2^2) + 2n\sin\theta(\chi_1^2 + \chi_2^2);$$

$$H_{A^3}(\chi, \psi) = \frac{4}{3}n^2\sin\theta + \frac{4\sqrt{2}}{3}n(\cos\theta - \sin\theta)(\psi_1^2 + \psi_2^2) - \sqrt{2}n\sin\theta(\chi_1^2 + \chi_2^2);$$

$$H_{A^2}(\chi, \psi) = n^2\sin\theta + n((\cos\theta - \sin\theta)\psi_1^2 + (2\cos\theta - \sin\theta)\psi_2^2 - 4\sin\theta\chi_1^2 + \sin\theta\chi_2^2);$$

$$H_{B^4}(\chi, \psi) = n^2\cos\theta + n(-4\cos\theta\psi_1^2 + \sin\theta\psi_2^2 - (\cos\theta - \sin\theta)\chi_1^2 + \sin\theta\chi_2^2);$$

$$H_{B^3}(\chi, \psi) = \tfrac{1}{3}n[(\sin\theta + 3\cos\theta)(n - (\psi_1^2 + \psi_2^2) - 2\sqrt{3}(\chi_1\psi_1 + \chi_2\psi_2)) \\ -12\cos\theta(\chi_1^2 + \chi_2^2) + 6(\sin\theta - \cos\theta)(\chi_1\psi_2 - \chi_2\psi_1)].$$

(Note that in agreement with remark 4, we can always find two such local charts for any two critical points in the same critical orbit, such that the respective local Hamiltonians are the same.) Using local quadratic Hamiltonians $H(\chi, \psi)$ we compute the Morse indices and the linear stability types in table 2. □

Remark 14. Besides the linear stability type of the stationary points of $\hat{H}$ we can also determine that in some cases the stationary point is orbitally stable. This is true when the Morse index of the point is either 0 or 4 and we can use the reduced Hamiltonian to get the appropriate estimate.

Corollary 1. *The* 27 *equilibria of the system with Hamiltonian $\hat{H}$ in (11), which are the critical points of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2$, satisfy Morse inequalities and give the right Euler*

Table 3. Local coordinates at critical points on $\boldsymbol{CP}^2(n)$. We use these relations in order to express the Hamiltonian in each local chart and to compute the Poisson brackets of the local coordinates.

A_4^x	$\chi_1 = -\dfrac{1}{\sqrt{2n}}\sigma_2$	$\chi_2 = \dfrac{1}{\sqrt{2n}}\sigma_3$
	$\psi_1 = \dfrac{1}{\sqrt{2n}}\tau_2$	$\psi_2 = \dfrac{1}{\sqrt{2n}}\tau_3$
	$\nu_1 = \dfrac{\delta}{2}$	$\nu_2 = \dfrac{\sigma_3^2+\tau_3^2}{2\delta}$
	$\nu_3 = \dfrac{\sigma_2^2+\tau_2^2}{2\delta}$	$\sigma_1 = \dfrac{\sigma_2\sigma_3-\tau_2\tau_3}{\delta}$
	$\tau_1 = \dfrac{-\sigma_2\tau_3-\tau_2\sigma_3}{\delta}$	where $\delta = n + (n^2-\sigma_2^2-\sigma_3^2-\tau_2^2-\tau_3^2)^{1/2}$
A_3^a	$\chi_1 = \dfrac{1}{2^{1/4}\sqrt{n}}(2n-2\sigma_2-\sigma_3)$	$\chi_2 = -\dfrac{1}{2^{1/4}\sqrt{3n}}(2n-3\sigma_3)$
	$\psi_1 = \dfrac{3}{2^{7/4}\sqrt{n}}\tau_3$	$\psi_2 = \dfrac{\sqrt{3}}{2^{7/4}\sqrt{n}}(2\tau_2+\tau_3)$
	$\nu_1 = \dfrac{\delta}{2}$	$\nu_2 = \dfrac{\sigma_3^2+\tau_3^2}{2\delta}$
	$\nu_3 = \dfrac{\sigma_2^2+\tau_2^2}{2\delta}$	$\sigma_1 = \dfrac{\sigma_2\sigma_3-\tau_2\tau_3}{\delta}$
	$\tau_1 = \dfrac{-\sigma_2\tau_3-\tau_2\sigma_3}{\delta}$	where $\delta = n - (n^2-\sigma_2^2-\sigma_3^2-\tau_2^2-\tau_3^2)^{1/2}$
A_2^z	$\chi_1 = \dfrac{1}{\sqrt{n}}(\nu_2-\dfrac{n}{2})$	$\chi_2 = \dfrac{1}{\sqrt{n}}\sigma_1$
	$\psi_1 = \dfrac{1}{\sqrt{n}}\tau_3$	$\psi_2 = \dfrac{1}{\sqrt{n}}\tau_1$
	$\nu_1 = \dfrac{\tau_3^2+\delta^2}{4\nu_2}$	$\nu_3 = \dfrac{\sigma_1^2+\tau_1^2}{4\nu_2}$
	$\sigma_2 = \dfrac{-\tau_1\tau_3+\sigma_1\delta}{2\nu_2}$	$\sigma_3 = \delta$
	$\tau_2 = \dfrac{-\sigma_1\tau_3-\tau_1\delta}{2\nu_2}$	where $\delta = (4n\nu_2-4\nu_2^2-\sigma_1^2-\tau_1^2-\tau_3^2)^{1/2}$
B_4^x	$\chi_1 = \dfrac{1}{\sqrt{n}}\sigma_1$	$\chi_2 = \dfrac{1}{\sqrt{n}}\sigma_2$
	$\psi_1 = \dfrac{1}{\sqrt{n}}\left(\nu_3-\dfrac{n}{2}\right)$	$\psi_2 = \dfrac{1}{\sqrt{n}}\tau_2$
	$\nu_1 = \dfrac{\sigma_2^2+\tau_2^2}{4\nu_3}$	$\nu_2 = \dfrac{\sigma_1^2+\delta}{4\nu_3}$
	$\sigma_3 = \dfrac{\sigma_1\sigma_2-\tau_2\delta}{2\nu_3}$	$\tau_1 = \delta$
	$\tau_3 = \dfrac{-\sigma_1\tau_2-\sigma_2\delta}{2\nu_3}$	where $\delta = (4n\nu_3-4\nu_3^2-\sigma_1^2-\sigma_2^2-\tau_2^2)^{1/2}$
B_3^a	$\chi_1 = \dfrac{1}{4\sqrt{n}}(\sqrt{3}(\sigma_2-\sigma_3)+\tau_2-\tau_3)$	$\chi_2 = \dfrac{1}{4\sqrt{n}}(3(\sigma_2+\sigma_3)+\sqrt{3}(\tau_2+\tau_3))$
	$\psi_1 = -\dfrac{1}{2\sqrt{n}}(4n+3\sigma_2-\sqrt{3}\tau_2-2\sqrt{3}\tau_3)$	$\psi_2 = -\dfrac{1}{2\sqrt{n}}(\sqrt{3}\sigma_2+2\sqrt{3}\sigma_3+3\tau_2)$
	$\nu_1 = \dfrac{\delta}{2}$	$\nu_2 = \dfrac{\sigma_3^2+\tau_3^2}{2\delta}$
	$\nu_3 = \dfrac{\sigma_2^2+\tau_2^2}{2\delta}$	$\sigma_1 = \dfrac{\sigma_2\sigma_3-\tau_2\tau_3}{\delta}$
	$\tau_1 = \dfrac{-\sigma_2\tau_3-\tau_2\sigma_3}{\delta}$	where $\delta = n - (n^2-\sigma_2^2-\sigma_3^2-\tau_2^2-\tau_3^2)^{1/2}$

characteristic for $\boldsymbol{CP}^2$ *only in the region IIb when* $\theta \in (\rho_1, \frac{1}{2}\pi)$. *In this region* $\hat{H}$ *can be the simplest* $T_d \times \mathcal{T}$*-invariant Morse function. For all other values of* θ *this system must have other equilibria.*

Proof. Use lemma 9, appendix B, and table 2. □

5. RE corresponding to non-critical points

In this section, we find additional equilibria of the system with Hamiltonian $\hat{H}$ in (11) predicted in corollary 1 for θ outside the closed interval $[\rho_1, \frac{1}{2}\pi]$. Outside this interval $\hat{H}$ does not satisfy the Morse inequalities. Additional equilibria are likely to be created in bifurcations which take place when the value of θ leaves the interval $(\rho_1, \frac{1}{2}\pi)$.

Remark 15. Let c be one of the critical points of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2$ described in theorem 2. By the theorem of Michel cited in section 1.3, c is a stationary point of $\hat{H}(\theta)$ in (11). Suppose that the new stationary point ξ is created in a bifurcation of c. When trying to locate where ξ can be found, we should take into account the following facts (see [34] for some of these statements).

(i) The isotropy group G_ξ of ξ is a subgroup of the isotropy group G_c of c.
(ii) If c belongs not only in the closure of the stratum with trivial stabilizer C_1 but also in the closure of one or several strata with nontrivial stabilizers G', G'', etc, then a generic one-parameter bifurcation of the stationary point c will not break the symmetry G_c all the way down to C_1 but G_ξ will become one of G', G'', etc.
(iii) If such a generic bifurcation takes place, the new stationary point ξ with non-trivial stabilizer G_ξ will remain on a continuous set $\mathfrak{M} \subset \boldsymbol{CP}^2$ of non-critical G_ξ-invariant points. $\mathfrak{M}$ in turn is a subset of a submanifold $M \subset \boldsymbol{CP}^2$ with isotropy group $G_M \subseteq G_\xi$.
(iv) The manifold $M \ni \xi$ can contain points c of higher isotropy; it can itself be a subspace of a larger manifold $M' \subset \boldsymbol{CP}^2$ with lower isotropy group $G_{M'} \subset G_M \subseteq G_\xi \subset G_c$.
(v) When looking for stationary points $\xi \in M \subset M' \subset \cdots \boldsymbol{CP}^2$ we should check if the Morse inequalities hold for all invariant submanifolds M, M', etc.
(vi) Particularly interesting are the situations when $P = M$, or M', etc are also dynamically invariant, because then G_P is symplectic. In that case P is symplectic, and we can restrict our system to P and look for its equilibria there.
(vii) The action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$ has several invariant submanifolds M with topology $\boldsymbol{S}^1$, $\boldsymbol{S}^2 \sim \boldsymbol{CP}^1$, and $\boldsymbol{RP}^2$. Information on these manifolds and their intersections can be found in [11] and appendix A. The 2-spheres with stabilizers of conjugacy class C_s and C_2 can serve as restricted phase spaces P.

