IOP PUBLISHING NONLINEARITY

Nonlinearity **21** (2008) 1385–1410 doi:10.1088/0951-7715/21/6/014

Heteroclinic cycles between unstable attractors

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Received 23 October 2007
Published 15 May 2008
Online at stacks.iop.org/Non/21/1385

Recommended by J A Glazier

Abstract
We consider networks of pulse coupled linear oscillators with non-zero delay where the coupling between the oscillators is given by the Mirollo–Strogatz function. We prove the existence of heteroclinic cycles between unstable attractors for a network of four oscillators and for an open set of parameter values.

Mathematics Subject Classification: 34D45, 37L30, 92B99

1. Introduction

In this study, we analyse heteroclinic cycles that occur in *global* networks of pulse-coupled oscillators. By definition, a *heteroclinic cycle* is a collection of orbits that connect sequences of saddle equilibria in a topological circle [1]. Robust heteroclinic cycles constitute a generic feature of certain dynamical systems with symmetry [1, 2]. They have been found to be relevant in a number of physical phenomena that include rotating convection [3], population dynamics [4], climate models [5] and coupled oscillator networks [6, 7].

The pulse coupled oscillator networks that we study in this work are used, among other things, to model the synchronization in the flashing patterns of fireflies [8, 9] and in biological neuron networks [6,10–12]. The primary motivation, however, lies in several studies [13–18] which propose that unstable attractors connected by heteroclinic cycles could be used to model information processing in neural systems.

In the model that we study, neurons are represented by linear oscillators and their *membrane potential* is related to the phase of the oscillator through a Mirollo–Strogatz function [8]. When the membrane potential reaches a particular threshold, the neuron *fires*, and the potential is reset to a lower value. As a consequence of the firing, the membrane potential (i.e. the phase) of all the other neurons (oscillators) is increased by a constant amount ε. In the original Mirollo–Strogatz model [8], this increase occurs simultaneously with the

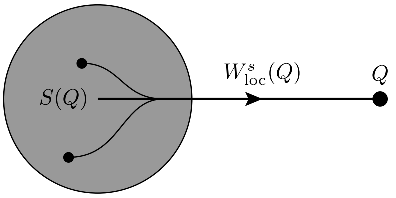

Figure 1. A schematic picture of an unstable attractor. Q is a saddle point whose stable set $W^s(Q)$ contains an open set S. Initial states from S collapse onto the local stable set $W^s_{\mathrm{loc}}(Q)$ and converge to Q. Since Q is a saddle point, almost all nearby initial states move away from Q.

firing. Here, following later investigations (for example [9, 12]), we assume that there is a *time delay* τ between the firing of an oscillator and the time the other oscillators receive the pulse.

In networks consisting of three or more such oscillators, previous works [6, 10–12, 19] have established the existence of unstable attractors for an open set of parameter values. In addition, numerical studies [6, 10, 11] show that for certain values of parameters and for a sufficiently large number of oscillators the network has heteroclinic cycles between unstable attractors. The main aim of this paper is to prove the existence of such heteroclinic cycles. In particular, we have the following theorem.

Theorem 1. *Global networks of four pulse coupled oscillators with delay where the coupling is given by the Mirollo–Strogatz function have heteroclinic cycles between two unstable attractors for an open set of parameters.*

A more detailed version of this statement is given in section 3 (theorem 2). In order to prove theorem 1 we use the metric in the infinite dimensional state space, introduced in [19], that allows us to study instability in a rigorous way.

The unstable attractors we study in this paper are saddle periodic orbits or saddle fixed points of a suitably defined Poincaré map. This means that they have local stable and unstable manifolds that are both non-zero dimensional. At the same time, there exists an open set of points in the state space that converges to the attractor. The situation is sketched in figure 1. The attractor Q is a saddle point and its stable set $W^s(Q)$ contains an open set $S(Q)$. Initial states from $S(Q)$ collapse onto the local stable set $W^s_{\mathrm{loc}}(Q)$ and converge to Q. Since Q is a saddle point there is a neighbourhood U of Q such that all initial states in $U \setminus W^s_{\mathrm{loc}}(Q)$ leave U after some time.

The unstable attractors in a system can be connected by heteroclinic cycles. Consider the case that a system has N unstable attractors $Q_1, \ldots, Q_N$ such that Q_j lies in the interior of the closure of the basin of Q_{j+1}, and Q_N lies in the interior of the closure of the basin of Q_1. The dynamics in this case is not very interesting because any initial state in the neighbourhood of Q_1 will end up to Q_2 and stay there forever. But if we add small noise to the system, then the system can leave Q_2 to reach Q_3 and so on. In this way the existence of heteroclinic connections together with some external noise can make the system move from one state to another.

The paper is organized as follows. In section 2 we discuss the setting of the problem by providing a description of the system and defining its state space. In section 3, we prove analytically the existence of heteroclinic cycles between unstable attractors in a network of four oscillators. Finally in section 4, we compare the theoretical results that we obtained with a numerical study of the system and we present our conclusions.

2. Definition of the dynamics

In this section we follow closely [12] and in particular [19]. We repeat only the definitions that are necessary for this paper. For more details we refer to [12, 19].

The system studied in this paper is a delay system [20]. The state space of such systems is an appropriate space $\mathcal{P}^n_\tau$ of functions (see definition 1) defined on the interval $(-\tau, 0]$, where $\tau > 0$ is the delay of the system, and taking values in an n-dimensional manifold N. The state space thus is infinite dimensional. In our case, points in N represent the phases of the n coupled oscillators, which implies that $N = \mathbb{T}^n$, the n-dimensional torus.

For a given $\phi \in \mathcal{P}^n_\tau$ and for each $t \in (-\tau, 0]$, $\phi(t) \in N$ represents the phases of the oscillators at time t. Using the dynamics of the system, ϕ can be *extended* to a unique function $\phi^+ : (-\tau, +\infty) \to N$, such that $\phi^+(t) = \phi(t)$ for $t \in (-\tau, 0]$ and $\phi^+(t) \in N$ represents the phases of the oscillators at any time $t \geqslant -\tau$. Then the *evolution operator* $\Phi^t : \mathcal{P}^n_\tau \to \mathcal{P}^n_\tau$ is defined by $\Phi^t(\phi)(s) = \phi^t(s) = \phi^+(t+s)$ for any $t \geqslant 0$ and $s \in (-\tau, 0]$. In other words, the evolution operator maps the initial state $\phi = \phi^0$ to the state ϕ^t of the system at time t. The latter is the restriction of ϕ^+ in $(t-\tau, t]$ shifted back to the interval $(-\tau, 0]$.

2.1. Pulse coupled oscillator networks with delay

We now specialize the above notions of the theory of delay equations to the current setting.

Definition 1 (State space, cf [12]). *The state space $\mathcal{P}^n_\tau$ of the system of n pulse coupled oscillators with delay $\tau > 0$ is the space of phase history functions*

$$\phi : (-\tau, 0] \to \mathbb{T}^n : s \mapsto \phi(s) = (\phi_1(s), \ldots, \phi_n(s)),$$

which satisfy the following conditions:

(i) Each ϕ_i is upper-semicontinuous, i.e. $\phi_i(s^+) := \lim_{t\to s^+} \phi_i(t) = \phi_i(s)$ and $\phi_i(s^-) := \lim_{t\to s^-} \phi_i(t) \leqslant \phi_i(s)$ for all $s \in (-\tau, 0]$.
(ii) Each ϕ_i is only discontinuous at a finite (or empty) set $S_i = \{s_{i,1}, \ldots, s_{i,k_i}\} \subset (-\tau, 0]$ with $k_i \in \mathbb{N}$ and $s_{i,1} > s_{i,2} > \cdots > s_{i,k_i}$.
(iii) $\mathrm{d}\phi_i(s)/\mathrm{d}s = 1$ for $s \notin S_i$.

The coupling between the n oscillators is defined using the *pulse response function*.

Definition 2 (Pulse response function, cf [12]). *A pulse response function is a map*

$$V : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto V(\theta, \varepsilon), \tag{1}$$

that satisfies the following conditions:

(i) V is smooth on $(\mathbb{T} \setminus \{0\}) \times \mathbb{R}_+$.
(ii) $\partial V(\theta, \varepsilon)/\partial\theta > 0$ on $(\mathbb{T} \setminus \{0\}) \times (\mathbb{R}_+ \setminus \{0\})$.
(iii) $\partial V(\theta, \varepsilon)/\partial\varepsilon > 0$ on $\mathbb{T} \times \mathbb{R}_+$.
(iv) $V(\theta, 0) = 0$ for all $\theta \in \mathbb{T}$.
(v) $0 < V(0, \varepsilon) < 1$ for all $\varepsilon \in (0, 1)$.
(vi) H, given by (4), satisfies

$$H_m(\theta) = H_1 \circ H_{m-1}(\theta) = \overbrace{H_1 \circ \ldots \circ H_1}^{m\text{-times}}(\theta). \tag{2}$$

Note that in the above definition $\partial V/\partial\theta > 0$, therefore V cannot be smooth everywhere on $\mathbb{T}$. This is reflected in condition (i) of the definition. The pulse response function depends on the parameter $\varepsilon \geqslant 0$, called *coupling strength*. As a shorthand notation we introduce

$$V_m(\theta) = V(\theta, m\hat{\varepsilon}), \qquad \text{for } m = 1, 2, 3, \ldots, \tag{3}$$

where $\hat{\varepsilon} = \varepsilon/(n-1)$. Given a pulse response function V we also define

$$H : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto H(\theta, \varepsilon) = \theta + V(\theta, \varepsilon) \tag{4}$$

and

$$H_m(\theta) = H(\theta, m\hat{\varepsilon}), \qquad \text{for } m = 1, 2, 3, \ldots. \tag{5}$$

Definition 3 (Dynamics, cf [12]). *A system of n pulse coupled oscillators with delay is a quadruple $\mathcal{D} = (n, V, \varepsilon, \tau)$, where V is as in definition 2, $\varepsilon \geqslant 0$ and $\tau \geqslant 0$. Given a system $\mathcal{D}$ and an initial state $\phi \in \mathcal{P}^n_\tau$, we extend ϕ to a function $\phi^+ : (-\tau, +\infty) \to \mathbb{T}^n$ using the following rules:*

(i) $\phi^+(t) = \phi(t)$ *for* $t \in (-\tau, 0]$.
(ii) $d\phi^+_i(t)/dt = 1$ *for* $t \geqslant 0$, *if* $\phi^+_j(t-\tau) \neq 0 \pmod{\mathbb{Z}}$ *for all* $j \neq i$.
(iii) $\phi^+_i(t) = \min\{1, H_m(\phi^+_i(t^-))\} \pmod{\mathbb{Z}}$, *if there are* $j_1, \ldots, j_m \neq i$ *such that* $\phi^+_{j_k}(t-\tau) = 0 \pmod{\mathbb{Z}}$ *for all* $k = 1, \ldots, m$.

The dynamics described in definition 3 can be interpreted in the following way. The phase ϕ_i of each oscillator O_i, $i = 1, \ldots, n$, increases linearly. When the phase reaches the value $1 = 0 \pmod{\mathbb{Z}}$, then the oscillator O_i *fires* and all the other oscillators O_j, $j \neq i$ receive a *pulse* after a time *delay* τ. In general, an oscillator O_j may receive m simultaneous pulses at time t if m oscillators $O_{i_1}, \ldots, O_{i_m}$ have fired simultaneously at time $t - \tau$. Then the phase of O_j is increased to $H(u_j, m\hat{\varepsilon}) = H_m(u_j)$ where $u_j = \phi^+_j(t^-)$, unless the pulse causes the oscillator to fire and then the phase becomes exactly 1.

The *evolution operator* Φ^t for $t \geqslant 0$ is then defined by

$$\Phi^t : \mathcal{P}^n_\tau \to \mathcal{P}^n_\tau : \phi \mapsto \Phi^t(\phi) = \phi^t = \phi^+|_{(t-\tau,t]} \circ T_t, \tag{6}$$

where T_t is the shift $s \mapsto s + t$ and the *positive semiorbit* of $\phi \in \mathcal{P}^n_\tau$ is given by

$$\mathcal{O}_+(\phi) = \{\Phi^t(\phi) : t \geqslant 0\}. \tag{7}$$

In [19] it was proven that the evolution operator Φ^t is well defined.

For a given system $\mathcal{D} = (n, V, \varepsilon, \tau)$, the *accessible state space* is $\mathcal{P}_\mathcal{D} = \Phi^\tau(\mathcal{P}^n_\tau)$. In other words, $\phi \in \mathcal{P}_\mathcal{D}$ if there is a state $\psi \in \mathcal{P}^n_\tau$ such that $\Phi^\tau(\psi) = \phi$, i.e. $\mathcal{P}_\mathcal{D}$ includes only those states that are dynamically accessible. From now on, we restrict our attention to $\mathcal{P}_\mathcal{D}$.