5.1. Existence and stability of the $C_s \wedge \mathcal{T}_2$ *RE*

Following remark 15(vi–vii), consider the C_s- and C_2-invariant spheres $\boldsymbol{S}^2 \subset \boldsymbol{CP}^2(n)$ described in appendix A. Critical points of type A_4 and A_2 occur at the points of intersection of the two types of spheres (see figure A2). This means that the new equilibrium points ξ created in a bifurcation of A_4 or A_2 can depart either on a C_2 or a C_s sphere (cf remark 15(iii)). On the C_s spheres we also find points of type B_4, while on the C_2 spheres we find points of type A_3.

Lemma 14. *The* A_2*,* A_4*, and* B_4 *equilibrium points of the system with Hamiltonian* $\hat{H}$ *in (11) alone do not give the right Euler characteristic for* $\boldsymbol{S}^2$ *on the* C_s *spheres when* $0 < \theta < \rho_1$ *and* $\frac{1}{2}\pi < \theta < \pi$*, i.e. outside the region IIb in table 2; they give the right characteristic when* $\rho_1 < \theta < \frac{1}{2}\pi$*. Furthermore, for the same values of* θ*, the two points* A_2 *and* A_4 *alone,* *which lie on the same $C_s \wedge \mathcal{T}_2$-invariant circle of the C_s sphere, do not give the correct Euler characteristic for $\boldsymbol{S}^1$.*

Proof. We can always split local coordinates in table 3 in order to select a C_s- or C_2-invariant symplectic pair. Then, checking the Morse inequalities and the Euler characteristic for the C_2 and C_s spheres is straightforward. □

Corollary 2. *In all regions in table 2 except IIb, the system with Hamiltonian (11) should have additional equilibria $\xi \in \boldsymbol{S}^1 \subset \boldsymbol{S}^2 \subset \boldsymbol{CP}^2(n)$, where the isotropy group of ξ and $\boldsymbol{S}^1$ is $C_s \wedge \mathcal{T}_2$ and $\boldsymbol{S}^2$ has isotropy group C_s.*

Lemma 15. *The $C_s \wedge \mathcal{T}_2$-invariant equilibria in corollary 2 exist only when*

$$0 < \theta < \rho_1 = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \qquad or \qquad \frac{1}{2}\pi < \theta < \pi,$$

and have stability type EH. By symmetry, there should be two such equilibria on each of the six C_s spheres. The original system with Hamiltonian (2) has 12 corresponding $C_s \wedge \mathcal{T}_2$-invariant nonlinear normal modes.

Proof. By an argument similar to that given in remark 4, it is sufficient to study one of the six equivalent C_s spheres. We choose the sphere C_s^{ab} which is defined in appendix A as the set of points on $\boldsymbol{CP}^2(n)$ whose coordinates $(\nu_1, \nu_2, \nu_3; \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3)$ are

$$(\nu; \sigma; \tau) = n\left(\frac{1+w}{4}, \frac{1+w}{4}, \frac{1-w}{2}; \frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}, \frac{1+w}{2}; \frac{v}{\sqrt{2}}, -\frac{v}{\sqrt{2}}, 0\right), \tag{14}$$

where $u^2+v^2+w^2 = 1$. The $C_s \wedge \mathcal{T}_2$-invariant circle on this sphere is defined by the additional equation $u = 0$. The reduced system restricted to this sphere corresponds to the original system restricted to the 4-plane in $T^*\boldsymbol{R}^3$ defined by $\{x = y,\ p_x = p_y\}$. The Poisson algebra generated by the functions (u, v, w) in (14) and restricted to the C_s^{ab} sphere is the algebra $so(3)$ with Casimir $u^2 + v^2 + w^2$. We find that the vector field of Hamiltonian (11) restricted to this sphere is

$$\begin{aligned}
\dot{u} &= -v(w+1)(\sin\theta - 4\cos\theta) - 4v\cos\theta,\\
\dot{v} &= u(1-3w)\sin\theta,\\
\dot{w} &= 4uv(\sin\theta - \cos\theta).
\end{aligned} \tag{15}$$

The constant level sets of this system are shown in figure 3 in the coordinate system of figure A3(*b*) with axis w aligned vertically. The equations $\dot{u} = \dot{v} = \dot{w} = 0$ for the equilibria on the sphere can be easily solved. Solutions $u = 0, v = 0, w = 1$ (north pole) and $u = 0, v = 0, w = -1$ (south pole) represent points A_2^z and A_4^z, respectively; the two points A_3^a and A_3^b correspond to $u = \pm\frac{2}{3}\sqrt{2}$, $v = 0$, $w = \frac{1}{3}$. The position of these critical points is shown in figure 3 for the example of $\theta = 5\pi/36$. The two new stationary points $\xi_\pm$ have coordinates

$$u = 0, \qquad v = \pm(1-w^2)^{1/2}, \qquad w = \frac{\sin\theta}{4\cos\theta - \sin\theta}.$$

This solution exists only for the values of θ specified in the lemma. From (14) we find the $\boldsymbol{CP}^2(n)$ coordinates of $\xi_\pm$

$$\xi_\pm = n(r, r, 1-2r; 0, 0, 2r; \pm 2\sqrt{r(1-2r)}, \pm 2\sqrt{r(1-2r)}, 0),$$

where $r = \cos\theta/(4\cos\theta - \sin\theta)$. In figure 3, $\theta = 5\pi/36$ (region I), $\xi_\pm$ can be seen as two deep minima, which lie on the $u = 0$ meridian slightly above the equatorial line and below the

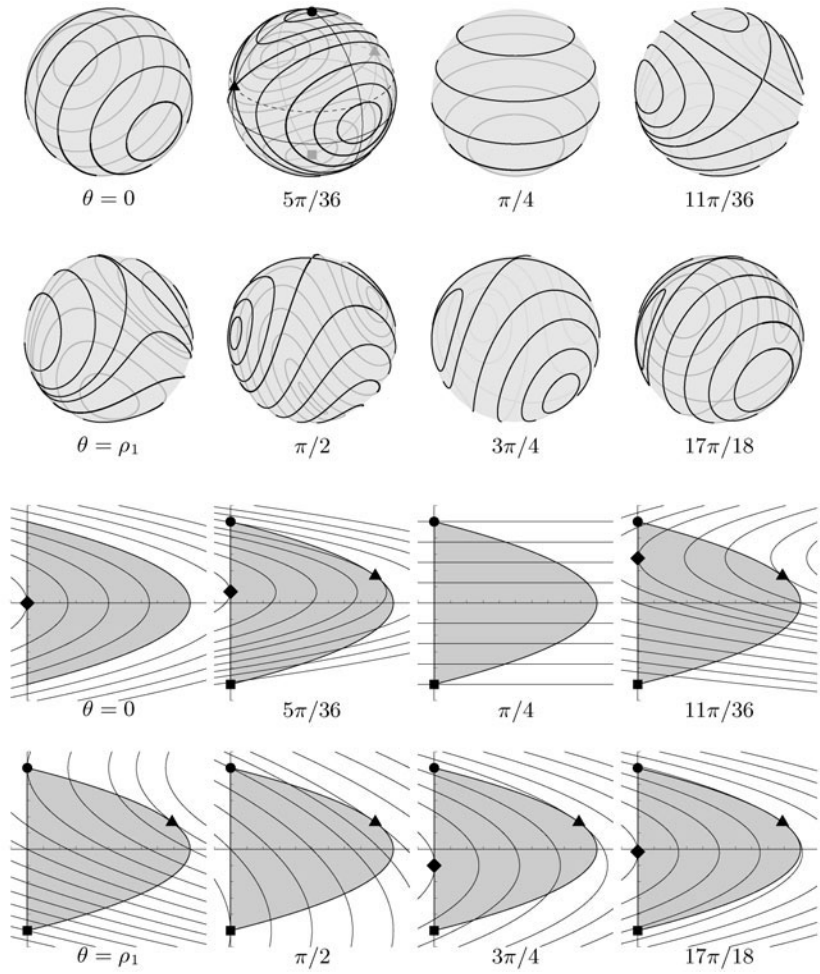

Figure 3. First part: phase portraits of the system with Hamiltonian $\hat{H}$ in (11) restricted to the C_s^{ab}-invariant sphere $\boldsymbol{S}^2 \subset \boldsymbol{CP}^2(n)$ for different values of the parameter θ. The sphere is oriented in the same way as in figure A3(b); location of the critical points A_2, A_3, and A_4 is marked for the $\theta = 5\pi/36$ case by a filled disc, triangle and square, respectively. Second part: the orbit space of the action of the discrete symmetry on the C_s sphere. The C_s stationary points are represented by a filled diamond. The level sets of the Hamiltonian $\hat{H}$ restricted on the C_s sphere and expressed in terms of $U = u^2$ and w are also depicted (for more details see appendix A.2).

latitude of the A_3 points shown by a dashed line. As θ increases $\xi_\pm$ move up (north). In the region IIa, they become unstable, see case $\theta = 11\pi/36$. In the region IIb $\xi_\pm$ do not exist, they reappear for $\theta > \frac{1}{2}\pi$ as minima. Using the same local analysis as for other stationary points, i.e. finding a local symplectic chart and linearizing $\hat{H}$ in this chart, we find that $\xi_\pm$ are always of type EH (elliptic–hyperbolic). In the regions I, IIIa, and IIIb the elliptic plane is tangent to the sphere and the hyperbolic plane is orthogonal to the sphere; in the IIa region the situation is reversed. □

Remark 16. The second part of figure 3 shows how to find the same results from the intersections of level sets of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ reduced Hamiltonian and the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbit space.

Theorem 5. *The reduced one-parameter Hamiltonian $\hat{H}$ (13) has exactly 27 stationary points in the region IIb $(\rho_1, \frac{1}{2}\pi)$, which are the critical points of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2$. In the other regions it has exactly 12 more stationary points with stabilizer $C_s \wedge \mathcal{T}_2$. The only exceptional values are $\theta = 0, \frac{1}{4}\pi, \rho_1, \frac{1}{2}\pi, \rho_2, \pi$, where $\hat{H}$ is not a Morse function.*

Proof. The vector field $X_{\hat{H}}$ of $\hat{H}$ expressed in terms of the invariants has nine components that are quadratic polynomials:

$$\begin{aligned}
\dot{\nu}_1 &= 2(\cos\theta - \sin\theta)(\sigma_2\tau_2 - \sigma_3\tau_3),\\
\dot{\nu}_2 &= 2(\cos\theta - \sin\theta)(\sigma_3\tau_3 - \sigma_1\tau_1),\\
\dot{\nu}_3 &= 2(\cos\theta - \sin\theta)(\sigma_1\tau_1 - \sigma_2\tau_2),\\
\dot{\sigma}_1 &= 2(\sin\theta - \cos\theta)(\sigma_3\tau_2 - \sigma_2\tau_3) + 4\cos\theta(\nu_2 - \nu_3)\tau_1,\\
\dot{\sigma}_2 &= 2(\sin\theta - \cos\theta)(\sigma_1\tau_3 - \sigma_3\tau_1) + 4\cos\theta(\nu_3 - \nu_1)\tau_2,\\
\dot{\sigma}_3 &= 2(\sin\theta - \cos\theta)(\sigma_2\tau_1 - \sigma_1\tau_2) + 4\cos\theta(\nu_1 - \nu_2)\tau_3,\\
\dot{\tau}_1 &= 4\sin\theta(\nu_3 - \nu_2)\sigma_1,\\
\dot{\tau}_2 &= 4\sin\theta(\nu_1 - \nu_3)\sigma_2,\\
\dot{\tau}_3 &= 4\sin\theta(\nu_2 - \nu_1)\sigma_3.
\end{aligned}$$

The equilibria of $X_{\hat{H}}$ are given by the common roots of these polynomials and the polynomials Σ_k, $k = 0, \ldots, 9$ (6). We solve this system of polynomial equations by finding its Gröbner basis using the lexicographic order $\tau_3 > \tau_2 > \tau_1 > \sigma_3 > \sigma_2 > \sigma_1 > \nu_3 > \nu_2 > \nu_1$. Such a basis can be constructed using the computer program Mathematica. Although the Gröbner basis consists of 89 polynomials it is straightforward to solve. There are two types of solutions: 27 solutions that do not depend on θ correspond to the critical stationary points of $\tilde{H}$; 12 solutions that depend on θ correspond to the extra non-critical stationary points and are valid only when $0 < \nu_1, \nu_2, \nu_3 < n$. This condition implies that the extra stationary points do not exist in the region IIb. □

5.2. Configuration space image of the $C_s \wedge \mathcal{T}_2$ RE

Remark 17. Evolution of the additional $C_s \wedge \mathcal{T}_2$ RE can be best seen on the interval $\theta \in [-\frac{1}{2}\pi, \rho_1]$. (The part $[-\frac{1}{2}\pi, 0]$ is equivalent to region III $[\frac{1}{2}\pi, \pi]$ in figure 2 and table 2 up to the sign of $\hat{H}$, see remark 10.) These RE branch off the A_4 RE at $\theta = -\frac{1}{2}\pi$ and then exist continuously until their merger with the A_2 RE at $\theta = \rho_1$.

The principles of the RE representation in the configuration space $\boldsymbol{R}^3_{x,y,z}$ are discussed in appendix A.3. The $C_s \wedge \mathcal{T}_2$ RE are not $\mathcal{T}$-invariant and, therefore, they appear in $\boldsymbol{R}^3$ as loops. The two $C_s \wedge \mathcal{T}_2$ stationary points on the same C_s sphere, such as, for example, in figure 3 with $\theta = 5\pi/36$, are mapped into each other by the $\mathcal{T}$ operation. These two points correspond to two loops running along the same closed curve in $\boldsymbol{R}^3$ but in different directions. According to remark 17, these loops branch off one of the three A_4 orbits and merge with an A_2 orbit. Take, for example, the three RE A_4^z, A_2^z, and $A_2^{\bar{z}}$. The A_4^z RE is represented by a segment on axis z, while images of A_2^z and $A_2^{\bar{z}}$ lie in the planes aOb and cOd (see figure A1 and figure 4, left). Note that axis z is the intersection $aOb \cap cOd$, and that the aOb and cOd planes are the configuration spaces of the restricted systems whose reduced phase spaces are the C_s^{ab} and C_s^{cd} spheres.

Without any loss in our present qualitative description, we can consider RE of the normalized system instead of those of the original system shown in figure 1. In the transformed

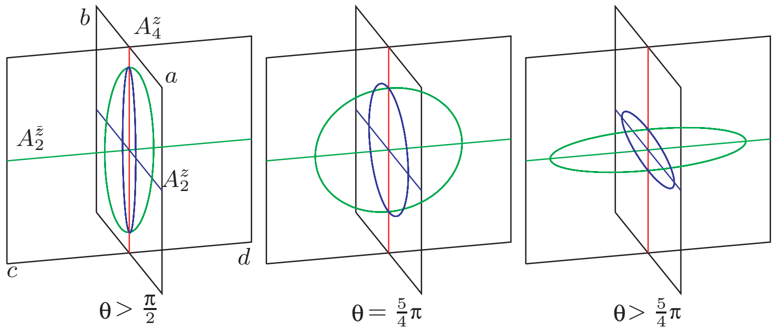

Figure 4. Schematic representation of the A_4^z, A_2^z, $A_2^{\bar z}$, and $C_s \wedge \mathcal{T}$ RE in the configuration space $\boldsymbol{R}^3_{x,y,z}$ of the normalized Hamiltonian (2).

space $\tilde{\boldsymbol{R}}^3$, the A_4^z RE remains a segment of axis z, while A_2^z and $A_2^{\bar z}$ become segments of straight lines $x = y$ and $x = -y$ in the horizontal plane $z = 0$ as shown in figure 4. At $\theta = \frac{1}{2}\pi$, four $C_s \wedge \mathcal{T}_2$ orbits bifurcate from A_4^z. These four RE project into two distinct closed curves in $\boldsymbol{R}^3$. When the value of θ is only slightly above $\frac{1}{2}\pi$, the loops have a highly eccentric elliptical shape stretched along A_4^z (see figure 4, left). The major axis of the ellipse is on the z-axis and the ellipses lie in the aOb and cOd planes, respectively. As θ increases, the eccentricity is reduced until the ellipses become circles at $\theta = \frac{5}{4}\pi$. At this point the elliptic and hyperbolic directions for the $C_s \wedge \mathcal{T}_2$ points $\xi_\pm \in \boldsymbol{CP}^2(n)$ are interchanged. For $\theta > \frac{5}{4}\pi$ the eccentricity increases again but now the major axes lie near the intersections $aOb \cap xOy$ and $cOd \cap xOy$, respectively. As θ approaches $\pi + \rho_1$, the two ellipses come closer to the orbits A_2^z and $A_2^{\bar z}$ and vanish exactly at $\theta = \pi + \rho_1$.

6. Bifurcations

Definition 7. *The three-dimensional Hénon–Heiles system with Hamiltonian (2) is called generic if it is ϵ^2-generic and the corresponding principal order of the reduced Hamiltonian $\hat{H}$ in (11) is a $T_d \times \mathcal{T}$-invariant Morse function on $\boldsymbol{CP}^2(n)$.*

Generic systems belong to one of the subfamilies in table 2. We have characterized the RE of these systems. In this section, we comment on some of the changes of linear stability of the RE in table 2 and the possible bifurcations that may be happening when the parameter θ is varied. A full study of non-Morse members of the family (11) which occur at the end points of the five intervals in the first column of table 2 and bifurcations of RE requires going to higher orders of the normal form and is beyond the scope of our present work. Some of these bifurcations cannot be fully understood using the single-parameter classification scheme of section 2.

As we saw in section 5.1, several bifurcations are related to the evolution of the $C_s \wedge \mathcal{T}_2$ RE. At $\theta = \frac{1}{2}\pi$ (or $-\frac{1}{2}\pi$) four $C_s \wedge \mathcal{T}_2$ RE are created in the bifurcation of each of the three A_4 RE. In this bifurcation, as we go from the region IIb to IIIa in table 2, the stability and Morse index of the A_4 RE change from 2E and 0 to 2H and 2, respectively. Since A_4 and B_4 share the same C_2-invariant subspace $\boldsymbol{S}^2$ (figure A3, left), the Morse index change of A_4 forces the change of the Morse index of B_4 in order to preserve the right Euler characteristic of $\boldsymbol{S}^2$. When $\theta = \pi$ and we enter region I from IIIb, the Morse index of A_4 changes from 2 to 0 (or 4, see remark 10). When $\theta = \rho_1$ the $C_s \wedge \mathcal{T}_2$ RE collide pairwise at the A_2 RE and vanish. At this moment the A_2 RE change the linear stability type from EE to EH and the Morse index from 2 to 1. This bifurcation can be considered as a collision of two stationary points on the C_s sphere; in systems with 1-DOF it is often called a 'pitchfork' bifurcation.