2.2. The Mirollo–Strogatz model

A pulse response function V that satisfies all the requirements of definition 2 is provided by the Mirollo–Strogatz model [8] where the pulse response function is

$$V_{\mathrm{MS}}(\theta, \varepsilon) = f^{-1}(f(\theta) + \varepsilon) - \theta, \tag{8}$$

and f is a function which is concave down ($f'' < 0$) and monotonically increasing ($f' > 0$). Moreover, $f(0) = 0$ and $f(1) = 1$. A concrete example is given by

$$f_b(\theta) = \frac{1}{b}\ln(1 + (\mathrm{e}^b - 1)\theta). \tag{9}$$

We present a sketch of the function f_b for various values of b in figure 2(a). For any given positive value of ε, the pulse response function $V_{\mathrm{MS}}(\theta, \varepsilon)$ for $f = f_b$ as in (9) is affine:

$$V_{\mathrm{MS}}(\theta, \varepsilon) = m_\varepsilon + K_\varepsilon\theta, \tag{10}$$

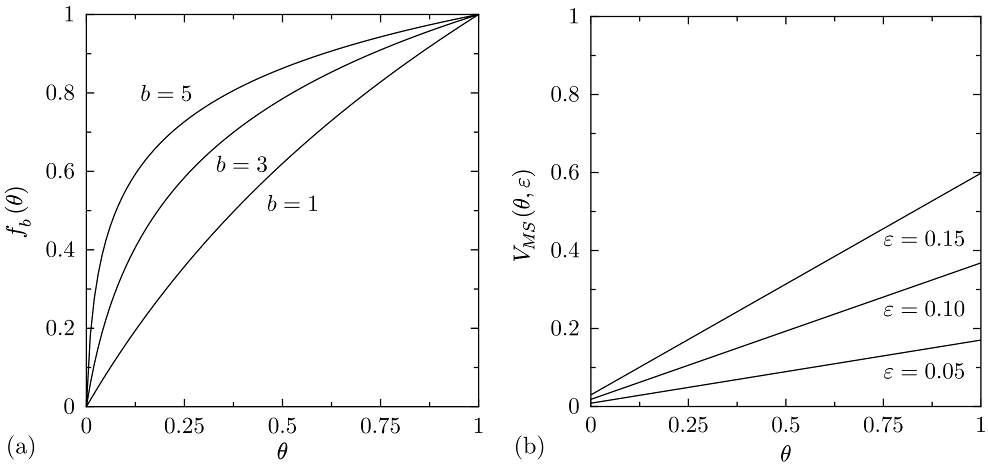

Figure 2. (*a*) Graph of f_b (9) as a function of θ for different values of b. (*b*) Graph of V_{MS} (10) as a function of θ for $f = f_b$, $b = 3$ and different values of ε.

where $m_\varepsilon = (\mathrm{e}^{b\varepsilon} - 1)/(\mathrm{e}^b - 1)$ and $K_\varepsilon = \mathrm{e}^{b\varepsilon} - 1$. The graph of V_{MS} (10) is depicted in figure 2(*b*) for different values of ε.

In the numerical computations in this paper, we use the Mirollo–Strogatz model with f_b as in (9) with fixed $b = 3$. After fixing b, the parameter space of the system is $\{(\varepsilon, \tau) : \varepsilon > 0, \tau > 0\} = \mathbb{R}^2_+$ where we recall that τ is the *delay* and ε is the *coupling strength*.

2.3. Metric

We introduce a metric d on $\mathcal{P}_\mathcal{D}$ which we later use to define a neighbourhood of states. Recall that given a phase history function $\phi \in \mathcal{P}^n_\tau$, we can define the extended phase history function ϕ^+.

We define a *lift* [21] of an extended phase history function ϕ^+ as any function $L_\phi : (-\tau, +\infty) \to \mathbb{R}^n$ such that

(i) $L_\phi(s) \pmod{\mathbb{Z}} = \phi^+(s)$ and
(ii) for any $s \in (-\tau, +\infty)$ and for $i = 1, \ldots, n$,

$$(L_\phi)_i(s) - (L_\phi)_i(s^-) = \phi^+_i(s) - \phi^+_i(s^-).$$

It follows from these properties that if $L^{(1)}_\phi$ and $L^{(2)}_\phi$ are two lifts of the same extended phase history function ϕ^+ then they differ by a constant integer vector, i.e. $L^{(1)}_\phi(s) - L^{(2)}_\phi(s) = k \in \mathbb{Z}^n$, for all $s \in (-\tau, \infty)$.

Definition 4 (Metric on $\mathcal{P}_\mathcal{D}$). *The metric $d : \mathcal{P}_\mathcal{D} \times \mathcal{P}_\mathcal{D} \to \mathbb{R}$ is given by*

$$d(\phi, \psi) = \min_{k \in \mathbb{Z}^n} \sum_{i=1}^{n} \int_{-\tau}^{\tau} |(L_\phi)_i(s) - (L_\psi)_i(s) - k_i| \, \mathrm{d}s, \tag{11}$$

where L_ϕ and L_ψ are arbitrary lifts of ϕ and ψ, respectively.

2.4. Other representations of the dynamics

It is often useful in what follows to use alternative representations of the dynamics. In this section we introduce, following [12], the *past firings* and the *event* representation.

2.4.1. The past firings representation. It follows from definition 3 that the evolution of an initial state $\phi \in \mathcal{P}_\mathcal{D}$ only depends on the values $\phi_i(0)$ and the firing sets $\Sigma_i(\phi)$ that are defined as follows:

Definition 5. *Given a phase history function $\phi \in \mathcal{P}_\mathcal{D}$, the firing sets $\Sigma_i(\phi) \subset (-\tau, 0]$, $i = 1, \ldots, n$ are the sets of solutions of the equation $\phi_i(s) = 0$ for $s \in (-\tau, 0]$. The total firing set is given by*

$$\Sigma(\phi) = \{(i, \sigma) : i = 1, \ldots, n, \sigma \in \Sigma_i(\phi)\}$$

Therefore, if we are interested only in the future evolution of the system we can consider the following equivalence relation in $\mathcal{P}_\mathcal{D}$.

Definition 6. *Two phase history functions ϕ_1, ϕ_2 in $\mathcal{P}_\mathcal{D}$ are equivalent, denoted by $\phi_1 \sim \phi_2$, if $\phi_1(0) = \phi_2(0)$ and $\Sigma(\phi_1) = \Sigma(\phi_2)$. Let $\mathbb{P}_\mathcal{D} = \mathcal{P}_\mathcal{D}/\sim$ the quotient set of equivalence classes and by $[\phi] \in \mathbb{P}_\mathcal{D}$ denote the equivalence class of $\phi \in \mathcal{P}_\mathcal{D}$.*

Points $[\phi] \in \mathbb{P}_\mathcal{D}$ are completely determined by the values of the phases $\phi_i(0)$ and the firing sets $\Sigma(\phi)$ (which may be empty). We denote the elements of $\Sigma_i(\phi)$ by $\sigma_{i,1} > \sigma_{i,2} > \cdots > \sigma_{i,k_i}$ where k_i is the cardinality of $\Sigma_i(\phi)$. Note that by definition, $\phi_i(0) \geqslant \sigma_{i,1}$, and $\phi_i(0) = 0$ if and only if $\sigma_{i,1} = 0$.

It is possible to give an equivalent description of the dynamics described by definition 3, using only the variables $\phi_i(0)$ and $\sigma_{i,j}$. For such a definition see [12]. Note also the following proposition 1.

Proposition 1. *If $\phi_1 \sim \phi_2$ then*

(i) $\Phi^t(\phi_1) \sim \Phi^t(\phi_2)$ *for* $t \geqslant 0$ *and*
(ii) $\Phi^t(\phi_1) = \Phi^t(\phi_2)$ *for* $t \geqslant \tau$.

2.4.2. Poincaré map. Given a network of n oscillators with dynamics defined by the pulse response function V, with pulse strength ε and with delay τ, we can simplify the study of the system $\mathcal{D} = (n, V, \varepsilon, \tau)$ by considering intersections of the positive semiorbits $\mathcal{O}_+(\phi)$ with the set

$$\boldsymbol{P} = \{\phi \in \mathcal{P}_\mathcal{D} : \phi_n(0) = 0\}. \tag{12}$$

The set $\boldsymbol{P}$ is called a (Poincaré) *surface of section* [22,23] and it inherits the metric d, see (11).

The evolution operator Φ, see (6), defines a map $R : \boldsymbol{P} \to \boldsymbol{P}$ in the following way. Consider any $\phi \in \boldsymbol{P}$, i.e. such that $\phi_n(0) = 0$. Since the phases of the oscillators are always increasing there is a minimum time $t(\phi)$ such that the phase of O_n becomes 0 again, i.e. such that $\Phi^{t(\phi)}(\phi)_n(0) = 0$. We define

$$R(\phi) = \Phi^{t(\phi)}(\phi). \tag{13}$$

The map R is called a *Poincaré (return) map*. Furthermore, we can define the quotient map

$$R_\sim : \boldsymbol{P}/\sim \to \boldsymbol{P}/\sim : [\phi] \mapsto [R(\phi)],$$

of the Poincaré (return) map R, where $\sim$ is the equivalence relation given by definition 6. By proposition 1 the map $R_\sim$ is well defined.

2.4.3. The event representation. Given a phase history function ϕ, the firing sets $\Sigma_i(\phi) = \{\sigma_{i,1}, \ldots, \sigma_{i,k_i}\}$ describe at which moments in the interval $(-\tau, 0]$ the oscillator O_i fires. Hence, they also describe at which instants in the interval $(0, \tau]$ the oscillators O_ℓ, for $\ell \neq i$ would receive a pulse from O_i, making ϕ_ℓ^+, $\ell \neq i$, discontinuous at $\sigma_{i,j} + \tau \in (0, \tau]$, for $j = 1, \ldots, k_i$. Also, note that if the phase of the oscillator O_i at time 0 is $\phi_i(0)$ then the oscillator will fire after time $1 - \phi_i(0)$, unless it receives a pulse before it fires. Hence, the numbers $\sigma_{i,j}$ and $\phi_i(0)$ where $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$ can completely describe the future evolution of the system.

The *event representation* is a symbolic description of the dynamics in which the state of the system is represented by a sequence of events consisting of firings and pulse receptions that would occur. Each event E in the sequence is characterized by a triplet $[K(E), O(E), T(E)]$ where $K(E)$ denotes the type of the event F or mP. The event F denotes a firing event and mP (m a natural number) stands for the simultaneous reception of m pulses. The event $K(E)$ is associated with oscillator $O(E) \in \{1, \ldots, n\}$. Finally, $T(E) \in [0, 1]$ denotes how much time is left for the event to occur. For example, the event denoted by $[F, 2, 0.4]$ signifies that the oscillator O_2 will fire after time 0.4 (and this means that its current phase is $1 - 0.4 = 0.6$), while the event denoted by $[P, 1, 0.3]$ signifies that O_1 is set to receive a pulse after time 0.3. We use the shorthand notation $[F, (i_1, \ldots, i_k), t]$ and $[mP, (i_1, \ldots, i_k), t]$ to indicate that the oscillators $O_{i_1}, \ldots, O_{i_k}$ fire or receive m pulses, respectively, after time t.

Given a particular initial state $\phi \in \mathcal{P}_\mathcal{D}$, such that its equivalence class $[\phi] \in \mathbb{P}_\mathcal{D}$ is characterized by the phases $\phi_i(0)$ and firing times $\sigma_{i,j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$, consider the space $\mathcal{A}$ of event sequences $(E_1, E_2, \ldots, E_k)$ of finite (but not fixed) length and the map

$$\mathcal{E} : \mathbb{P}_\mathcal{D} \to \mathcal{A} : [\phi] \to \mathcal{E}([\phi]), \tag{14}$$

which maps $[\phi]$ to the event sequence $\mathcal{E}([\phi])$ constructed in the following way. First, consider the set Y consisting of the following events:

(i) $[F, i, 1 - \phi_i(0)]$ for $i = 1, \ldots, n$ and
(ii) $[P, \ell, \tau + \sigma_{i,j}]$ for $\ell = 1, \ldots, n$ with $\ell \neq i$.

Then, impose time ordering on Y (i.e. order the events so that events that occur earlier appear first) and in the case that there are $m > 1$ identical events $[P, i, t]$ collect them together to $[mP, i, t]$ to obtain $\mathcal{E}([\phi])$. It follows that $\mathcal{E}$ is injective and hence the inverse map $\mathcal{E}^{-1} : \mathcal{E}(\mathbb{P}_\mathcal{D}) \subset \mathcal{A} \to \mathbb{P}_\mathcal{D}$ is well defined.

Next, define the map

$$\Phi_\mathcal{A} : \mathcal{E}(\mathbb{P}_\mathcal{D}) \to \mathcal{E}(\mathbb{P}_\mathcal{D}) \tag{15}$$

using the following algorithm:

(i) For $Z \in \mathcal{E}(\mathbb{P}_\mathcal{D})$, consider the first event $E_1 \in Z$ and let $\mathfrak{t} = T(E_1)$. If $T_1 \neq 0$ then set $T(E)$ to $T(E) - \mathfrak{t}$ for all $E \in Z$.
(ii) Take the sequence Z_0 of events $E \in Z$ with $T(E) = 0$ and define $Z_+ = Z \setminus Z_0$. For each event $E \in Z_0$ do the following:
   (a) If $K(E) = F$, then
      1. append to Z_+ the event $[F, O(E), 1]$;
      2. append to Z_+ the events $[P, \ell, \tau]$ for all $\ell \in \{1, \ldots, n\}$ with $\ell \neq O(E)$.