The stability type of the B_3 points changes between elliptic–elliptic (EE) and complex hyperbolic (CH) at $\theta = \pi/4$ and ρ_2. At these values of θ the four eigenvalues of the respective Hamiltonian matrices move along the imaginary axis, collide pairwise, and then move off the axis into the complex plane. Such EE $\leftrightarrow$ CH change of linear stability is called linear Hamiltonian Hopf bifurcation[1]. It suggests that a nonlinear Hamiltonian Hopf bifurcation might also be taking place [35]. This important codimension-one bifurcation happens in Hamiltonian systems with two or more degrees of freedom. The EE $\leftrightarrow$ CH change is necessary but not sufficient for the nonlinear bifurcation. The latter occurs when a family of periodic orbits either detaches from the bifurcating stationary point or shrinks to this point and disappears. We should take the nonlinearity of the system into account in order to find out if this takes place.

Unfortunately, standard theorems on the Hamiltonian Hopf bifurcation do not apply directly in our case. When $\theta = \rho_2$ we can prove that the B_3 points lie on an invariant two-dimensional manifold. Consequently, the bifurcation remains degenerate in all orders. A preliminary study using normal form techniques shows that a bifurcation of short periodic orbits which differs slightly from the Hamiltonian Hopf bifurcation takes place at $\theta = \rho_2$. At $\theta = \frac{1}{4}\pi$ the eigenvalues of the Hamiltonian matrix of the linearized equations near the B_3 equilibrium meet at 0 and then jump off to the complex plane. This means that at the moment of bifurcation the quadratic part of the local Hamiltonian is nilpotent and hence the local Hamiltonian cannot be normalized in the standard way. We believe that both the degeneracy of the $\theta = \rho_2$ case and the nilpotency of the $\theta = \frac{1}{4}\pi$ case is removed in the sixth (or higher) order normal form of (2).

A generalization of the linear Hamiltonian Hopf bifurcation is proposed in [36]. This paper describes a bifurcation of short periodic orbits that happens when a stationary point with isotropy group $SO(2) \times \mathcal{T}$ changes linear stability type from degenerate elliptic (2E) to degenerate hyperbolic (2H). In our system, the A_3 and A_4 RE change stability type from 2E to 2H at $\theta = \frac{1}{4}\pi$ and $\theta = \frac{1}{2}\pi$, respectively. Since by theorem 3 the A_4 and A_3 RE can only be of type 2E or 2H, the eigenvalues must become simultaneously zero when their stability changes from 2E to 2H. This degeneracy is robust under $T_d \times \mathcal{T}$ symmetric perturbations.

7. Conclusion: classification of three-dimensional Hénon–Heiles systems

If we now consider all stationary points of $\hat{H}$ in (11) that we have found, we can verify that Morse inequalities (B.2) are now satisfied on $\boldsymbol{CP}^2(n)$ and on all invariant subspaces of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2(n)$ for all values of θ. Studying the restrictions of $\hat{H}$, we can also verify that there are no other stationary points on the subspaces.

Proposition 1. *Table 2 gives a complete set of stationary points of the reduced Hamiltonian* $\hat{H}$ *in (11) in all regions of the values of the parameter* θ *where* $\hat{H}$ *is a Morse function on* $\boldsymbol{CP}^2$.

Corollary 3. *The family of systems with Hamiltonian (2) and its further generalizations mentioned in remark 1 has five qualitatively different subfamilies. Each subfamily is characterized by the number and stability of nonlinear normal modes which correspond to the stationary points described in table 2. There is only one subfamily with minimal possible number (27) of nonlinear normal modes described in theorem 1. Other subfamilies have a set of* 12 *additional equivalent modes of the type* $C_s \wedge \mathcal{T}_2$.

[1] The name is because this is reminiscent of the Hopf bifurcation in dissipative systems.

This concludes the classification of all systems with Hamiltonian (2) generic in the sense of definition 7.

Acknowledgments

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Appendix A. Action of the group $T_d \times \mathcal{T}$ on the spaces $\boldsymbol{R}^3$, $T^*\boldsymbol{R}^3$, and $\boldsymbol{CP}^2$

The symmetry group $T_d \subset O(3)$ of the tetrahedron is a group of point transformations of the physical 3-space (see figure A1). As an abstract group it is isomorphic to the permutation group of four elements. We assume that the coordinate functions (x, y, z) in the configuration space $\boldsymbol{R}^3$ of (2) span a three-dimensional vector representation of T_d, that is, T_d acts on (x, y, z) as on the coordinates in the physical 3-space. Let O be the origin $(0, 0, 0) \in \boldsymbol{R}^3$, and let Ox, Oy, and Oz be the directed semi-axes of the coordinate system in $\boldsymbol{R}^3$. Consider also the four directed semi-axes Oa, Ob, Oc, and Od in figure A1, where $a = (1, 1, 1)$, $b = (-1, -1, 1)$, $c = (1, -1, -1)$, and $d = (-1, 1, -1)$. Any pair of semi-axes $(O\alpha, O\beta)$ defines a 2-plane $\alpha O\beta$ passing through O. Table A1 gives explicit definitions of some basic operations in T_d, which we further explain below.

S_4 Operation S_4^x combines the counterclockwise rotation by $2\pi/4 = \frac{1}{2}\pi$ about Ox and the reflection in the plane $yOz \perp Ox$. Similar operations S_4^y and S_4^z involve axes Oy and Oz,

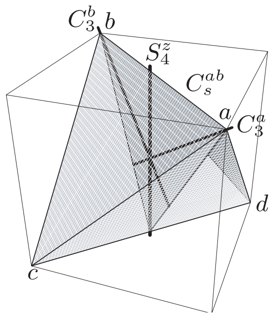


Figure A1. Symmetry axes and planes of a tetrahedron.

Table A1. The action of some elements of T_d on the representation spanned by x, y, z.

R	Rx	Ry	Rz
S_4^x	$-x$	$-z$	y
S_4^y	z	$-y$	$-x$
S_4^z	$-y$	x	$-z$

R	Rx	Ry	Rz
C_3^a	z	x	y
C_3^b	$-z$	x	$-y$
C_3^c	$-z$	$-x$	y
C_3^d	z	$-x$	$-y$

R	Rx	Ry	Rz
C_s^{ab}	y	x	z
C_s^{cd}	$-y$	$-x$	z
C_s^{ad}	z	y	x
C_s^{bc}	$-z$	y	$-x$
C_s^{ac}	x	z	y
C_s^{bd}	x	$-z$	$-y$

respectively. The conjugacy class S_4 in T_d also contains the elements $(S_4^\alpha)^{-1}$, $\alpha = x, y, z$; elements $C_2^\alpha = (S_4^\alpha)^2$ form the conjugacy class C_2.

C_3 Operations C_3^a, C_3^b, C_3^c, and C_3^d are counterclockwise rotations by $2\pi/3$ about axes Oa, Ob, Oc, and Od, respectively. The conjugacy class C_3 also includes $(C_3^a)^2$, $(C_3^b)^2$, $(C_3^c)^2$, and $(C_3^d)^2$.

C_s Reflection in each of the six planes $\{aOb, cOd, aOd, bOc, aOc, bOd\}$, which we denote as C_s^{ab}, C_s^{cd}, C_s^{ad}, C_s^{bc}, C_s^{ac}, and C_s^{bd}, leaves the tetrahedron invariant. These operations form the conjugacy class C_s.

Extension of the T_d action described above to the phase space $T^*\boldsymbol{R}^3$ of (2) uses the following lemma.

Lemma A.1. *If matrices M_q and M_p in $GL(\boldsymbol{R}, 3)$ acting on the coordinates $q = (x, y, z)$ and the conjugate momenta $p = (p_x, p_y, p_z)$ define a linear symplectic transformation in $T^*\boldsymbol{R}^3$, then $M_p = (M_q^{-1})^T$.*

It follows that $M_q = M_p$ for $M_q \in T_d \subset O(3)$; that is, (x, y, z) and (p_x, p_y, p_z) transform according to the same representation of T_d. Action of the full symmetry group $T_d \times \mathcal{T}$ on $T^*\boldsymbol{R}^3$ is obtained by combining the action of T_d and the momentum reversal $T : (q, p) \to (q, -p)$.

Appendix A.1. Fixed points of the action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$

Projection of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2$ has been discussed in detail in [20, 15, 11, 10]. We give only the information that is useful for our study. The action of $T_d \times \mathcal{T}$ on the invariants (5) can be found straightforwardly using the action of $T_d \times \mathcal{T}$ on $T^*\boldsymbol{R}^3$. Table A2 gives the results. Zhilinskií described the critical orbits of the T_d action on $\boldsymbol{CP}^2$ in [20]. The action of the full group $T_d \times \mathcal{T}$ has the same five critical orbits [11, 10] which we characterize in tables 1 and A3. Observe that points of types A and B transform differently with respect to T: A_4, A_3, and A_2 are T-invariant, while B_4 and B_3 are not, because T maps each B-type point to another: for example, $B_4^z \to B_4^{\bar{z}}$.

Appendix A.2. Subspaces of $\boldsymbol{CP}^2$-invariant under the action of $T_d \times \mathcal{T}$

The action of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$ has a number of invariant subspaces M of topology $\boldsymbol{RP}^2$, $\boldsymbol{CP}^1 \sim \boldsymbol{S}^2$, and $\boldsymbol{S}^1$ [11]. Points of M are non-isolated fixed points of the action of the stabilizer $G_M \subset T_d \times \mathcal{T}$ of M. The invariant manifolds of points with stabilizers C_2 and C_s are 2-spheres $\boldsymbol{S}^2$ which are symplectic. Moreover, these spheres remain invariant under the flow of any $T_d \times \mathcal{T}$-invariant Hamiltonian $\hat{H}$. According to [20], the 27 critical points and the spheres intersect in $\boldsymbol{CP}^2$ as shown in figure A2. We discuss these spheres in more detail.

Consider specifically the action of $C_2^x \subset T_d$ on $\boldsymbol{CP}^2(n)$. Using table A2 we find that fixed points of this action are of the form $(\nu_1, \nu_2, \nu_3; \sigma_1, 0, 0; \tau_1, 0, 0)$. Taking relations (6) into

Table A2. Action of some elements of $T_d \times \mathcal{T}$ on $\boldsymbol{CP}^2$.

R	$R\nu_1$	$R\nu_2$	$R\nu_3$	$R\sigma_1$	$R\sigma_2$	$R\sigma_3$	$R\tau_1$	$R\tau_2$	$R\tau_3$
C_2^x	ν_1	ν_2	ν_3	σ_1	$-\sigma_2$	$-\sigma_3$	τ_1	$-\tau_2$	$-\tau_3$
S_4^x	ν_1	ν_3	ν_2	$-\sigma_1$	$-\sigma_3$	σ_2	τ_1	τ_3	$-\tau_2$
C_3^a	ν_3	ν_1	ν_2	σ_3	σ_1	σ_2	τ_3	τ_1	τ_2
C_s^{ab}	ν_2	ν_1	ν_3	σ_2	σ_1	σ_3	$-\tau_2$	$-\tau_1$	$-\tau_3$
$\mathcal{T}$	ν_1	ν_2	ν_3	σ_1	σ_2	σ_3	$-\tau_1$	$-\tau_2$	$-\tau_3$

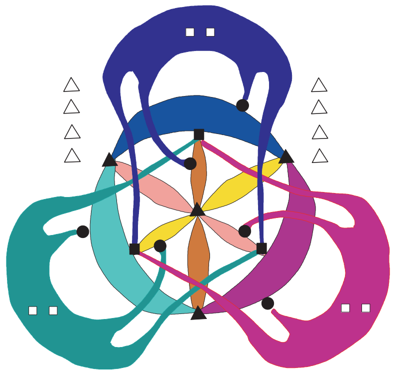
Table A3. Critical points of the $T_d \times \mathcal{T}$ action on $\boldsymbol{CP}^2(n)$. In the second column $\mathcal{T}_2 = \{1, C_2 T\}$ and $\mathcal{T}_s = \{1, C_s T\}$.

Point	Isotropy	
A_4^x	$D_{2d}^x \times \mathcal{T}$	$\{1, S_4^x, C_2^x, (S_4^x)^{-1}, C_2^y, C_2^z, C_s^{ac}, C_s^{bd}\} \times \mathcal{T}$
A_4^y	$D_{2d}^y \times \mathcal{T}$	$\{1, S_4^y, C_2^y, (S_4^y)^{-1}, C_2^x, C_2^z, C_s^{ad}, C_s^{bc}\} \times \mathcal{T}$
A_4^z	$D_{2d}^z \times \mathcal{T}$	$\{1, S_4^z, C_2^z, (S_4^z)^{-1}, C_2^x, C_2^y, C_s^{ab}, C_s^{cd}\} \times \mathcal{T}$
A_3^a	$C_{3v}^a \times \mathcal{T}$	$\{1, C_3^a, (C_3^a)^2, C_s^{ab}, C_s^{ac}, C_s^{ad}\} \times \mathcal{T}$
A_3^b	$C_{3v}^b \times \mathcal{T}$	$\{1, C_3^b, (C_3^b)^2, C_s^{ab}, C_s^{bc}, C_s^{bd}\} \times \mathcal{T}$
A_3^c	$C_{3v}^c \times \mathcal{T}$	$\{1, C_3^c, (C_3^c)^2, C_s^{ac}, C_s^{bc}, C_s^{cd}\} \times \mathcal{T}$
A_3^d	$C_{3v}^d \times \mathcal{T}$	$\{1, C_3^d, (C_3^d)^2, C_s^{ad}, C_s^{bd}, C_s^{cd}\} \times \mathcal{T}$
A_2^x	$C_{2v}^x \times \mathcal{T}$	$\{1, C_2^x, C_s^{ac}, C_s^{bd}\} \times \mathcal{T}$
$A_2^{\bar{x}}$	$C_{2v}^x \times \mathcal{T}$	$\{1, C_2^x, C_s^{ac}, C_s^{bd}\} \times \mathcal{T}$
A_2^y	$C_{2v}^y \times \mathcal{T}$	$\{1, C_2^y, C_s^{ad}, C_s^{bc}\} \times \mathcal{T}$
$A_2^{\bar{y}}$	$C_{2v}^y \times \mathcal{T}$	$\{1, C_2^y, C_s^{ad}, C_s^{bc}\} \times \mathcal{T}$
A_2^z	$C_{2v}^z \times \mathcal{T}$	$\{1, C_2^z, C_s^{ab}, C_s^{cd}\} \times \mathcal{T}$
$A_2^{\bar{z}}$	$C_{2v}^z \times \mathcal{T}$	$\{1, C_2^z, C_s^{ab}, C_s^{cd}\} \times \mathcal{T}$
B_4^x	$S_4^x \wedge \mathcal{T}_2^y$	$\{1, S_4^x, C_2^x, (S_4^x)^{-1}, C_2^y T, C_2^z T, C_s^{ac} T, C_s^{bd} T\}$
$B_4^{\bar{x}}$	$S_4^x \wedge \mathcal{T}_2^y$	$\{1, S_4^x, C_2^x, (S_4^x)^{-1}, C_2^y T, C_2^z T, C_s^{ac} T, C_s^{bd} T\}$
B_4^y	$S_4^y \wedge \mathcal{T}_2^z$	$\{1, S_4^y, C_2^y, (S_4^y)^{-1}, C_2^x T, C_2^z T, C_s^{ad} T, C_s^{bc} T\}$
$B_4^{\bar{y}}$	$S_4^y \wedge \mathcal{T}_2^z$	$\{1, S_4^y, C_2^y, (S_4^y)^{-1}, C_2^x T, C_2^z T, C_s^{ad} T, C_s^{bc} T\}$
B_4^z	$S_4^z \wedge \mathcal{T}_2^x$	$\{1, S_4^z, C_2^z, (S_4^z)^{-1}, C_2^x T, C_2^y T, C_s^{ab} T, C_s^{cd} T\}$
$B_4^{\bar{z}}$	$S_4^z \wedge \mathcal{T}_2^x$	$\{1, S_4^z, C_2^z, (S_4^z)^{-1}, C_2^x T, C_2^y T, C_s^{ab} T, C_s^{cd} T\}$
B_3^a	$C_3^a \wedge \mathcal{T}_s^{ab}$	$\{1, C_3^a, (C_3^a)^2, C_s^{ab} T, C_s^{ac} T, C_s^{ad} T\}$
$B_3^{\bar{a}}$	$C_3^a \wedge \mathcal{T}_s^{ab}$	$\{1, C_3^a, (C_3^a)^2, C_s^{ab} T, C_s^{ac} T, C_s^{ad} T\}$
B_3^b	$C_3^b \wedge \mathcal{T}_s^{ab}$	$\{1, C_3^b, (C_3^b)^2, C_s^{ab} T, C_s^{bc} T, C_s^{bd} T\}$
$B_3^{\bar{b}}$	$C_3^b \wedge \mathcal{T}_s^{ab}$	$\{1, C_3^b, (C_3^b)^2, C_s^{ab} T, C_s^{bc} T, C_s^{bd} T\}$
B_3^c	$C_3^c \wedge \mathcal{T}_s^{cd}$	$\{1, C_3^c, (C_3^c)^2, C_s^{ac} T, C_s^{bc} T, C_s^{cd} T\}$
$B_3^{\bar{c}}$	$C_3^c \wedge \mathcal{T}_s^{cd}$	$\{1, C_3^c, (C_3^c)^2, C_s^{ac} T, C_s^{bc} T, C_s^{cd} T\}$
B_3^d	$C_3^d \wedge \mathcal{T}_s^{cd}$	$\{1, C_3^d, (C_3^d)^2, C_s^{ad} T, C_s^{bd} T, C_s^{cd} T\}$
$B_3^{\bar{d}}$	$C_3^d \wedge \mathcal{T}_s^{cd}$	$\{1, C_3^d, (C_3^d)^2, C_s^{ad} T, C_s^{bd} T, C_s^{cd} T\}$

account we find that the subset of $\boldsymbol{CP}^2(n)$ with stabilizer C_2^x is the disjoint union of A_2^x and the $\boldsymbol{S}^2$ sphere $(0, \nu_2, n - \nu_2; \sigma_1, 0, 0; \tau_1, 0, 0)$ with $\sigma_1^2 + \tau_1^2 + (2\nu_2 - n)^2 = n^2$. In the coordinates $u = \sigma_1 n^{-1}$, $v = \tau_1 n^{-1}$, and $w = 2\nu_2 n^{-1} - 1$, its equation is $u^2 + v^2 + w^2 = 1$. There are three C_2 spheres corresponding to the three axes C_2. On each sphere we find six critical points, two of type A_4, two of type A_2, and two of type B_4. Specifically, on the C_2^x sphere we find the points A_4^y, A_4^z, A_2^x, $A_2^{\bar{x}}$, B_4^x, and $B_4^{\bar{x}}$ (figure A3(*a*)).

The same analysis for the action of $C_s^{ab} \subset T_d$ shows that the set of $\boldsymbol{CP}^2(n)$ points fixed under this action is the disjoint union of $A_2^{\bar{z}}$ and the $\boldsymbol{S}^2$ sphere $(\nu_1, \nu_1, n - 2\nu_1; \sigma_1, \sigma_1, 2\nu_1; \tau_1, -\tau_1, 0)$ with $2\sigma_1^2 + 2\tau_1^2 + (4\nu_1 - n)^2 = n^2$ and coordinates $u = \sqrt{2}\sigma_1 n^{-1}$, $v = \sqrt{2}\tau_1 n^{-1}$, and $w = 4\nu_1 n^{-1} - 1$. There are six such spheres, one for each C_s plane. On each C_s sphere we find four critical points, one of type A_2, one of type A_4, and two of type A_3. Specifically, on the C_s^{ab} sphere we find the points A_2^z, A_4^z, A_3^a, and A_3^b (figure A3(*b*)).