(b) If $K(E) = mP$, then
 1. find the (unique) event $E' \in Z_+$ with $K(E') = F$ and $O(E') = O(E)$;
 2. set $T(E')$ to $\max\{T(E') - V(1 - T(E'), m\hat{\varepsilon}), 0\}$.

(iii) Impose time ordering on Z_+ and collect together identical pulse events.

(iv) Set $\Phi_{\mathcal{A}}(Z) = Z_+$.

It follows from the definition of $\Phi_{\mathcal{A}}$ that we have proposition 2.

Proposition 2.

(i) The map $\Phi_{\mathcal{A}} : \mathbb{P}_{\mathcal{D}} \to \mathcal{E}(\mathbb{P}_{\mathcal{D}})$ *is well defined.*

(ii) $[\Phi^{\mathfrak{t}}(\phi)] = \mathcal{E}^{-1}(\Phi_{\mathcal{A}}(Z))$ *where* $Z = \mathcal{E}([\phi])$ *and* $\mathfrak{t}$ *is determined at the first step of the algorithm.*

(iii) Consider an initial state $\phi \in \mathcal{P}_{\mathcal{D}}$ *and the corresponding event sequence* $\mathcal{E}([\phi])$*. If we apply* $\Phi_{\mathcal{A}}$*, m times to* $\mathcal{E}([\phi])$ *and the time that elapses at the* j*th* ($j = 1, \ldots, m$) *application is* $\mathfrak{t}_j$ *with* $\mathfrak{t} = \sum_j \mathfrak{t}_j$*, then it is possible to reconstruct the extended phase history function* ϕ^+ *on the interval* $[0, \mathfrak{t}]$.

The last part of proposition 2 implies that if $\mathfrak{t} \geqslant \tau$ then it is possible to obtain from the sequence $\{Z, \Phi_{\mathcal{A}}(Z), \Phi^2_{\mathcal{A}}(Z), \ldots, \Phi^m_{\mathcal{A}}(Z)\}$, where $Z = \mathcal{E}([\phi])$, not only the equivalence class $[\Phi^t(\phi)]$ but also the phase history function $\Phi^t(\phi) = \phi^+|_{(t-\tau,t]} \circ T_t$ for any time $t \in [\tau, \mathfrak{t}]$.

3. Heteroclinic cycles

We begin by defining,

$$g_1(\tau) = H_2(H_1(2\tau) + \tau),$$

$$g_2(\tau) = H_1(\alpha + \tau + \beta),$$

$$g_3(\tau) = 1 - H_2(W_1 + \tau),$$

$$g_4(\tau) = H_1(W_2 + \tau),$$

where

$$\alpha = H_1(H_1(\tau) + \tau),$$

$$\beta = 1 - H_1(H_2(2\tau) + \tau),$$

$$W_1 = 1 + H_1(\tau) - H_2(H_1(\tau) + \tau) + \tau - H_1(2\tau),$$

$$W_2 = 1 + H_1(\tau) - H_2(H_1(\tau) + \tau).$$

Using the terminology introduced in section 2, we restate the main theorem of this paper (theorem 1) as follows.

Theorem 2. *Consider a system* $\mathcal{D} = (n, V, \varepsilon, \tau)$ *such that*

(i) $n = 4$,

(ii) V *is given by a Mirollo–Strogatz model (section 2.2),*

(iii) $g_1(\tau) < 1$,

(iv) $g_2(\tau) < 1$,

(v) $g_3(\tau) < \tau$,

(vi) $g_4(\tau) < 1$.

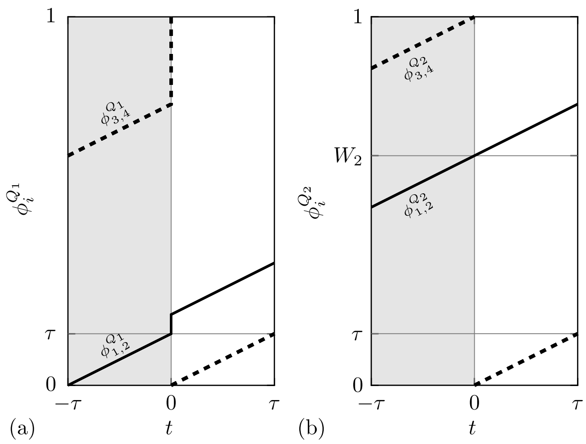

Figure 3. Graphs of the phase history functions ϕ^{Q_1} and ϕ^{Q_2}.

Then, there exist two unstable attractors $\phi^{Q_1}, \phi^{Q_2} \in \boldsymbol{P}$ (figure 3) with a heteroclinic cycle between them and conditions (i)–(vi) define an open set in the parameter space (b, ε, τ).

The fixed points ϕ^{Q_1} and ϕ^{Q_2} are defined in the following way.

Definition 7. *$\phi^{Q_1} \in \boldsymbol{P}$ for $H_2(\tau + H_1(\tau)) < 1$ is defined by*

$$\phi_i^{Q_1}(s) = \begin{cases} \tau + s, & \text{for } i = 1, 2, \\ \tau + W_2 + s, & \text{for } i = 3, 4, \end{cases} \tag{16a}$$

for $s \in (-\tau, 0)$, while for $s = 0$,

$$\phi_i^{Q_1}(0) = \begin{cases} H_1(\tau), & \text{for } i = 1, 2, \\ 0, & \text{for } i = 3, 4, \end{cases} \tag{16b}$$

where $W_2 = 1 + H_1(\tau) - H_2(\tau + H_1(\tau))$.

Definition 8. *$\phi^{Q_2} \in \boldsymbol{P}$ for $H_2(\tau + H_1(\tau)) < 1$ is defined by*

$$\phi_i^{Q_2}(s) = \begin{cases} W_2 + s, & \text{for } i = 1, 2, \\ s, & \text{for } i = 3, 4, \end{cases} \tag{17}$$

for $s \in (-\tau, 0]$, where $W_2 = 1 + H_1(\tau) - H_2(\tau + H_1(\tau))$.

The parameter region in which theorem 2 is valid for $b = 3$ is represented by the grey region in figure 4. This represents an intersection of the plane $b = 3$ and the open parameter region in the space (b, ε, τ) in which theorem 2 holds.

3.1. Proof of existence of heteroclinic cycles

In this section we prove theorem 2. We show that almost all points in an open neighbourhood of ϕ^{Q_1} are mapped in finitely many iterations to ϕ^{Q_2}. Finally, using a symmetry argument we show that almost all points in an open neighborhood of ϕ^{Q_2} are mapped in finitely many iterations to ϕ^{Q_1}, thus establishing the existence heteroclinic cycle between ϕ^{Q_1} and ϕ^{Q_2}.

Proposition 3. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ that satisfies the conditions of theorem 2, the point ϕ^{Q_1} (16a) and (16b) is a fixed point of the Poincaré map R.*

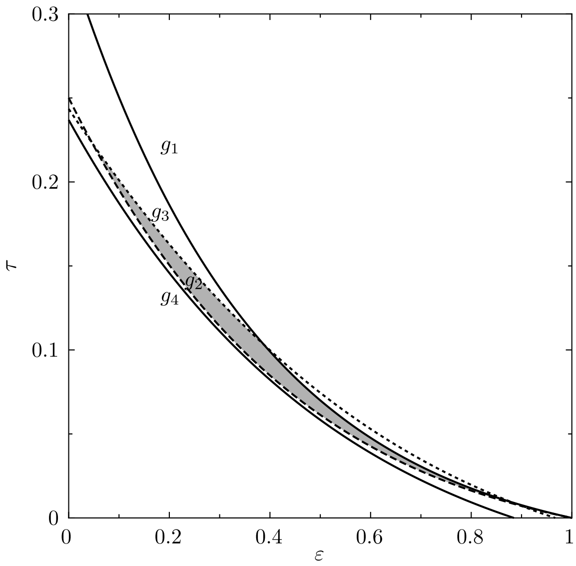

Figure 4. The region defined by the inequalities in theorem 2 is depicted by grey. In this figure $b = 3$.

Proof. ϕ^{Q_1} evolves in the event sequence representation as follows:

$$([2P, (1, 2), \tau], [P, (3, 4), \tau], [F, (1, 2), 1 - H_1(\tau)], [F, (3, 4), 1])$$
$$\xrightarrow{1} ([2P, (1, 2), 0], [P, (3, 4), 0], [F, (1, 2), 1-\tau - H_1(\tau)], [F, (3, 4), 1-\tau]).$$

Since $H_2(\tau + H_1(\tau)) < 1$, the evolution is

$$\xrightarrow{2} ([F, (1, 2), 1 - H_2(\tau + H_1(\tau))], [F, (3, 4), 1 - H_1(\tau)])$$
$$\xrightarrow{3} ([F, (1, 2), 0], [F, (3, 4), H_2(\tau + H_1(\tau)) - H_1(\tau)])$$
$$\xrightarrow{4} ([2P, (3, 4), \tau], [P, (1, 2), \tau], [F, (3, 4), H_2(\tau + H_1(\tau)) - H_1(\tau)], [F, (1, 2), 1])$$
$$\xrightarrow{5} ([2P, (3, 4), 0], [P, (1, 2), 0], [F, (3, 4), H_2(\tau + H_1(\tau)) - H_1(\tau) - \tau], [F, (1, 2), 1 - \tau])$$
$$\xrightarrow{6} ([F, (3, 4), 0], [F, (1, 2), 1 - H_1(\tau)])$$
$$\xrightarrow{7} ([2P, (1, 2), \tau], [P, (3, 4), \tau], [F, (1, 2), 1 - H_1(\tau)], [F, (3, 4), 1]).$$

Transition 6 requires $H_2(W_2 + \tau) \geqslant 1$ which will be proven in proposition 16. Since we return to the initial state, ϕ^{Q_1} is a fixed point of R. □

Proposition 4. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ that satisfies the conditions of theorem 2, the point ϕ^{Q_2} (17) is a fixed point of the Poincaré map R.*

Proof. ϕ^{Q_2} evolves in the event representation as follows:

$$([2P,(1,2),\tau],[P,(3,4),\tau],[F,(1,2),H_2(\tau+H_1(\tau))-H_1(\tau)],[F,(3,4),1])$$

$$\xrightarrow{1}([2P,(1,2),0],[P,(3,4),0],[F,(1,2),H_2(\tau+H_1(\tau))-H_1(\tau)-\tau],$$
$$[F,(3,4),1-\tau]).$$

Since $H_2(W_2+\tau)\geqslant 1$,

$$\xrightarrow{2}([F,(1,2),0],[F,(3,4),1-H_1(\tau)])$$

$$\xrightarrow{3}([2P,(3,4),\tau],[P,(1,2),\tau],[F,(3,4),1-H_1(\tau)],[F,(1,2),1])$$

$$\xrightarrow{4}([2P,(3,4),0],[P,(1,2),0],[F,(3,4),1-H_1(\tau)-\tau],[F,(1,2),1-\tau]).$$

Since $H_2(\tau+H_1(\tau))=H_1(\alpha)<g_2(\tau)<1$ (where $\alpha=H_1(H_1(\tau)+\tau)$),

$$\xrightarrow{5}([F,(3,4),1-H_2(\tau+H_1(\tau))],[F,(1,2),1-H_1(\tau)])$$

$$\xrightarrow{6}([F,(3,4),0],[F,(1,2),H_2(\tau+H_1(\tau))-H_1(\tau)]),$$

$$\xrightarrow{7}([2P,(1,2),\tau],[P,(3,4),\tau],[F,(1,2),H_2(\tau+H_1(\tau))-H_1(\tau)],$$
$$[F,(3,4),1]).$$

We note that we return to the initial state and that ϕ^{Q_2} is a fixed point of R. □

Lemma 5. *If the assumptions stated in theorem 2 hold, then there is an open neighbourhood $U\subseteq \boldsymbol{P}$ of ϕ^{Q_1} such that $U/\sim$ is one-dimensional and all states $[\phi]\in(U/\sim)\setminus\{[\phi^{Q_1}]\}$ converge to ϕ^{Q_2} in finitely many iterations of the Poincaré map R.*

Proof. According to proposition 7 there is an open neighbourhood $U\subset\boldsymbol{P}$ of ϕ^{Q_1} such that the equivalence class $[\phi]$ of each state $\phi\in U$ is characterized by the event sequence

$$([2P,(1,2),\tau],[P,(3,4),\tau],[F,1,1-v],[F,2,1-w],[F,(3,4),1]), \tag{18}$$

where v and w can be made to be arbitrarily close to $H_1(\tau)$ and moreover if $v>w$ then $A_\varepsilon w+(A_\varepsilon-1)v=(2A_\varepsilon-1)H_1(\tau)$, while if $v<w$ then $A_\varepsilon v+(A_\varepsilon-1)w=(2A_\varepsilon-1)H_1(\tau)$. This shows that $U/\sim$ is one-dimensional. Since all the oscillators are identical, the system is invariant under the permutation $1\leftrightarrow 2$, therefore it is enough to consider only the case $v>w$.