The action of the group $T_d \times \mathcal{T}$ on each C_s sphere is reduced to the action of a $C_{2v} = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ group generated by the transformations $u \to -u$ and $v \to -v$. The orbit space of this action is defined by the invariants $U = u^2$, $V = v^2$, and w subject to the relations $U + V + w = 1$, $U > 0$, and $V > 0$. Because of the linear relation between the invariants we can use only two of them to describe the orbit space. We choose U and w. The orbit space is depicted in figure A3(*c*).



fixed point	stabilizer in $T_d \times \mathcal{T}$
■	$D_{2d} \times \mathcal{T}$
▲	$C_{3v} \times \mathcal{T}$
□	$S_4 \wedge \mathcal{T}_2$
△	$C_3 \wedge \mathcal{T}_s$
•	$C_{2v} \times \mathcal{T}$

Figure A2. Orbits of the T_d and $T_d \times \mathcal{T}$ group action on $\boldsymbol{CP}^2$ according to [20]. Coloured areas represent the three C_2-invariant and the six C_s-invariant spheres.

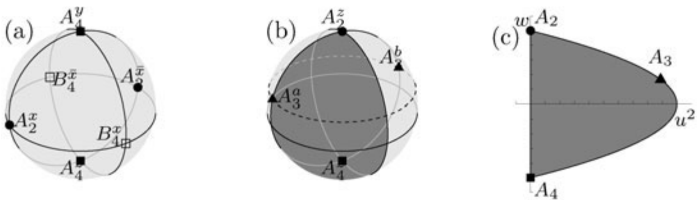


Figure A3. The C_2^x-invariant sphere (a) and the C_s^{ab}-invariant sphere (b) in the ambient space $\boldsymbol{R}^3$ with coordinates (u, v, w) adapted for each case (see text). Solid lines represent the intersections of the spheres with the planes $\{u = 0\}$, $\{v = 0\}$, and $\{w = 0\}$. In (b) the dashed line represents the intersection of the sphere with the plane $\{w = 1/3\}$. (c) Orbit space of the $C_{2v} = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ action on the C_s sphere, which is used as a chart in figure 3.

Appendix A.3. Action of $T_d \times \mathcal{T}$ on the projections of nonlinear normal modes in the configuration space $\boldsymbol{R}^3$

Like in the two-dimensional Hénon–Heiles system, it is quite convenient to represent the nonlinear normal modes of the three-dimensional system with Hamiltonian (2) by their projections in the configuration space $\boldsymbol{R}^3_{x,y,z}$ shown in figure 1. The qualitative 'shape' of each projection can be derived from the symmetry properties (isotropy group) of the mode using a set of simple principles, which we formulate below as lemmas.

Lemma A.2. *Projections $\Gamma \subset \boldsymbol{R}^3_{x,y,z}$ of periodic orbits of the system with Hamiltonian H in (2) can be of two types: (i) closed curves; (ii) curved line segments (degenerate closed curves), which begin and end orthogonally at the equipotential surface, the boundary of the projection of the constant level set of H in $\boldsymbol{R}^3_{x,y,z}$.*

Lemma A.3. *The action of an element $g \in T_d$ on projection Γ is found straightforwardly from the action of g on each point $m \in \Gamma \subset \boldsymbol{R}^3$.*

Lemma A.4. *In order to study the action of the time reversal operation $\mathcal{T}$ on closed curve projections Γ, we should consider the latter as directed closed curves, or loops. The two periodic orbits which project into the same closed curve Γ correspond to two loops Γ_+ and Γ_- with different directions. The $\mathcal{T}$ operation changes direction, i.e. $\mathcal{T} : \Gamma_+ \leftrightarrow \Gamma_-$. A segment projection represents one $\mathcal{T}$-invariant periodic orbit.*

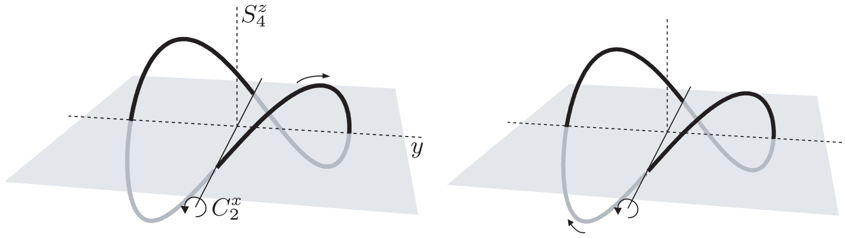


Figure A4. Two B_4^z nonlinear normal modes related by the $\mathcal{T}$ and C_2 operations. Compared to figure 1 the z-axis (vertical) scale is zoomed.

Lemma A.5. *Let* Γ *be an image of a periodic orbit defined according to lemma A.4, and* $G \subset T_d \times \mathcal{T}$ *be its isotropy group. Then each operation* $g \in G$ *maps* Γ *into itself as a whole, but points* $m \in \Gamma$ *are not necessarily fixed points of* g*. On the other hand, if* $g \notin G$ *(but* $g \in T_d \times \mathcal{T}$*), then* g *defines a 1 : 1 map* $\Gamma \to \Gamma'$ *where* Γ' *is an image of another periodic orbit in the same group orbit.*

It follows from lemma A.4 that the $\mathcal{T}$-invariant modes A_4, A_3, and A_2 project into segments (degenerate loops). Furthermore, action of the D_{2d} stabilizers on $\boldsymbol{R}^3$ is such that the A_4 modes must project onto the symmetry axes (Ox, Oy, Oz), e.g. A_4^z is represented by a segment of axis Oz (see figures 1 and A1). Similarly, the A_3 modes project onto the C_3 axes (Oa, Ob, Oc, Od). The spatial isotropy group of the A_2 modes is the group C_{2v}, whose C_2 axis is one of the (Ox, Oy, Oz). These modes project into curved line segments lying in the symmetry planes of the C_{2v} group. For example, the images of the periodic orbits A_2^z and $A_2^{\bar{z}}$ lie in the planes aOb (the plane $x = y$) and cOd (the plane $x = -y$), respectively, near the intersections of these planes with the horizontal plane xOy. The images do not intersect: A_2^z passes above the xOy plane while $A_2^{\bar{z}}$ lies below it (see figure 1).

On the other hand, the modes B_3 and B_4 are not $\mathcal{T}$-invariant. They project into closed curves in figure 1. According to lemma A.4, each such closed curve accommodates two orbits. As an instructive example, consider the two B_4^z modes in figure A4. In accordance with the spatial symmetry of these orbits S_4^z, their projection resembles a wobbled square whose two pairs of opposing smoothed vertices are lifted and lowered out of the plane xOy. It is easy to see from figure A4 that both operations C_2^x and T preserve this projection geometrically but change the direction of the mode, so that $B_{4+}^z \leftrightarrow B_{4-}^z$. At the same time, the modes are invariant with regard to the combination $T_2 = C_2 \circ T$ where $C_2 = C_2^x$ or C_2^y.

Appendix B. Local properties of RE

Appendix B.1. Morse inequalities and Euler characteristic

A function f defined on a manifold M is called a Morse function if all its stationary points $m \in M$ are nondegenerate, i.e. the determinant of the Hessian at m is not zero, $\det D^2 f(m) \neq 0$. The Morse index j of a nondegenerate stationary point m of f is defined as the number of negative eigenvalues of $D^2 f(m)$. Stationary points of a Morse function f must obey certain relations, called Morse inequalities, that are expressed in terms of the Betti numbers of M. The $\dim M + 1$ Betti numbers b_j, $j = 0, \ldots, \dim M$, are non-negative integers that depend only on topological properties of M. More precisely, b_j is the dimension of the jth homology group of M with integer coefficients. These numbers and the Euler characteristic

$B_{\dim M} = \sum_{j=0}^{\dim M}(-1)^j b_j$ for the spaces encountered in our work are given below.

Manifold M	dim M	Betti numbers	$B_{\dim M}$
$\boldsymbol{CP}^2$	4	$b_0 = 1, b_1 = 0, b_2 = 1, b_3 = 0, b_4 = 1$	3
$\boldsymbol{CP}^1 \sim \boldsymbol{S}^2$	2	$b_0 = 1, b_1 = 0, b_2 = 1$	2
$\boldsymbol{S}^1$	1	$b_0 = 1, b_1 = 1$	0

If c_j is the number of stationary points of f with Morse index j, and

$$C_j = c_j - C_{j-1} \quad \text{for } j = 1, \ldots, \dim M \quad \text{and} \quad C_0 = c_0,$$
$$B_j = b_j - B_{j-1} \quad \text{for } j = 1, \ldots, \dim M \quad \text{and} \quad B_0 = b_0,$$

then the Morse inequalities hold, namely

$$C_j \geqslant B_j \qquad \text{for } j = 0, \ldots, n-1 \quad \text{and} \quad C_{\dim M} = B_{\dim M}. \tag{B.1}$$

In the case of $\boldsymbol{CP}^2$ the inequalities (B.1) become

$$\begin{aligned} &c_0 \geqslant 1, \qquad c_1 - c_0 \geqslant -1, \qquad c_2 - c_1 + c_0 \geqslant 2, \\ &c_3 - c_2 + c_1 - c_0 \geqslant -2, \qquad c_4 - c_3 + c_2 - c_1 + c_0 = 3. \end{aligned} \tag{B.2}$$

Remark B.1. The minimal number of stationary points of a Morse function h on $\boldsymbol{CP}^2$ in the absence of symmetries is three. When h has just three stationary points, Morse inequalities (B.2) become equalities and h is called a perfect Morse function.

Appendix B.2. Linear stability types

Consider a Hamiltonian $H : \boldsymbol{R}^4 \to \boldsymbol{R}$ of a k degrees of freedom (k-DOF) system which has a stationary point (equilibrium) m. The spectral stability of point m is determined by the eigenvalues of the Hamiltonian matrix $\mathcal{H} \in \mathrm{sp}(2k, \boldsymbol{R})$ of the linearized equations of motion at m. Recall that if $\lambda \in \boldsymbol{C}$ is one of the eigenvalues of the Hamiltonian matrix $\mathcal{H}$, then $-\lambda$ and $\bar{\lambda}$ are also eigenvalues of $\mathcal{H}$. In a 1-DOF system there are two cases, elliptic (E) or stable with two imaginary eigenvalues $\pm \mathrm{i}\omega$ and hyperbolic (H) or unstable with two real eigenvalues $\pm\lambda$. A generic equilibrium of a 2-DOF Hamiltonian system can be of one of the four possible linear stability types.

- EE Elliptic–elliptic when eigenvalues $\pm \mathrm{i}\omega_1, \pm \mathrm{i}\omega_2$, where $\omega_{1,2} \in \boldsymbol{R}\backslash\{0\}$, lie on the imaginary axis. The quantities ω_1 and ω_2 are called frequencies.
- EH Elliptic–hyperbolic when two of the eigenvalues are real and two are imaginary.
- HH Hyperbolic–hyperbolic when all eigenvalues are real.
- CH Complex–hyperbolic when $\lambda \neq 0$ is neither real nor imaginary and the eigenvalues form a quadruplet $\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}$.

There are also two cases that become generic in the presence of symmetry.

- 2E Degenerate elliptic when two pairs of equal eigenvalues ($\pm \mathrm{i}\omega$) and ($\pm \mathrm{i}\omega$) with $\omega \in \boldsymbol{R}\backslash\{0\}$ lie on the imaginary axis.
- 2H Degenerate hyperbolic when two pairs of equal eigenvalues ($\pm\lambda$) and ($\pm\lambda$) with $\lambda \in \boldsymbol{R}\backslash\{0\}$ lie on the real axis.

All these case are illustrated in figure B1. In a 1-DOF system the correspondence between the two stability types and the Morse index is simple: a stable point can be of index 0 or 2, while an unstable point has Morse index 1.

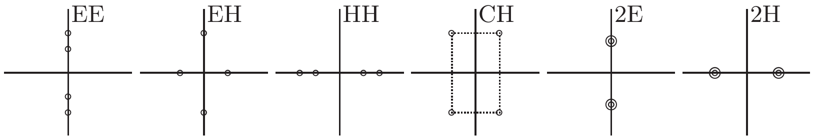


Figure B1. Types of linear stability of an equilibrium of a 2-DOF Hamiltonian system.

Lemma B.1. *In* 2*-DOF Hamiltonian systems we have the following relation of possible linear stability types and Morse indices.*

Morse index	*0 or 4*	*1 or 3*	*2*
Stability type	*EE*	*EH*	*EE, HH, CH*

Proof. Consider normal forms of quadratic Hamiltonians for 2-DOF systems [37] and compute possible Morse indices for each one of them. □

Appendix B.3. Linearization near stationary points on $\boldsymbol{CP}^2$

We explain how to compute linearized equations of motion in a local symplectic chart $T^*\boldsymbol{R}^2(\xi)$ at the stationary point $\xi \in \boldsymbol{CP}^2(n)$ in order to determine the linear stability of ξ. Note that even though different local charts can be chosen, the linear stability type of ξ or the Morse index of ξ do not depend on the choice of coordinates.

We denote invariants in (5) as π_j, $j = 1, \ldots, 9$, and use four of these invariants as coordinates α_k, $k = 1, \ldots, 4$, in $T^*\boldsymbol{R}^2(\xi)$. If the α are chosen correctly then it should be possible to express the remaining five invariants β_l, $l = 1, \ldots, 5$ near ξ in terms of the α and n using relations Σ_i, $i = 0, \ldots, 9$ in (6). We ensure this requirement is met by means of the implicit function theorem. We take the 9×10 Jacobian matrix $\partial\Sigma_i/\partial\pi_j$ evaluated at ξ, where $i = 0, \ldots, 9$ and $j = 1, \ldots, 9$, and select five rows and five columns of this matrix so that the determinant of the resulting 5×5 submatrix is non-zero. Invariants $\beta_1, \ldots, \beta_5$ correspond to the selected columns, and relations $\tilde{\Sigma}_m(\beta; \alpha, n)$, $m = 1, \ldots, 5$, correspond to the selected rows; note that $\Sigma_0 \in \{\tilde{\Sigma}\}$. We can now solve the relations $\{\tilde{\Sigma}_m\}$ for β_l in terms of α_k and n. If the choice of $\{\beta\}$ and $\{\tilde{\Sigma}\}$ is not unique, we aim at such choice that yields the simplest possible expressions $\beta_l(\alpha, n)$.

In order to study the system near ξ, we introduce the displacements $\delta\alpha_k$ of α_k from their values $\alpha_k(\xi)$, i.e. $\delta\alpha_k = \alpha_k - \alpha_k(\xi)$. The local coordinates $\delta\alpha = (\delta\alpha_1, \ldots, \delta\alpha_4)$ are not necessarily canonical coordinates in $T^*\boldsymbol{R}^2(\xi)$. However, it is always possible to find a linear transformation

$$(\chi, \psi) = (\chi_1, \chi_2, \psi_1, \psi_2) = B \cdot \delta\alpha$$

such that the variables (χ, ψ) are canonical at $(\chi, \psi) = 0$. The Poisson brackets of these variables evaluated near $(\chi, \psi) = 0$ are $\{\chi_1, \chi_2\} = \{\psi_1, \psi_2\} = \{\chi_1, \psi_2\} = \{\chi_2, \psi_1\} = \mathcal{O}(\chi, \psi)$ and $\{\chi_1, \psi_1\} = \{\chi_2, \psi_2\} = 1 + \mathcal{O}(\chi, \psi)$.

Appendix C. Classification of the two-dimensional Hénon–Heiles systems

Consider the two-dimensional Hénon–Heiles system with Hamiltonian (1). The finite symmetry group of this system is $D_3 \times \mathcal{T}$, where D_3 is the dihedral group of transformations of the configuration 2-plane of (1) with coordinates (x, y). Near the linearization limit $\epsilon \to 0$ we can normalize (1) with regard to the Hamiltonian

$$H_0 = \tfrac{1}{2}(\bar{z}_x z_x + \bar{z}_y z_y)$$

of the 1 : 1 resonant harmonic oscillator. The reduced system has the Poisson algebra $so(3)$ generated by the three quadratic invariants of the respective $\boldsymbol{S}^1$ action

$$\boldsymbol{j} = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} z_x\bar{z}_y + z_y\bar{z}_x \\ iz_x\bar{z}_y - iz_y\bar{z}_x \\ z_x\bar{z}_x - z_y\bar{z}_y \end{pmatrix}, \tag{C.1}$$

with Casimir $H_0 = 2j$. For all $j > 0$ the reduced phase space is the 2-sphere $\boldsymbol{S}^2_j \sim \boldsymbol{CP}^1$ of radius j defined by the equation $j_1^2 + j_2^2 + j_3^2 = j^2$ in the ambient space $\boldsymbol{R}^3$ with coordinates (j_1, j_2, j_3) (see [38, 39]).

Symmetrization of (C.1) with respect to the $D_3 \times \mathcal{T}$ action is described by the generating function

$$g(\lambda) = \frac{1}{(1-\lambda^2)(1-\lambda^3)},$$

where λ stands for any angular momentum component in (j_1, j_2, j_3). The ring of all $\boldsymbol{S}^1 \times D_3 \times \mathcal{T}$ polynomials is generated freely by j and two other invariants, which can be chosen as

$$\mu = j^2 - (j_1^2 + j_3^2) = j_2^2, \qquad \xi = \tfrac{1}{2} j_3(3j_1^2 - j_3^2). \tag{C.2}$$

It follows that the normal form of (1) can be expressed as

$$\tilde{H}_\epsilon = 2j + \epsilon^2(c_m\mu + c_0 j^2) + \epsilon^4(c_3\xi + c_{0m}\mu j + c_{00} j^3) + \mathcal{O}(\epsilon^5) \tag{C.3}$$

with constants $c_m, c_0, c_3, c_{0m}, c_{00}$. Note that μ is $SO(2)$ symmetric and we must go to order ϵ^4 in order to reproduce correctly the discrete symmetry of the two-dimensional Hénon–Heiles system. Rescaling (C.3) by ϵ^2 and dropping constants and terms that depend only on j, we obtain the principal terms of the reduced Hamiltonian

$$\hat{H}_\epsilon : \boldsymbol{S}^2_j \to \boldsymbol{R}, \qquad \hat{H}_\epsilon = K_m\,\mu + \epsilon^2 K_3 \xi, \tag{C.4a}$$

where K_m and K_3 are constants of the same order of magnitude which can always be expressed using $(c_m, c_0, c_3, c_{0m}, c_{00})$ and parameters (j, ϵ). In particular, K_m is a linear function of j. In the simplest case of (1) with a 'classic' Hénon–Heiles cubic potential scaled by ϵ these constants are fixed. Since (C.4a) is not a homogeneous polynomial in $(z, \bar{z})$, the analysis for $j > 0$ is simplified if we parameterize (C.4a) after normalizing μ and ξ by j so that

$$\hat{H} = K_m\, j^2 \frac{\mu}{j^2} + \epsilon^2 K_3\, j^3 \frac{\xi}{j^3} = \cos\alpha \frac{\mu}{j^2} + \sin\alpha \frac{\xi}{j^3}, \qquad \tan\alpha = \epsilon^2 j \frac{K_3}{K_m}. \tag{C.4b}$$

(In other words, we project on a sphere of radius 1.) The obvious important difference of (C.4b) with the three-dimensional Hénon–Heiles system studied in the main body of this paper is that here α depends on the ratio of coefficients in front of the terms of different orders, and is, therefore, determined by the perturbation smallness parameter ϵ and the value of the dynamical parameter j. A direct normal form computation for (1) gives

$$\alpha = \tan^{-1}\left[\frac{16\epsilon^2 j}{6 - \epsilon^2 j}\right].$$

At first sight this suggests that even though we have no external parameters in the system, we can move between different members of the family (C.4b) by increasing j. In reality, however, $\epsilon^2 j$ should be small. In fact, we can see from (C.3) that values larger than $\epsilon^2 j \approx \frac{1}{12}$ give energies above the saddle point energy $(6\epsilon^2)^{-1}$ of the potential in (1). This means that physical values of α are $0 \leqslant \alpha \lesssim 0.07\pi$.