We denote by $[\phi^{v,w}]$ the equivalence class that corresponds to the event sequence (18), where now v and w take any value in $[\tau,1)$, and by Λ the set of equivalence classes $\{[\phi^{v,w}]\,:\,v\in[\tau,1),w\in[\tau,1)\}$. Note that $[\phi^{Q_2}]=[\phi^{W_2,W_2}]\in\Lambda$. Let $\tilde{\Lambda}$ be the subset of $[\tau,1)^2$ such that if $(v,w)\in\tilde{\Lambda}$ then $R([\phi^{v,w}])\in\Lambda$. Since there is a one-to-one correspondence between equivalence classes $[\phi^{v,w}]$ and pairs (v,w) we can define a map $R_\Lambda:\tilde{\Lambda}\subseteq[\tau,1)^2\to[\tau,1)^2$ given by $[\phi^{R_\Lambda(v,w)}]=R([\phi^{v,w}])$. Therefore, we can follow the evolution of an initial state $[\phi^{v,w}]$ on $[\tau,1)^2$ with coordinates v and w as long as $R^m_\Lambda(v,w)\in\tilde{\Lambda}$ for $m\in\mathbb{N}$. In the course of the proof we show that all the equivalence classes that we consider belong in $\tilde{\Lambda}$.

The next step of the proof is to divide the space $[\tau,1)^2$ into different regions for which we can solve the dynamics and show that the initial state with v and w close to $H_1(\tau)$ goes through a succession of regions until it reaches the point $Q_2=(W_2,W_2)$ that corresponds to $[\phi^{Q_2}]$.

These regions are shown in figure 5. First, define the line segment ℓ, given by $v>w$, $H_1(\tau)<v<H_1(2\tau)$, $w>\tau$ and $A_\varepsilon w+(A_\varepsilon-1)v=(2A_\varepsilon-1)H_1(\tau)$. Define also ℓ_1 as the subset of ℓ for which $H_2(v+\tau)-H_2(w+\tau)<\tau$ and ℓ_2 as the subset of ℓ for which

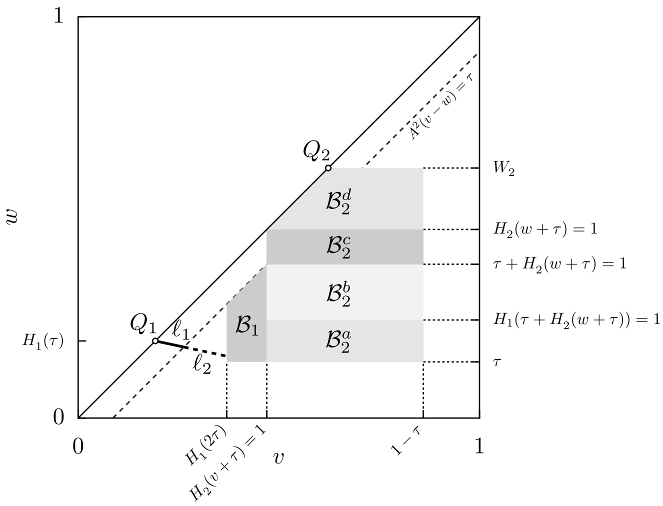

Figure 5. The regions used in the proof of lemma 5. The line segments ℓ_1 and ℓ_2 are indicated by the thick solid and dashed lines, respectively. The line segment ℓ_1 begins at Q_1 and is separated from ℓ_2 by the thin dashed line given by $H_2(v+\tau) - H_2(w+\tau) = A^2(v-w) = \tau$ which is parallel to the diagonal that joins Q_1 and Q_2. Points on ℓ_1 are mapped to ℓ_2, and next in $\mathcal{B}_1$ or $\mathcal{B}_2 = \mathcal{B}_2^a \cup \mathcal{B}_2^b \cup \mathcal{B}_2^c \cup \mathcal{B}_2^d$. Points in $\mathcal{B}_1$ are mapped to $\mathcal{B}_2$ and finally, points in $\mathcal{B}_2$ are mapped in finite iterations to Q_2.

$H_2(v+\tau) - H_2(w+\tau) \geqslant \tau$. Then, proposition 7 states that the part of the open neighbourhood U of ϕ^{Q_1} for which $v > w$ lies in ℓ_1.

Next, according to proposition 9 the point $(v, w) \in \ell_1$ is mapped in finite iterations of R_Λ to ℓ_2 and according to proposition 10, points in ℓ_2 are mapped either in the region $\mathcal{B}_1$ defined by the relations $v > w$, $w \geqslant \tau$, $v \geqslant H_1(2\tau)$, $H_2(v+\tau) < 1$ and $H_2(v+\tau) - H_2(w+\tau) \geqslant \tau$ or in the region $\mathcal{B}_2$ defined by the relations $v > w$, $v < 1-\tau$, $w > \tau$, $H_2(v+\tau) \geqslant 1$, $w < W_2$.

Denote by $\mathcal{B}_2^a$ the subset of $\mathcal{B}_2$ given by $H_1(\tau + H_2(w+\tau)) < 1$, by $\mathcal{B}_2^b$ the set given by $H_1(\tau + H_2(w+\tau)) \geqslant 1$ and $\tau + H_2(w+\tau) < 1$, by $\mathcal{B}_2^c$ the set given by $\tau + H_2(w+\tau) \geqslant 1$ and $H_2(w+\tau) \geqslant 1$, and finally by $\mathcal{B}_2^d$ the set given by $H_2(w+\tau) > 1$. Note that $\mathcal{B}_2 = \mathcal{B}_2^a \cup \mathcal{B}_2^b \cup \mathcal{B}_2^c \cup \mathcal{B}_2^d$.

Next, according to proposition 11 all points in $\mathcal{B}_1$ are mapped in $\mathcal{B}_2$. Proposition 12 states that all points in $\mathcal{B}_2^a$ are mapped in finite iterations in $\mathcal{B}_2^b \cup \mathcal{B}_2^c \cup \mathcal{B}_2^d$. Next, proposition 13 states that all points in $\mathcal{B}_2^b$ are mapped to a single point in $\mathcal{B}_2^c$. According to proposition 14 all points in $\mathcal{B}_2^c$ are mapped in $\mathcal{B}_2^d$ and finally, proposition 15 states that all points in $\mathcal{B}_2^d$ are mapped to $[\phi^{Q_2}]$.

Therefore, the initial states $\phi \in U$ near ϕ^{Q_1} are mapped in a finite number of iterations of R to $[\phi^{Q_2}]$. Since $R([\phi^{Q_2}]) = \phi^{Q_2}$ (from proposition 4) we conclude that $\phi \in U$ is mapped in a finite number of iterations of R to ϕ^{Q_2}. □

Lemma 6. *Under the assumptions of theorem 2 there is an open set $W \subseteq \boldsymbol{P}$ of initial states around ϕ^{Q_2} such that $W/\sim$ is three-dimensional and all states $[\phi] \in W/\sim$, except those in a two-dimensional subset, converge to ϕ^{Q_1} in finitely many iterations of the Poincaré map R.*

Proof. According to proposition 8 there is an open neighbourhood $W \subset \boldsymbol{P}$ of ϕ^{Q_2} such that the equivalence class $[\phi]$ of each state $\phi \in W$ is characterized by the event sequence (making the assumption that $v \geqslant w$)

$$([P,(1,2,4),\tau-u],[P,(1,2,3),\tau],[F,1,1-W_2-v],$$
$$[F,2,1-W_2-w],[F,3,1-u],[F,4,1]) \tag{19}$$

where $0 < u \ll 1$ or by

$$([F,3,-u],[P,(1,2,3),\tau],[F,1,1-v],[F,2,1-w],[F,4,1]) \tag{20}$$

where $-1 \ll u < 0$. Note that in the case $u = 0$ the event sequences (19) and (20) are essentially identical and the initial state is mapped in one iteration to ϕ^{Q_2}. Therefore, states in the neighbourhood W can be characterized by three small parameters (u, v, w) and we show that except states with $u = 0$, all other states in W are mapped in finite iterations to ϕ^{Q_1}.

Consider the surface of section $\boldsymbol{Q} = \{\phi \in \mathcal{P}_{\mathcal{D}} : \phi_2(0) = 0\}$ and the maps $T_1 : \boldsymbol{P} \to \boldsymbol{Q}$ and $T_2 : \boldsymbol{Q} \to \boldsymbol{P}$ defined so that for $\phi \in \boldsymbol{P}$, $T_1(\phi)$ is the first intersection of $\{\Phi^t(\phi)\}$ with $\boldsymbol{Q}$ and for $\psi \in \boldsymbol{Q}$, $T_2(\psi)$ is the first intersection of $\{\Phi^t(\psi)\}$ with $\boldsymbol{P}$. We consider $\phi \in W$ and compute $T_1(\phi)$. In particular, we consider the event sequence (19), so the evolution is

$$\begin{aligned}
&\xrightarrow{1}([P,(1,2,4),0],[P,(1,2,3),u],[F,1,1-W_2-v+u-\tau],\\
&\quad [F,2,1-W_2-w+u-\tau],[F,3,1-\tau],[F,4,1+u-\tau])\\
&\xrightarrow{2}([P,(1,2,3),u],[F,1,1-H_1(W_2+v-u+\tau)],\\
&\quad [F,2,1-H_1(W_2+w-u+\tau)],[F,3,1-\tau],[F,4,1-H_1(\tau-u)])\\
&\xrightarrow{3}([P,(1,2,3),0],[F,1,1-H_1(W_2+v-u+\tau)-u],\\
&\quad [F,2,1-H_1(W_2+w-u+\tau)-u],[F,3,1-\tau-u],\\
&\quad [F,4,1-H_1(\tau-u)-u])\\
&\xrightarrow{4}([F,1,0],[F,2,0],[F,3,1-H_1(\tau+u)],[F,4,1-H_1(\tau-u)-u])
\end{aligned}$$

where in the last transition we used the fact that $H_1(H_1(W_2 + v - u + \tau) + u) \geqslant 1$ and $H_1(H_1(W_2 + w - u + \tau) + u) \geqslant 1$ for (u, v, w) small enough, since $H_2(W_2 + \tau) \geqslant 1$. The situation for $w > v$ is identical up to interchanging oscillators O_1 and O_2. The case for the event sequence (19) is also similar and we do not analyse it separately. Therefore, we observe that $T_1(W) = U'$ where U' is the set of $\psi \in \boldsymbol{Q}$ with $\psi_1(0) = 0$, $\psi_2(0) = 0$, $\psi_3(0) = H_1(\tau+u)$, $\psi_4(0) = H_1(\tau - u) + u$, $\Sigma_1(\psi) = \Sigma_2(\psi) = \{0\}$ and $\Sigma_3(\psi) = \Sigma_4(\psi) = \emptyset$ and $u > 0$ can be chosen to be arbitrarily small.

Consider the map $\mathcal{K} : \mathcal{P}_{\mathcal{D}} \to \mathcal{P}_{\mathcal{D}}$ defined by $\mathcal{K}(\phi) = (\phi_3, \phi_4, \phi_1, \phi_2)$, i.e. $\mathcal{K}$ corresponds to a permutation of the oscillators (note also that $\mathcal{K}^{-1} = \mathcal{K}$). Define also $\psi^{Q_1} = \mathcal{K}(\phi^{Q_1})$ and $\psi^{Q_2} = \mathcal{K}(\phi^{Q_2})$ and note that the neighbourhood U' of ψ^{Q_1} is mapped by $\mathcal{K}$ to the neighbourhood $\mathcal{K}(U') = U$ of ϕ^{Q_1}, where U is defined in the proof of lemma 5. Moreover, following the evolution of ψ^{Q_2} one can show that $T_2(\psi^{Q_2}) = \phi^{Q_1}$.

Since all the oscillators are identical, we have that $\mathcal{K}\circ\Phi^t = \Phi^t\circ\mathcal{K}$. Moreover, if we denote by $R_{\boldsymbol{P}}$ the Poincaré map on $\boldsymbol{P}$ and $R_{\boldsymbol{Q}}$ the Poincaré map on $\boldsymbol{Q}$ we have that $R_{\boldsymbol{P}} = \mathcal{K}^{-1}\circ R_{\boldsymbol{Q}}\circ\mathcal{K}$. Then it follows that for $\psi \in U'$

$$R^m_{\boldsymbol{Q}}(\psi) = \mathcal{K}^{-1}(R^m_{\boldsymbol{P}}(\mathcal{K}(\psi))).$$

According to lemma 5 for every $\phi = \mathcal{K}(\psi) \in U$ (except ϕ^{Q_1}) there is $m > 0$ such that $R^m_{\boldsymbol{P}}(\phi) = \phi^{Q_2}$. Therefore,

$$R^m_{\boldsymbol{Q}}(\psi) = \mathcal{K}(\phi^{Q_2}) = \psi^{Q_2}.$$

Therefore, for $\phi \in W$ we have that

$$T_2(R_Q^m(T_1(\phi))) = T_2(\psi^{Q_2}) = \phi^{Q_1}$$

i.e., there is some $t > 0$ such that $\Phi^t(\phi) = \phi^{Q_1}$ for $\phi \in W \subset \boldsymbol{P}$. This implies that there is $m' > 0$ such that $R_{\boldsymbol{P}}^{m'}(\phi) = \phi^{Q_1}$ for $\phi \in W$. □

4. Discussion

4.1. Numerical simulations and comparisons with the theoretical results

In section 2, we established, that since the system studied in this paper is a delay system, the state space is the set of functions $\mathcal{P}_\tau^n$ that represent the values of the phases of the oscillators in the time interval $(-\tau, 0]$.

Nevertheless, it is often the case (see for example [11, 12]) that in numerical simulations of such systems, only the phases θ_i, for $i = 1, \ldots, n-1$, of the oscillators at time $t = 0$ are given as initial data and then the times $\sigma_{i,j}$ when the oscillators have fired in the interval $(-\tau, 0]$ are determined through a set of rules and $\theta_n = 0$, since we consider states on the surface of section. These rules essentially determine a map $G : \mathbb{T}^{n-1} \to \boldsymbol{P}/\sim$ and they define an $n-1$ dimensional subset $\mathcal{S} = G(\mathbb{T}^{n-1})$ of the infinite dimensional space $\boldsymbol{P}/\sim$.

If $\theta, \theta' \in \mathbb{T}^{n-1}$ then it appears natural to define the distance between the points $G(\theta), G(\theta') \in \mathcal{S}$ by the 'Euclidean' distance between θ and θ'. We call this metric, the $\mathbb{T}^{n-1}$ metric. It would be interesting to know whether studying a pulse coupled oscillator network using the $\mathbb{T}^{n-1}$ metric gives the same results as studying the network using the metric d, given by (11).

For this reason, in this section we numerically study a pulse coupled four-oscillator network with delay by giving a map $G : \mathbb{T}^3 \to \boldsymbol{P}/\sim$ and using the corresponding $\mathbb{T}^3$ metric and we compare these numerical results with the results obtained in section 3. In particular, we consider a four-oscillator network with the dynamics as described in section 2.1 and coupling given by a Mirollo–Strogatz pulse response function V_{MS} (8) for $b = 3$.

Given $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3$, we consider in the event representation the state given (up to time ordering) by

$$G(\theta) = ([P, (1, 2, 3), \tau], [F, 1, 1-\theta_1], [F, 2, 1-\theta_2], [F, 3, 1-\theta_3], [F, 4, 1])$$

if $\theta_i \geqslant \tau$ for $i = 1, 2, 3$. If $\theta_1 < \tau$ then O_1 must have fired in the interval $[-\theta_1, 0)$. We make the extra assumption that in this case O_1 has fired exactly at time $-\theta_1$. Therefore, we add to $G(\theta)$ the events $[P, (2, 3, 4), \tau - \theta_1]$. Similarly, if $\theta_2 < \tau$ we add to $G(\theta)$ the events $[P, (1, 3, 4), \tau - \theta_2]$ and if $\theta_3 < \tau$ we add the events $[P, (1, 2, 4), \tau - \theta_3]$. In each case we time-order $G(\theta)$. This construction defines the mapping $G : \mathbb{T}^3 \to \boldsymbol{P}/\sim$. By definition, G is a bijection on $\mathcal{S} = G(\mathbb{T}^3)$. Note that the Poincaré map $R_\sim$ does not define a map on $\mathbb{T}^3$ by $G^{-1} \circ R_\sim \circ G$, because the image of $R_\sim \circ G$ contains points that do not belong to $\mathcal{S}$.

The fixed states $[\phi^{Q_i}]$ $(i = 1, 2)$ of the Poincaré map belong to $\mathcal{S}$, therefore there exist $Q_1, Q_2 \in \mathbb{T}^3$ such that $G(Q_i) = [\phi^{Q_i}]$ for $i = 1, 2$. We study numerically which states $G(\theta)$ converge to $[\phi^{Q_1}]$ or $[\phi^{Q_2}]$ for parameter values $\varepsilon = 0.1$ and $\tau = 0.2$. The result is depicted in figure 6 where the intersection of the basins of the unstable attractors with planes $\theta_3 =$ const is shown. The basin of attraction of Q_1 is represented by dark grey and that of Q_2 by light grey. From figure 6 we conclude that there is an open, in $\mathbb{T}^3$, ball B_1 around Q_1 that belongs in the basin of Q_2 (except for points on the plane $\theta_1 = \theta_2$). Moreover, there is an open, in $\mathbb{T}^3$, ball B_2 of points around Q_2 contained (except points on the plane $\theta_3 = 0$) in the basin of Q_1. This is harder to see in figure 6 but a magnification of the region near Q_2 reveals that the situation

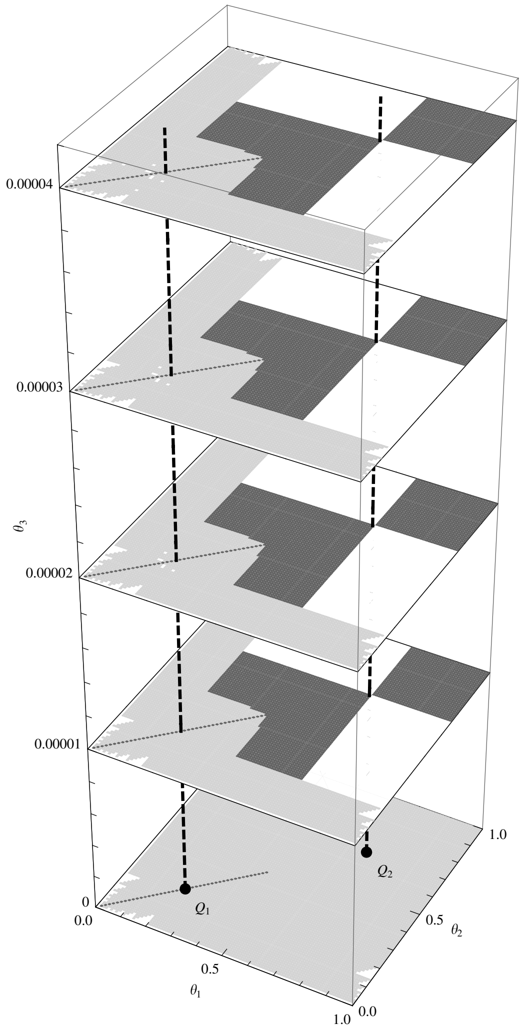

Figure 6. Basin of attraction, projected to $\mathcal{S}$, of the unstable attractors Q_1 and Q_2 for a network of four oscillators with $b = 3$, $\varepsilon = 0.1$, $\tau = 0.2$. Both fixed point attractors lie on the intersection of the plane $\theta_3 = 0$ (bottom plane) with the plane $\theta_1 = \theta_2$. Initial states that converge to the attractors Q_1 and Q_2 are shown in dark and light grey, respectively. Observe that initial states near the attractor Q_1 belong to the basin of Q_2 and vice versa, which demonstrates that there exists a heteroclinic cycle between Q_1 and Q_2.

is as just described. Therefore, from the numerical results we conclude that if we restrict our attention to $\mathcal{S}$ with the $\mathbb{T}^3$ metric, the fixed points Q_1 and Q_2 are unstable attractors since their basins have interior points and there is a heteroclinic cycle between them. This is exactly the result that we obtained for ϕ^{Q_1} and ϕ^{Q_2} in section 3.

It is an important question whether one can infer, in all cases, from such numerical results using the $\mathbb{T}^{n-1}$ metric, the existence of unstable attractors and heteroclinic cycles on the Poincaré surface of section $\boldsymbol{P}$ with the metric d, given by (11). Note first that the $\mathbb{T}^{n-1}$ metric is not equivalent to d. For example, we have obtained in lemma 5 that the open neighbourhood U of ϕ^{Q_1}, with respect to the metric d, is characterized by 1 dynamically significant parameter, i.e. $U/\sim$ is one dimensional. On the other hand, the open neighbourhood of $Q_1 \in \mathbb{T}^3$ with

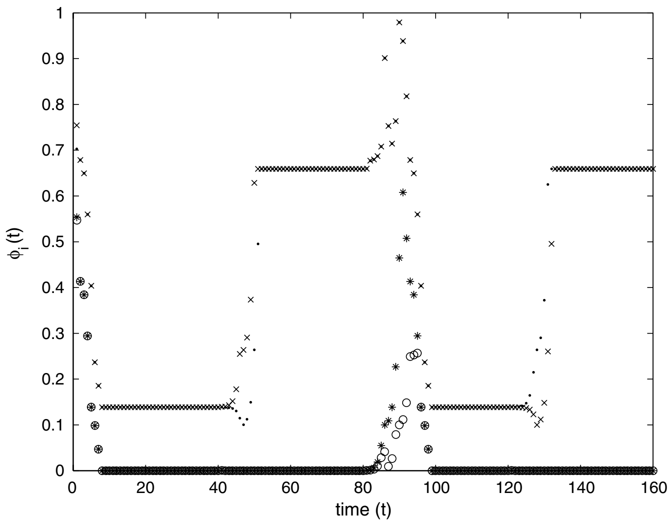

Figure 7. Heteroclinic connections for five oscillators: the phases $\phi_i(t)$ of the oscillators are shown against the time t at which the nth oscillator fires. Random noise is generated every 40 time units. The reconfiguration of the oscillators after the noise corresponds to switching between unstable attractors. The parameter values are $b = 3$, $\varepsilon = 0.3$ and $\tau = 0.1$.

respect to the $\mathbb{T}^3$ metric is three-dimensional and moreover $G^{-1}(U/\sim)$ is a one-dimensional *closed* subset of $\mathbb{T}^3$. In order to show that from the numerical results on the existence of unstable attractors and heteroclinic cycles we can infer similar results for $\boldsymbol{P}$, it would be enough to show that all open sets in $\mathbb{T}^{n-1}$ correspond to open sets in $\boldsymbol{P}$. Whether this is true remains an open question.

4.2. Conclusions

In this paper we proved the existence of heteroclinic cycles between unstable attractors in a global network consisting of four oscillators. Such heteroclinic cycles occur for an open set of parameter values in the class of systems that we considered. For this purpose we used the mathematical framework introduced in [12] and extended in [19], which permits us to study analytically the evolution of the system and define the neighbourhood of a state in the (infinite dimensional) state space.

A natural question is whether similar cycles occur in networks with more than four oscillators. In figure 7 we illustrate the presence of heteroclinic cycles for a five oscillator network. Moreover, numerical simulations in [10] suggest that such cycles exist for networks with $n = 100$ oscillators. Another question is how the existence of heteroclinic cycles is affected if we consider instead of the Mirollo–Strogatz model other pulse response functions.

The importance of heteroclinic connections such as those considered in this work, is that they provide flexibility to the system because it is possible to switch between unstable attractors. Furthermore, they can also be used to perform computational tasks, such as design of a multibase counter [7] and sequence learning [24]. We are not aware of any work that answers the question whether heteroclinic connections persist for non-global networks or for non-identically coupled networks. It would be worthwhile to further explore the dynamics of pulse coupled oscillators and the existence of heteroclinic cycles between unstable attractors for varied non-global networks, namely, regular, random [25], small-world [26] and fractal [27,28] networks.

Acknowledgments

The authors would like to thank Marc Timme, Gert Vegter and Sijbo Holtman for useful discussions. KE acknowledges support from the NWO cluster NDNS+.

Appendix A. Propositions used in the proofs of lemmas 5 and 6

Proposition 7. *There is* $\rho_1 > 0$ *and* $C_2 > 0$ *such that if* $d(\phi^{Q_1}, \phi) = \epsilon < \rho$ *for some* $\phi \in \boldsymbol{P}$ *then there is some* x *with* $0 < x < C_2\epsilon$ *such that either*

(i) $\phi_1(0) = H_1(\tau + x)$, $\phi_2(0) = H_1(\tau - x) + x$, $\phi_3(0) = \phi_4(0) = 0$ *or*

(ii) $\phi_1(0) = H_1(\tau - x) + x$, $\phi_2(0) = H_1(\tau + x)$, $\phi_3(0) = \phi_4(0) = 0$ *or*

(iii) $\phi_1(0) = H_1(\tau)$, $\phi_2(0) = H_1(\tau)$, $\phi_3(0) = \phi_4(0) = 0$.

In all cases, $\Sigma_i(\phi) = \emptyset$ *for* $i = 1, 2$ *and* $\Sigma_3(\phi) = \Sigma_4(\phi) = \{0\}$.