Analysis of RE of the two-dimensional Hénon–Heiles systems is quite similar to the one we conducted in this paper. Of course, this analysis is simpler since it deals with an action

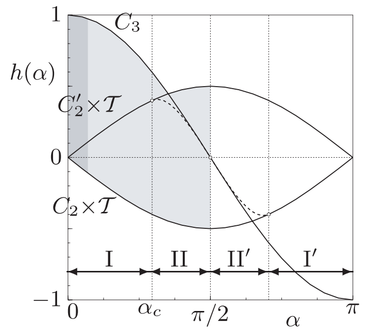


Figure C1. Energy h of the stationary points of the reduced Hamiltonian $\hat{H}$ in (C.4*b*) as a function of the parameter α. The dashed curve marks the energy of additional stationary points with symmetry C_2; the darkest shadowed area represents reduced energies of physically possible two-dimensional Hénon–Heiles systems with Hamiltonian (1); $\alpha_c = \tan^{-1}\frac{4}{3}$.

of a smaller group $D_3 \times \mathcal{T}$ on a smaller phase space $\boldsymbol{S}_j^2$. Results are well-known [3, 5, 6]. At small α (C.4*b*) is a $D_3 \times \mathcal{T}$-invariant function on $\boldsymbol{S}_j^2$ of the simplest kind with eight stationary points in three different critical orbits characterized below.

Traditional notation	Conjugacy class of stabilizers	Values of μ	ξ	Linear stability
$\Pi_{1,2,3}$	$C_2 \times \mathcal{T}$	0	$j^3/2$	H (+−)
$\Pi_{3,4,5}$	$C_2' \times \mathcal{T}$	0	$-j^3/2$	E (++)
$\Pi_{7,8}$	C_3	j^2	0	E (−−)

These points correspond to the eight nonlinear normal modes of the two-dimensional Hénon–Heiles system with Hamiltonian (1). All such systems are of this simplest type. Extending the analysis of (C.4*b*) to non-physical values of α (see figure C1) we find that on the interval $(\alpha_c,\ \pi-\alpha_c)$ this function has six extra stationary points with stabilizer C_2. As we increase α, these new points appear pairwise in the 'pitchfork' bifurcation with broken $\mathcal{T}$ symmetry of the three points $\Pi_{3,4,5}$ which happens when $\alpha = \alpha_c = \tan^{-1}\frac{4}{3}$. Then, the new points participate in a threefold 'touch-and-go' bifurcation of $\Pi_{7,8}$ at $\alpha = \frac{\pi}{2}$, and disappear in a pitchfork bifurcation of $\Pi_{1,2,3}$ at $\alpha = \pi - \alpha_c$. There are, therefore, four qualitatively different classes of reduced systems with Hamiltonian (C.4*b*). These classes are shown in figure C1; only class I can be realized in systems with Hamiltonian (1).

References

[1] Hénon M and Heiles C 1964 The applicability of the third integral of motion: some numerical experiments *Astron. J.* **69** 69–73

[2] Rod D L and Churchill R C 1985 A guide to the Hénon-Heiles Hamiltonian *Singularities and Dynamical Systems* ed S N Pnevmatikos (New York: Elsevier) pp 385–95

[3] Sadovskií D A 2001 Normal forms, geometry, and dynamics of atomic and molecular systems with symmetry *Symmetry and Perturbation Theory: Proc. International Conf. SPT2001* ed D Bambusi *et al* (Singapore: World Scientific)

[4] Fulton N, Tennyson J, Sadovskií D A and Zhilinskií B I 1993 Nonlinear normal modes and localized vibrations of H_3^+ and D_3^+ *J. Chem. Phys.* **99** 906–18

[5] Churchill R C, Kummer M and Rod D L 1983 On averaging, reduction and symmetry in Hamiltonian systems *J. Diff. Eqns* **49** 359–414

[6] Montaldi J, Roberts M and Stewart I 1988 Periodic solutions near equilibria of symmetric Hamiltonian systems *Phil. Trans. R. Soc. Ser.* A **325** 237–93

[7] Montaldi J, Roberts M and Stewart I 1990 Existence of nonlinear normal modes of symmetric Hamiltonian systems *Nonlinearity* **3** 695–730

[8] Landau L D and Lifshitz E M 1958 *Quantum Mechanics* (Oxford: Pergamon)

[9] Hammermesh M 1962 *Group Theory and its Application to Physical Problems* (Oxford: Pergamon)

[10] Efstathiou K, Sadovskií D A and Cushman R H 2003 Linear Hamiltonian Hopf bifurcation for point group invariant perturbations of the 1:1:1 resonance *Proc. R. Soc.* A at press

[11] Efstathiou K, Sadovskií D A and Zhilinskií B I 2003 Analysis of rotation-vibration relative equilibria on the example of a tetrahedral four atom molecule *SIAM J. Appl. Dyn. Syst. (SIADS)* submitted

[12] $\widehat{\text{Z}}$hilinskií B I 1989 *Teoriy͡a slo$\widehat{\text{zh}}$nyk͡h molekuly͡arnyk͡h spektrov* (Moscow: Moscow University Press) (English title: *Theory of Complex Molecular Spectra*) (in Russian)

[13] Michel L and Zhilinskií B I 2001 Symmetry, invariants, topology. Basic tools *Phys. Rep.* **341** 11–84

[14] Weinstein A 1973 Normal modes for nonlinear hamiltonian systems *Invent. Math.* **20** 47–57

[15] Sadovskií D A and Zhilinskií B I 1993 Group theoretical and topological analysis of localized vibration-rotation states *Phys. Rev.* A **47** 2653–71

[16] Ferrer S, Lara M, Palacián J, San Juan J F, Viartola A and Yanguas P 1998 The Hénon and Heiles problem in three dimensions. I. Periodic orbits near the origin *Int. J. Bifurc. Chaos* **8** 1199–213

[17] Ferrer S, Lara M, Palacián J, San Juan J F, Viartola A and Yanguas P 1998 The Hénon and Heiles problem in three dimensions. II. Relative equilibria and bifurcations in the reduced system *Int. J. Bifurc. Chaos* **8** 1215–29

[18] Yanguas P 1998 Integrability, normalization and symmetries of Hamiltonian systems in 1–1–1 resonance *PhD Thesis* Universidad Pública de Navarra

[19] Cushman R, Ferrer R and Hanßmann H 1999 Singular reduction of axially symmetric perturbations of the isotropic harmonic oscillator *Nonlinearity* **12** 389–410

[20] Zhilinskií B I 1989 Qualitative analysis of vibrational polyads: N mode case *Chem. Phys.* **137** 1–13

[21] Mumford D 1976 Algebraic geometry: I. Complex projective varieties *Grundlehren* vol 221 (Berlin: Springer)

[22] Ferrer S, Palacian J and Yanguas P 2000 Hamiltonian oscillators in 1–1–1 resonance: normalization and integrability *J. Nonlinear Sci.* **10** 145–74

[23] Schwarz G 1975 Smooth functions invariant under the action of a compact Lie group *Topology* **14** 63–8

[24] Arnol'd Vl I 1989 Mathematical methods of classical mechanics *Graduated Texts in Mathematics* 2nd edn, vol 60 (New York: Springer) (Transl. K Vogtmann and A Weinstein, original Russian edition *Matematic͡heskie Metody Klassic͡heskoí Mek͡haniki* (Moscow: Nauka) 1974)

[25] Duistermaat J J 1983 Bifurcations of periodic solutions near equilibrium points of Hamiltonian systems *Preprint* 300, Department of Mathematics, University of Utrecht, August 1983. 6 lectures given at the CIME course on *Bifurcation Theory and Applications* (Montecatini Terme, Italy)

[26] Michel L 1971 Points critiques des fonctions invariantes sur une G-varieté *C. R. Acad. Sci. Paris* **272** 433–6

[27] Milnor J W 1969 *Morse Theory* (Princetion, NJ: Princetion University Press)

[28] Morse M and Cairns S S 1969 *Critical Point Theory and Differential Topology* (New York: Academic)

[29] Hecht K T 1960 Vibration-rotation energies of tetrahedral XY_4 molecules: I. Theory of spherical top molecules *J. Mol. Spectrosc.* **5** 355–89

[30] Patterson C W 1985 Quantum and semiclassical description of a triply degenerate anharmonic oscillator *J. Chem. Phys.* **83** 4618–32

[31] Gröbner W 1960 *Die Lie-Reihen und ihre Anwendungen* (Berlin: Deutscher Verlag der Wissenschaftern)

[32] Deprit A 1969 Canonical transformations depending on a small parameter *Celest. Mech.* **1** 12–30

[33] Meyer K R and Hall G R 1991 Introduction to Hamiltonian dynamical systems and the N-body problem *Applied Mathematical Sciences* vol 69 (New York: Springer)

[34] Golubitsky M, Stewart I and Schaeffer D 1988 Singularities and groups in bifurcation theory *Applications of Mathematical Sciences* vol 69 (Berlin: Springer)

[35] van der Meer J C 1985 *The Hamiltonian Hopf Bifurcation* (*Lecture Notes in Mathematics* vol 1160) (Berlin: Springer)

[36] Hanßmann H and van der Meer J C 2002 On the Hamiltonian Hopf bifurcations in the 3D Hénon-Heiles family *J. Dyn. Diff. Eqns* **14** 657–95

[37] Williamson J 1936 On an algrbraic problem concerning the normal forms of linear dynamical systems *Am. J. Math.* **58** 141–63

[38] Cushman R H 1982 Geometry of the bifurcations of the Hénon-Heiles family *Proc. R. Soc.* A **382** 361–71

[39] Cushman R and Rod D L 1982 Reduction of the semisimple 1 : 1 resonance *Physica* D **6** 105–12