Proof. In this proof we use results from [19]. Since the phases $\phi_i^{Q_1}$, $i = 1, \ldots, 4$ do not have any discontinuities in the intervals $(-\tau, 0)$ and $(0, \tau)$ there is $\rho_1 > 0$ and constants $C_1, C_2 > 0$ such that if $d(\phi, \phi^{Q_1}) = \epsilon < \rho_1$ then $|\phi_j(s) - \phi_j^{Q_1}(s)| < C_1\epsilon$ for $j = 1, \ldots, 4$ and for $s \in (-\tau + C_2\epsilon, -C_2\epsilon) \cup (C_2\epsilon, \tau - C_2\epsilon)$. This also implies that no oscillator fires in the interval $(-\tau + C_2\epsilon, -C_2\epsilon)$, because then the phase of the other oscillators would have a discontinuity in the interval $(C_2\epsilon, \tau - C_2\epsilon)$. Given that the size of discontinuity has to be larger than $V_1(0) > 0$ we conclude that by making ρ_1 (and consequently ϵ) small enough, the condition $|\phi_j(s) - \phi_j^{Q_1}(s)| < C_1\epsilon$ would not hold, and therefore we have a contradiction.

Moreover, for similar reasons and because $\phi_4(0) = 0$ we conclude that $\phi_4(s) = s$ for all $s \in [0, \tau - C_2\epsilon)$. This in turn implies that the oscillators O_1, O_2 and O_3 do not fire in $(-\tau, -C_2\epsilon)$.

We have established that at time $-C_2\epsilon$, the phase $\phi_1(-C_2\epsilon)$ of the oscillator O_1 is $O(\epsilon)$ close to τ, while $\phi_1(C_2\epsilon)$ is $O(\epsilon)$ close to $H_1(\tau)$. From this we deduce that O_1 must receive exactly one pulse in the interval $(-C_2\epsilon, C_2\epsilon)$. The same is also true for O_2. On the other hand, $\phi_3(-C_2\epsilon)$ is $O(\epsilon)$ close to $\tau + W_2$ and $\phi_3(C_2\epsilon)$ is $O(\epsilon)$ close to 0. This means that the oscillator O_3 must receive enough pulses to fire in the interval $(-C_2\epsilon, C_2\epsilon)$. Given that $H_1(W_2 + \tau) < 1$ and that $H_2(W_2 + \tau) > 1$ and also that ϵ can be chosen arbitrarily small we conclude that O_3 must receive exactly two pulses in the interval $(-C_2\epsilon, C_2\epsilon)$. Similar arguments show that O_4 must also receive exactly two pulses in the interval $(-C_2\epsilon, 0]$.

The only possibility for this combination of pulses to happen is if the oscillators O_1 and O_2 fire at moments $t_1 = -\tau - x$ and $t_2 = -\tau$ such that $x < C_2\epsilon$. We should consider the cases that O_1 fires at t_1 and O_2 at t_2, that O_2 fires at t_1 and O_1 at t_2, or finally that O_2 and O_1 fire simultaneously at $t_1 = t_2 = 0$.

Consider first the case that O_1 fires at $t_1 = -\tau - x$ and that O_2 fires at $t_2 = -\tau$. Then, at time $-x$ the oscillators O_2, O_3 and O_4 receive a pulse from O_1 and at time 0 the oscillators O_1, O_3 and O_4 receive a pulse from O_2. Since, O_3 and O_4 receive the second pulse at time 0, we have $\phi_3(0) = \phi_4(0) = 0$. Moreover, when O_2 receives the pulse from O_1 its phase is $\phi_2(-x^-) = \tau - x$, so $\phi_2(0) = H_1(\tau - x) + x$. Finally, when O_1 receives the pulse from O_2 its phase is $\phi_1(0^-) = \tau + x$ so its phase becomes $\phi_1(0) = H_1(\tau + x)$.

In the case that O_2 fires before O_1 we can use similar arguments to show that $\phi_1(0) = H_1(\tau - x) + x$ and $\phi_2(0) = H_1(\tau + x)$. Finally, if O_1 and O_2 fire simultaneously at $-\tau$, then $\phi_1(0) = \phi_2(0) = H_1(\tau)$. □

Proposition 8. *There is $\rho_1 > 0$ and $C_1 > 0$ such that if $d(\phi^{Q_2}, \phi) = \epsilon < \rho$ for some $\phi \in \boldsymbol{P}$ then $|\phi_i(0) - W_2| < C_1\epsilon$ and $\Sigma_i(\phi) = \emptyset$ for $i = 1, 2$, $\phi_4(0) = 0$ with $\Sigma_4(\phi) = \{0\}$ and either*

(i) $0 \leqslant \phi_3(0) < C_2\epsilon$ *and* $\Sigma_3(\phi) = \{-\phi_3(0)\}$*, or*

(ii) $-C_2\epsilon < \phi_3(0) < 0$ *and* $\Sigma_3(\phi) = \emptyset$*.*

Proof. Since the phases $\phi_i^{Q_2}$, $i = 1, \ldots, 4$ do not have any discontinuities in the intervals $(-\tau, \tau)$ there is $\rho_1 > 0$ and constants $C_1, C_2 > 0$ such that if $d(\phi, \phi^{Q_2}) = \epsilon < \rho_1$ then $|\phi_j(s) - \phi_j^{Q_2}(s)| < C_1\epsilon$ for $j = 1, \ldots, 4$ and for $s \in (-\tau + C_2\epsilon, \tau - C_2\epsilon)$. This also implies that no oscillator receives a pulse in the interval $(-\tau + C_2\epsilon, \tau - C_2\epsilon)$ which means that no oscillator fires in the interval $[-\tau, -C_2\epsilon)$. For the oscillators O_1 and O_2 we can conclude that they do not fire also in $[-C_2\epsilon, 0]$ since their phases are $O(\epsilon)$ close to $\phi_1(0) = \phi_2(0) = W_2$ in this time interval. Therefore, $|\phi_j(0) - W_2| < C_1\epsilon$ for $j = 1, 2$ and $\Sigma_j(\phi) = \emptyset$. Moreover, $\phi_4(0) = 0$ since $\phi \in \boldsymbol{P}$ and therefore $\Sigma_4(\phi) = \{0\}$. Finally, for the oscillator O_3 we have that $\phi_3(0)$ is $O(\epsilon)$ close to 0 (mod 1), and therefore either it fires at time u with $-C_2\epsilon < u \leqslant 0$ and $\phi_3(0) = -u > 0$, $\Sigma_3(\phi) = \{u\}$ or it fires at time u with $C_2\epsilon > u > 0$ and $\phi_3(0) = 1 - u$, $\Sigma_3(\phi) = \emptyset$. □

Proposition 9. *If $\phi \in \ell_1$ then $R(\phi) \in \ell$ and there is a finite number m of iterations such that $R^m(\phi) \in \ell_2$.*

Proof. The initial event sequence in ℓ_1 with $H_2(v + \tau) - H_2(w + \tau) < \tau$ evolves as

$$([2P, (1, 2), \tau], [P, (3, 4), \tau], [F, 1, 1 - v], [F, 2, 1 - w], [F, (3, 4), 1])$$

$$\xrightarrow{1} ([2P, (1, 2), 0], [P, (3, 4), 0], [F, 1, 1 - v - \tau], [F, 2, 1 - w - \tau],$$
$$[F, (3, 4), 1 - \tau])$$
$$\xrightarrow{2} ([F, 1, 1 - H_2(v + \tau)], [F, 2, 1 - H_2(w + \tau)], [F, (3, 4), 1 - H_1(\tau)])$$
$$\xrightarrow{3} ([F, 1, 0], [F, 2, H_2(v + \tau) - H_2(w + \tau)], [F, (3, 4), H_2(v + \tau) - H_1(\tau)]).$$

Let $\nu = H_2(v + \tau) - H_2(w + \tau)$. Since $\nu < \tau < H_2(w + \tau) - H_1(\tau) < H_2(v + \tau) - H_1(\tau)$ (which follows from equation (B.2) in proposition 16), the evolution continues as

$$\xrightarrow{4} ([F, 2, \nu], [P, (2, 3, 4), \tau], [F, (3, 4), H_2(v + \tau) - H_1(\tau)], [F, 1, 1])$$
$$\xrightarrow{5} ([F, 2, 0], [P, (2, 3, 4), \tau - \nu], [F, (3, 4), H_2(v + \tau) - H_1(\tau) - \nu],$$
$$[F, 1, 1 - \nu])$$
$$\xrightarrow{6} ([P, (2, 3, 4), \tau - \nu], [P, (1, 3, 4), \tau], [F, (3, 4), H_2(v + \tau) - H_1(\tau) - \nu],$$
$$[F, 1, 1 - \nu], [F, 2, 1])$$
$$\xrightarrow{7} ([P, (2, 3, 4), 0], [P, (1, 3, 4), \nu], [F, (3, 4), H_2(v + \tau) - H_1(\tau) - \tau],$$
$$[F, 1, 1 - \tau], [F, 2, 1 - \tau + \nu]).$$

Since $H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau)) < H_1(W_2 + \tau) < 1$ and $\nu < 1 - H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau)$ (equation (B.3) in proposition 16), we get

$$\xrightarrow{8} ([P, (1, 3, 4), \nu], [F, (3, 4), 1 - H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau)],$$
$$[F, 1, 1 - \tau], [F, 2, 1 - H_1(\tau - \nu)])$$
$$\xrightarrow{9} ([P, (1, 3, 4), 0], [F, (3, 4), 1 - H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau) - \nu],$$
$$[F, 1, 1 - \tau - \nu], [F, 2, 1 - H_1(\tau - \nu) - \nu]).$$

For $\nu < \tau$, one can show that $H_1(H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau) + \nu) \geqslant 1$ (equation (B.4) in proposition 16). Therefore,

$$\xrightarrow{10} ([F, (3, 4), 0], [F, 1, 1 - H_1(\tau + \nu)], [F, 2, 1 - H_1(\tau - \nu) - \nu])$$

$$\xrightarrow{11} ([2P, (1, 2), \tau], [P, (3, 4), \tau], [F, 1, 1 - H_1(\tau + \nu)],$$
$$[F, 2, 1 - H_1(\tau - \nu) - \nu], [F, (3, 4), 1]).$$

Let $v'' = H_1(\tau + \nu)$ and $w'' = H_1(\tau - \nu) + \nu$ be the phases of the oscillators O_1 and O_2. Then, since $0 < \nu < \tau$ we have that $H_1(\tau) < v'' < H_1(2\tau)$ and $w'' = H_1(\tau) + (1 - A_\varepsilon)\nu$. Since $A_\varepsilon > 1$ we obtain that $w'' > H_1(\tau) + (1 - A_\varepsilon)\tau = m_\varepsilon + \tau > \tau$. Furthermore,

$$A_\varepsilon w'' + (A_\varepsilon - 1)v'' = A_\varepsilon(H_1(\tau) + (1 - A_\varepsilon)\nu) + (A_\varepsilon - 1)(H_1(\tau) + A_\varepsilon \nu)$$
$$= (2A_\varepsilon - 1)H_1(\tau).$$

Finally,

$$\nu'' = H_2(v'' + \tau) - H_2(w'' + \tau) = A_\varepsilon^2(v'' - w'')$$
$$= A_\varepsilon^2(H_1(\tau) + A_\varepsilon \nu - H_1(\tau) + (A_\varepsilon - 1)\nu) = A_\varepsilon^2(2A_\varepsilon - 1)\nu.$$

This implies that in finite iterations the initial state is mapped to a state characterized by v', w' such that $H_2(v' + \tau) - H_2(w' + \tau) \geqslant \tau$. □

Proposition 10. *If $\phi \in \ell_2$ then $R(\phi) \in \mathcal{B}_1 \cup \mathcal{B}_2$.*

Proof. Observe that for $v < H_1(2\tau)$, we have that $H_2(v + \tau) < g_1(\tau) < 1$. Let

$$\mu = H_1(1 + \tau + H_2(w + \tau) - H_2(v + \tau))$$

and

$$\kappa = H_1(1 + \tau + H_1(\tau) - H_2(v + \tau))$$

and note that

$$\mu - \kappa = H_1(H_2(w + \tau) - H_1(\tau)) = H_2(H_1(w + \tau) - \tau).$$

We skip the first three transitions in the evolution of $[\phi^{v,w}]$ which are the same as in proposition 9. Then we have

$$\xrightarrow{4} ([P, (2, 3, 4), \tau], [F, 2, H_2(v + \tau) - H_2(w + \tau)],$$
$$[F, (3, 4), H_2(v + \tau) - H_1(\tau)], [F, 1, 1])$$

$$\xrightarrow{5} ([P, (2, 3, 4), 0], [F, 2, H_2(v + \tau) - H_2(w + \tau) - \tau],$$
$$[F, (3, 4), H_2(v + \tau) - H_1(\tau) - \tau], [F, 1, 1 - \tau]).$$

We distinguish now two cases based on the value of μ. First, for $\mu < 1$, the evolution is

$$\xrightarrow{6} ([F, 2, 1 - \mu], [F, (3, 4), 1 - \kappa], [F, 1, 1 - \tau])$$

$$\xrightarrow{7} ([F, 2, 0], [F, (3, 4), \mu - \kappa], [F, 1, \mu - \tau]).$$

Since $\mu - \kappa > \tau$ (equation (B.5) in proposition 16)

$$\xrightarrow{8} ([P, (1,3,4), \tau], [F, (3,4), \mu - \kappa], [F, 1, \mu - \tau], [F, 2, 1])$$

$$\xrightarrow{9} ([P, (1,3,4), 0], [F, (3,4), \mu - \kappa - \tau], [F, 1, \mu - 2\tau], [F, 2, 1 - \tau])$$

$$\xrightarrow{10} ([F, (3,4), 1 - H_1(1 + \kappa + \tau - \mu)], [F, 1, 1 - H_1(2\tau + 1 - \mu)],$$
$$[F, 2, 1 - \tau])$$

$$\xrightarrow{11} ([F, (3,4), 0], [F, 1, H_1(1 + \kappa + \tau - \mu) - H_1(2\tau + 1 - \mu)],$$
$$[F, 2, H_1(1 + \kappa + \tau - \mu) - \tau])$$

$$\xrightarrow{12} ([2P, (1,2), \tau], [P, (3,4), \tau], [F, 1, H_1(1 + \kappa + \tau - \mu) - H_1(2\tau + 1 - \mu)],$$
$$[F, 2, H_1(1 + \kappa + \tau - \mu) - \tau], [F, (3,4), 0]).$$

Therefore,

$$v' = 1 - H_1(1 + \kappa + \tau - \mu) + H_1(2\tau + 1 - \mu) > H_1(2\tau + 1 - \mu) > H_1(2\tau)$$

and

$$w' = 1 + \tau - H_1(1 + \kappa + \tau - \mu) > \tau.$$

For the case $\mu \geqslant 1$, we have

$$\xrightarrow{6} ([F, 2, 0], [F, (3,4), 1 - \kappa], [F, 1, 1 - \tau]).$$

Since $1 - \kappa > \tau$ when $v > v^* = H_1(\tau) + (\tau / A_\varepsilon(2A_\varepsilon - 1))$ (equation (B.6) in proposition 16), we obtain

$$\xrightarrow{7} ([P, (1,3,4), \tau], [F, (3,4), 1 - \kappa], [F, 1, 1 - \tau], [F, 2, 1])$$

$$\xrightarrow{8} ([P, (1,3,4), 0], [F, (3,4), 1 - \kappa - \tau], [F, 1, 1 - 2\tau], [F, 2, 1 - \tau])$$

$$\xrightarrow{9} ([F, (3,4), 1 - H_1(\kappa + \tau)], [F, 1, 1 - H_1(2\tau)], [F, 2, 1 - \tau])$$

$$\xrightarrow{10} ([F, (3,4), 0], [F, 1, H_1(\kappa + \tau) - H_1(2\tau)], [F, 2, H_1(\kappa + \tau) - \tau])$$

where we made the assumption that $H_1(\kappa + \tau) < 1$ and from which we conclude that

$$v' = 1 + H_1(2\tau) - H_1(\kappa + \tau) \geqslant H_1(2\tau)$$

and

$$w' = 1 + \tau - H_1(\kappa + \tau) \geqslant \tau.$$

Finally, it is possible for $H_1(\kappa + \tau) \geqslant 1$ to have the evolution

$$\xrightarrow{9} ([F, (3,4), 0], [F, 1, 1 - H_1(2\tau)], [F, 2, 1 - \tau])$$

which gives that $v' = H_1(2\tau)$ and $w' = \tau$. □

Proposition 11. *States in $\mathcal{B}_1$ are mapped in finite iterations of R into $\mathcal{B}_2$.*

Proof. As before, we let $\mu = H_1(1+\tau+H_2(w+\tau)-H_2(v+\tau))$ and $\kappa = H_1(1+\tau+H_1(\tau)-H_2(v+\tau))$ and the evolution of $[\phi^{v,w}]$ is the same as in the previous proposition until transition 5. Transition 6 depends on the value of μ. For the case, $\mu \geqslant 1$,

$$\xrightarrow{6}([F,2,0],[F,(3,4),1-\kappa],[F,1,1-\tau])$$

$$\xrightarrow{7}([P,(1,3,4),\tau],[F,(3,4),1-\kappa],[F,1,1-\tau],[F,2,1])$$

$$\xrightarrow{8}([P,(1,3,4),0],[F,(3,4),1-\kappa-\tau],[F,1,1-2\tau],[F,2,1-\tau]).$$

Since $H_1(\kappa+\tau) < g_2(\tau) < 1$ for $v \geqslant H_1(2\tau)$

$$\xrightarrow{9}([F,(3,4),1-H_1(\kappa+\tau)],[F,1,1-H_1(2\tau)],[F,2,1-\tau])$$

$$\xrightarrow{10}([F,(3,4),0],[F,1,H_1(\kappa+\tau)-H_1(2\tau)],[F,2,H_1(\kappa+\tau)-\tau]).$$

Let $R(v) = 1+H_1(2\tau)-H_1(\kappa+\tau) = 1+H_1(\tau)-H_1(\kappa)$. The function,

$$\Delta(v) = R(v)-v = V_2(v+\tau)-V_1(1+\tau+H_1(\tau)-H_2(v+\tau))$$

is an increasing function of v. Also, from step 9, one can conclude that $\Delta(H_1(2\tau)) > 0$.

On the other hand if $\mu < 1$ then,

$$\xrightarrow{6}([F,2,1-\mu],[F,(3,4),1-\kappa],[F,1,1-\tau])$$

$$\xrightarrow{7}([F,2,0],[F,(3,4),\mu-\kappa],[F,1,\mu-\tau])$$

$$\xrightarrow{8}([P,(1,3,4),\tau],[F,(3,4),\mu-\kappa],[F,1,\mu-\tau],[F,2,1])$$

$$\xrightarrow{9}([P,(1,3,4),0],[F,(3,4),\mu-\kappa-\tau],[F,1,\mu-2\tau],[F,2,1-\tau]).$$

Since $H_1(1+\kappa+\tau-\mu) < g_2(\tau) < 1$ for $v \geqslant H_1(2\tau)$; $w > \tau$, (equation (B.7) in proposition 16)

$$\xrightarrow{10}([F,(3,4),1-H_1(1+\kappa+\tau-\mu)],[F,1,1-H_1(1+2\tau-\mu)],$$
$$[F,2,1-\tau])$$

$$\xrightarrow{11}([F,(3,4),0],[F,1,H_1(1+\kappa+\tau-\mu)-H_1(1+2\tau-\mu)],$$
$$[F,2,H_1(1+\kappa+\tau-\mu)-\tau]).$$

Also in this case, we have $R(v) = 1+H_1(1+2\tau-\mu)-H_1(1+\kappa+\tau-\mu) = 1+H_1(\tau)-H_1(\kappa)$ and the function

$$\Delta(v) = R(v)-v = V_2(v+\tau)-V_1(1+\tau+H_1(\tau)-H_2(v+\tau))$$

which is an increasing function of v with $\Delta(H_1(2\tau)) > 0$.

Note that, under the assumptions of this lemma, with every application of the Poincaré map R, the sequences $R(\phi^{v,w})_1(0)$, $R^2(\phi^{v,w})_1(0)$, $R^3(\phi^{v,w})_1(0), \cdots$ are an increasing sequence. Therefore, after some finite iterations m we must have $R^m(\phi_1^{v',w'}(0)) = v'$ such that $v' > H_1(2\tau)$; $w' > \tau$ with $H_2(v'+\tau) \geqslant 1$. □

Proposition 12. *Initial states in $\mathcal{B}_2^a$ are mapped in finite iterations of R in $\mathcal{B}_2^b \cup \mathcal{B}_2^c \cup \mathcal{B}_2^d$.*

Proof. Let $\eta = 1 - H_1(H_2(w+\tau)+\tau)$. The evolution in this case is given by

$$\xrightarrow{2}([F,1,0],[F,2,1-H_2(w+\tau)],[F,(3,4),1-H_1(\tau)])$$

$$\xrightarrow{3}([P,(2,3,4),\tau],[F,2,1-H_2(w+\tau)],[F,(3,4),1-H_1(\tau)],[F,1,1])$$

$$\xrightarrow{4}([P,(2,3,4),0],[F,2,1-H_2(w+\tau)-\tau],[F,(3,4),1-H_1(\tau)-\tau],$$
$$[F,1,1-\tau])$$

$$\xrightarrow{5}([F,2,\eta],[F,(3,4),1-\alpha],[F,1,1-\tau])$$

$$\xrightarrow{6}([F,2,0],[F,(3,4),1-\alpha-\eta],[F,1,1-\tau-\eta])$$

$$\xrightarrow{7}([P,(1,3,4),\tau],[F,(3,4),1-\alpha-\eta],[F,1,1-\tau-\eta],[F,2,1])$$

$$\xrightarrow{8}([P,(1,3,4),0],[F,(3,4),1-\alpha-\eta-\tau],[F,1,1-2\tau-\eta],$$
$$[F,2,1-\tau]).$$

Since $H_1(\alpha+\eta+\tau) < g_2(\tau) < 1$, (equation (B.8) in proposition 16)

$$\xrightarrow{9}([F,(3,4),1-H_1(\alpha+\eta+\tau)],[F,1,1-H_1(2\tau+\eta)],[F,2,1-\tau])$$

$$\xrightarrow{10}([F,(3,4),0],[F,1,H_1(\alpha)-H_1(\tau)],[F,2,H_1(\alpha+\eta+\tau)-\tau]).$$

Define $R(w) = 1+\tau - H_1(\alpha+\eta+\tau)$. The function,

$$\Delta(w) = R(w) - w = (A_\varepsilon^4 - 1)w + (A_\varepsilon - 1)^2(A_\varepsilon^2 + A_\varepsilon + 1)\tau + (A_\varepsilon^3 - 1)v_1 - A_\varepsilon + 1,$$

is an increasing function of w. Moreover, $\Delta(\tau) = 1 - g_2(\tau) > 0$. This means that beginning with a state $[\phi^{v,w}] \in \mathcal{B}_2^a$ the w-coordinate of successive iterations increases at an increasing rate, therefore, there is a finite number of iterations m such that $R^m([\phi^{v,w}])$ gets outside $\mathcal{B}_2^a$. Furthermore, since $R(w) < W_2$ and $v > w$ we conclude that $R^m([\phi^{v,w}]) \in \mathcal{B}_2^b \cup \mathcal{B}_2^c \cup \mathcal{B}_2^d$. □

Proposition 13. *Initial states in $\mathcal{B}_2^b$ are mapped in one iteration of R to a single state in $\mathcal{B}_2^c$.*

Proof. The evolution is given by

$$\xrightarrow{2}([F,1,0],[F,2,1-H_2(w+\tau)],[F,(3,4),1-H_1(\tau)])$$

$$\xrightarrow{3}([P,(2,3,4),\tau],[F,2,1-H_2(w+\tau)],[F,(3,4),1-H_1(\tau)],[F,1,1])$$

$$\xrightarrow{4}([P,(2,3,4),0],[F,2,1-H_2(w+\tau)-\tau],[F,(3,4),1-H_1(\tau)-\tau],$$
$$[F,1,1-\tau])$$

$$\xrightarrow{5}([F,2,0],[F,(3,4),1-\alpha],[F,1,1-\tau])$$

$$\xrightarrow{6}([P,(1,3,4),\tau],[F,(3,4),1-\alpha],[F,1,1-\tau],[F,2,1])$$

$$\xrightarrow{7}([P,(1,3,4),0],[F,(3,4),1-\alpha-\tau],[F,1,1-2\tau],[F,2,1-\tau]).$$

Since $H_1(\alpha+\tau) < g_2(\tau) < 1$,

$$\xrightarrow{8}([F,(3,4),1-H_1(\alpha+\tau)],[F,1,1-H_1(2\tau)],[F,2,1-\tau])$$

$$\xrightarrow{9}([F,(3,4),0],[F,1,H_1(\alpha)-H_1(\tau)],[F,2,H_1(\alpha+\tau)-\tau]).$$

Therefore, the new phases are $v' = 1 + H_1(\tau) - H_1(\alpha) = W_2$ and $w' = 1 + \tau - H_1(\tau + \alpha) = W_1$ (defined in theorem 2). Recall that $g_3(\tau) = 1 - H_2(W_1 + \tau) < \tau$, therefore $[\phi^{W_2,W_1}] \in \mathcal{B}_2^c$. □

Proposition 14. *Initial states in $\mathcal{B}_2^c$ are mapped in finite iterations of R in $\mathcal{B}_2^d$.*

Proof. Let $\xi = 1 - H_2(w + \tau) < \tau$. The evolution is given by

$$\begin{aligned}
&([2P, (1, 2), \tau], [P, (3, 4), \tau], [F, 1, 1 - v], [F, 2, 1 - w], [F, (3, 4), 1])\\
&\quad\xrightarrow{1}([2P, (1, 2), 0], [P, (3, 4), 0], [F, 1, 1 - v - \tau], [F, 2, 1 - w - \tau],\\
&\quad\quad [F, (3, 4), 1 - \tau])\\
&\quad\xrightarrow{2}([F, 1, 0], [F, 2, 1 - H_2(w + \tau)], [F, (3, 4), 1 - H_1(\tau)])\\
&\quad\xrightarrow{3}([F, 2, \xi], [P, (2, 3, 4), \tau], [F, (3, 4), 1 - H_1(\tau)], [F, 1, 1])\\
&\quad\xrightarrow{4}([F, 2, 0], [P, (2, 3, 4), \tau - \xi], [F, (3, 4), 1 - H_1(\tau) - \xi], [F, 1, 1 - \xi])\\
&\quad\xrightarrow{5}([P, (2, 3, 4), \tau - \xi], [P, (1, 3, 4), \tau], [F, (3, 4), 1 - H_1(\tau) - \xi],\\
&\quad\quad [F, 1, 1 - \xi], [F, 2, 1])\\
&\quad\xrightarrow{6}([P, (2, 3, 4), 0], [P, (1, 3, 4), \xi], [F, (3, 4), 1 - H_1(\tau) - \tau],\\
&\quad\quad [F, 1, 1 - \tau], [F, 2, 1 - \tau + \xi])\\
&\quad\xrightarrow{7}([P, (1, 3, 4), \xi], [F, (3, 4), 1 - \alpha], [F, 1, 1 - \tau], [F, 2, 1 - H_1(\tau - \xi)])\\
&\quad\xrightarrow{8}([P, (1, 3, 4), 0], [F, (3, 4), 1 - \alpha - \xi], [F, 1, 1 - \tau - \xi],\\
&\quad\quad [F, 2, 1 - H_1(\tau - \xi) - \xi])
\end{aligned}$$

where $\alpha = H_1(H_1(\tau) + \tau)$. In transition 5 we used the fact that $1 - H_1(\tau) - \xi > \tau$. Since $H_1(\alpha + \xi) < H_1(\alpha + \tau) < g_2(\tau) < 1$,

$$\begin{aligned}
&\xrightarrow{9}([F, (3, 4), 1 - H_1(\alpha + \xi)], [F, 1, 1 - H_1(\tau + \xi)],\\
&\quad [F, 2, 1 - H_1(\tau - \xi) - \xi])\\
&\xrightarrow{10}([F, (3, 4), 0], [F, 1, H_1(\alpha) - H_1(\tau)], [F, 2, H_1(\alpha + \xi) - H_1(\tau - \xi) - \xi]).
\end{aligned}$$

Therefore the value of v in one iteration becomes

$$v' = 1 + H_1(\tau) - H_2(H_1(\tau) + \tau) = W_2$$

while the value of w becomes

$$\begin{aligned}
w' &= 1 + \xi + H_1(\tau - \xi) - H_1(\alpha + \xi)\\
&= 1 + H_1(\tau) - H_1(\alpha) + (1 - 2A_\varepsilon)\xi = W_2 + (1 - 2A_\varepsilon)\xi.
\end{aligned}$$

The function

$$\Delta(w) = R(w) - w = W_2 + (A_\varepsilon - 1)(1 + A_\varepsilon + 2A_\varepsilon^2)w + (2A_\varepsilon - 1)(v_1 + A_\varepsilon v_1 + A_\varepsilon^2 \tau - 1),$$

is an increasing function of w. Moreover, if we denote by w_* the solution of $\tau + H_2(w_* + \tau) = 1$ we obtain that

$$\begin{aligned}\Delta(w_*) &= 1 + H_1(\tau) - H_2(H_1(\tau) + \tau) + (1 - 2A_\varepsilon)\tau - w_* \\ &= -\frac{A_\varepsilon^2 - 1}{A_\varepsilon^2}((1 + A_\varepsilon + A_\varepsilon^2)v_1 + A_\varepsilon^2(1 + A_\varepsilon)\tau - 1) \\ &= -\frac{A_\varepsilon^2 - 1}{A_\varepsilon^2}(g_1(\tau) - A_\varepsilon^3\tau - 1) > 0,\end{aligned}$$

which implies that $\Delta(w) > 0$ for all $w \geqslant w_*$. This means that beginning with a state $[\phi^{v,w}] \in \mathcal{B}_2^c$ the w-coordinate of successive iterations increases at an increasing rate; therefore, there is a finite number of iterations m such that $R^m([\phi^{v,w}]) \in \mathcal{B}_2^d$. □

Proposition 15. $R(\mathcal{B}_2^d) = \phi^{Q_2}$.

Proof. When $(v, w) \in \mathcal{B}_2^d$ the evolution of $[\phi^{v,w}]$ is given by

$$\begin{aligned}&\xrightarrow{1}([2P, (1, 2), 0], [P, (3, 4), 0], [F, 1, 1 - v - \tau], [F, 2, 1 - w - \tau], \\ &\quad [F, (3, 4), 1 - \tau]) \\ &\xrightarrow{2}([F, (1, 2), 0], [F, (3, 4), 1 - H_1(\tau)])\end{aligned}$$

because $H_2(v + \tau) \geqslant 1$ and $H_2(w + \tau) \geqslant 1$. Comparing the last event sequence with the event sequence after transition 2 in proposition 4 we observe that they are identical. This means that the semiorbit again intersects the Poincaré section $\boldsymbol{P}$ at ϕ^{Q_2}. □

Appendix B. Some useful inequalities

Proposition 16. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ that satisfies the conditions of theorem 2 the following inequalities hold:*

$$H_2(W_2 + \tau) \geqslant 1 \tag{B.1}$$

$$\tau < H_2(w + \tau) - H_1(\tau) < H_2(v + \tau) - H_1(\tau), \qquad \text{for } \tau < w < v \tag{B.2}$$

$$v < 1 - H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau), \qquad \text{for } v > H_1(\tau) \text{ and } w > \tau \tag{B.3}$$

$$H_1(H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau) + v) \geqslant 1$$
$$\text{for } v > H_1(\tau),\ w > \tau \text{ and } A_\varepsilon w = (1 - A_\varepsilon)v - (1 - 2A_\varepsilon)H_1(\tau) \tag{B.4}$$

$$\mu - \kappa > \tau \qquad \text{for } w > \tau \tag{B.5}$$

$$1 - \kappa > \tau \qquad \text{for } v > H_1(\tau) + \frac{\tau}{A_\varepsilon(2A_\varepsilon - 1)} \tag{B.6}$$

$$H_1(\alpha + 1 - \mu + \tau) < g_2(\tau) \qquad \text{for } w > \tau \tag{B.7}$$

$$H_1(\alpha + \eta + \tau) < g_2(\tau) \qquad \text{for } w > \tau. \tag{B.8}$$

Proof. Recall that $H_1(\theta) = m_\varepsilon + A_\varepsilon\theta$ and $H_j(\theta) = H_{j-1}(H_1(\theta))$ where $A_\varepsilon > 1$ and $m_\varepsilon > 0$.

(B.1) To prove $H_2(W_2 + \tau) > 1$, it suffices to show that $1 - H_2(W_2 + \tau) < 0$. We have $1 - H_2(W_2 + \tau) = (-1 + A_\varepsilon^2)(-1 + (1 + A_\varepsilon + A_\varepsilon^2)m_\varepsilon + A_\varepsilon^2(1 + A_\varepsilon)\tau)$. Since $A_\varepsilon > 1$, by rearranging terms we need to prove, $m_\varepsilon + A_\varepsilon m_\varepsilon + A_\varepsilon^2 m_\varepsilon + A_\varepsilon^2\tau + A_\varepsilon^3\tau - 1 < 0$. This inequality follows by noting that $g_1(\tau) - 1 = m_\varepsilon + A_\varepsilon m_\varepsilon + A_\varepsilon^2 m_\varepsilon + A_\varepsilon^2\tau + 2A_\varepsilon^3\tau - 1 < 0$.

(B.2) Since $w < v$, it follows that $H_2(w + \tau) < H_2(v + \tau)$ and hence $H_2(w + \tau) - H_1(\tau) < H_2(v + \tau) - H_1(\tau)$. Since, $H_2(w + \tau) - H_1(\tau) = H_2(w + \tau) - H_1(0 + \tau)$ and $w > \tau$ we obtain that $\tau < H_2(w + \tau) - H_1(\tau)$.

(B.3) We show that $H_2(v + \tau) - H_2(w + \tau) + H_1(1 - H_2(v + \tau) + H_1(\tau) + \tau) < 1$. Expanding the left-hand side, with $w = ((1 - A_\varepsilon)v - (1 - 2A_\varepsilon)H_1(\tau))/A_\varepsilon$, we have,

$$A_\varepsilon - A_\varepsilon v + 2A_\varepsilon^2 v - A_\varepsilon^3 v + m_\varepsilon + A_\varepsilon m_\varepsilon - 3A_\varepsilon^2 m_\varepsilon + A_\varepsilon\tau + 2A_\varepsilon^2\tau - 3A_\varepsilon^3\tau.$$

Since $-A_\varepsilon + 2A_\varepsilon^2 - A_\varepsilon^3 < 0$, the above expression is a decreasing function of v. Substituting $v = H_1(\tau)$, which is the lower bound on v, then

$$A_\varepsilon + m_\varepsilon - A_\varepsilon^2 m_\varepsilon - A_\varepsilon^3 m_\varepsilon + A_\varepsilon\tau + A_\varepsilon^2\tau - A_\varepsilon^3\tau - A_\varepsilon^4\tau = g_4(\tau) < 1.$$

(B.4) The expression $H_1(H_1(1-H_2(v+\tau)+H_1(\tau)+\tau)+H_2(v+\tau)-H_2(w+\tau))$ when expanded by substituting $w = ((1-A_\varepsilon)v-(1-2A_\varepsilon)H_1(\tau))/A_\varepsilon$ and $v = H_1(\tau)+(\tau/A_\varepsilon(2A_\varepsilon - 1))$ yields $A_\varepsilon+2A_\varepsilon^2+m_\varepsilon-2A_\varepsilon^2 m_\varepsilon-3A_\varepsilon^3 m_\varepsilon-2A_\varepsilon^4 m_\varepsilon+A_\varepsilon\tau-3A_\varepsilon^4\tau-2A_\varepsilon^5\tau$. Since $g_1(\tau) < 1$, we have

$$m_\varepsilon + A_\varepsilon m_\varepsilon + A_\varepsilon^2 m_\varepsilon + A_\varepsilon^2\tau + 2A_\varepsilon^3\tau < 1$$

and therefore,

$$m_\varepsilon < (1 - (A_\varepsilon^2 + 2A_\varepsilon^3)\tau)/(1 + A_\varepsilon + A_\varepsilon^2).$$

And for $m_\varepsilon < (1 - (A_\varepsilon^2 + 2A_\varepsilon^3)\tau)/(1 + A_\varepsilon + A_\varepsilon^2)$, we have,

$$A_\varepsilon + 2A_\varepsilon^2 + m_\varepsilon - 2A_\varepsilon^2 m_\varepsilon - 3A_\varepsilon^3 m_\varepsilon - 2A_\varepsilon^4 m_\varepsilon + A_\varepsilon\tau - 3A_\varepsilon^4\tau - 2A_\varepsilon^5\tau > 1.$$

(B.5) It follows by noting that $\mu - \kappa = A_\varepsilon^2(H_1(w + \tau) - \tau) > \tau$ for $w > \tau$.

(B.6) $\kappa + \tau = A_\varepsilon - A_\varepsilon^3 v + m_\varepsilon - A_\varepsilon^2 m_\varepsilon + \tau + A_\varepsilon\tau + A_\varepsilon^2\tau - A_\varepsilon^3\tau$ is a decreasing function of v. For $v = H_1(\tau) + (\tau/A_\varepsilon(2A_\varepsilon - 1))$, which is the lower bound for v, we have $\kappa+\tau = A_\varepsilon+m_\varepsilon-A_\varepsilon^2 m_\varepsilon-A_\varepsilon^3 m_\varepsilon+\tau+A_\varepsilon\tau+A_\varepsilon^2\tau-A_\varepsilon^3\tau-A_\varepsilon^4\tau-(A_\varepsilon^2\tau/(-1+2A_\varepsilon))$. Since for $A_\varepsilon > 1$, $1-A_\varepsilon^2/(-1+2A_\varepsilon) < 0$, we have $A_\varepsilon+m_\varepsilon-A_\varepsilon^2 m_\varepsilon-A_\varepsilon^3 m_\varepsilon+\tau+A_\varepsilon\tau+A_\varepsilon^2\tau-A_\varepsilon^3\tau-A_\varepsilon^4\tau-(A_\varepsilon^2\tau/(-1+2A_\varepsilon)) < A_\varepsilon+m_\varepsilon-A_\varepsilon^2 m_\varepsilon-A_\varepsilon^3 m_\varepsilon+A_\varepsilon\tau+A_\varepsilon^2\tau-A_\varepsilon^3\tau-A_\varepsilon^4\tau = g_4(\tau) < 1$.

(B.7) It follows from noting that $H_1(\alpha + 1 - \mu + \tau) - g_2(\tau) = A_\varepsilon^4\tau - A_\varepsilon^4 w < 0$ for $w > \tau$.

(B.8) It follows from noting that $H_1(\alpha + \eta + \tau) - g_2(\tau) = A_\varepsilon^4\tau - A_\varepsilon^4 w < 0$ for $w > \tau$. □

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