IOP PUBLISHING NONLINEARITY

Nonlinearity **21** (2008) 13–49 doi:10.1088/0951-7715/21/1/002

Robustness of unstable attractors in arbitrarily sized pulse-coupled networks with delay

Henk Broer, Konstantinos Efstathiou and Easwar Subramanian

Institute of Mathematics and Computer Science, University of Groningen, PO Box 407, 9700 AK Groningen, The Netherlands

E-mail: H.W.Broer@math.rug.nl, K.Efstathiou@math.rug.nl and easwar@cs.rug.nl

Received 29 May 2007, in final form 7 November 2007
Published 3 December 2007
Online at stacks.iop.org/Non/21/13

Recommended by J A Glazier

Abstract
We consider arbitrarily large networks of pulse-coupled oscillators with non-zero delay where the coupling is given by the Mirollo–Strogatz function. We prove that such systems have unstable attractors (saddle periodic orbits whose stable set has non-empty interior) in an open parameter region for three or more oscillators. The evolution operator of the system can be discontinuous and we propose an improved model with continuous evolution operator.

Mathematics Subject Classification: 34D45, 37L30, 92B99

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this paper we study unstable attractors that appear in global networks of pulse-coupled linear oscillators with non-zero delay. Such networks are used for modelling, for example, the activity in biological neuron networks [1–4] or the synchronization processes in the flashing of fireflies [5, 6].

In order to give an example and motivate the terminology to be used later, we briefly describe the relation between the system we study and neuron networks. In a network, each neuron has a *membrane potential* which increases with time. When the potential reaches a particular *threshold*, the neuron *fires*, and the potential is reset to a lower value. As a consequence of firing, a *pulse* is sent to several other neurons. The pulse relays a positive or negative charge to the neurons and this transfer of charge defines the coupling in the system. This type of interaction, in which neurons communicate by firing sudden pulses, is called *episodic* or *pulse-coupled* [7, 8].

In the model we study, neurons are represented by linear oscillators which are connected in a *global* network, i.e. each neuron interacts with all the other neurons in the network. The membrane potential of the neuron is related to the *phase* of the oscillator through the Mirollo–Strogatz function [5]. When a neuron fires, the membrane potential of all the other oscillators is increased by a constant amount ε. In the original Mirollo–Strogatz model [5], this increase happens simultaneously with the firing. Here, following later work (for example [4, 6]), we make the more realistic assumption that there is a *time delay* τ between the moment an oscillator fires and the moment the other oscillators receive the resulting pulse and increase their membrane potentials.

For this type of global pulse-coupled network with delay it has been observed numerically [1–3] that for certain values of the parameters and for a large enough number of oscillators the network has *unstable attractors* that coexist with stable attractors. Moreover, in [4] the authors prove the existence of an unstable attractor in a network of four oscillators, for an open set of the parameters of the system. The unstable attractor in [4] corresponds to the synchronization of the four oscillators in two clusters with two oscillators in each cluster.

The main result of this paper is a proof of the following statement.

Theorem 1. *There is a class of pulse-coupled oscillator networks with delay such that if the network has $n \geqslant 3$ oscillators then there is an open non-empty parameter region in which the system has a linearly unstable attractor.*

A more detailed version of theorem 1 is given in section 3 (theorem 2). The main improvement with respect to previous works is that we establish the existence of unstable attractors for an arbitrary number of oscillators $n \geqslant 3$. In order to prove theorem 1 we introduce a *metric* in the phase space that allows us to study questions such as instability in a rigorous way. Also, our methods permit us to predict analytically the parameter regions for which these unstable attractors exist. Furthermore, we observe that the evolution operator of the system can be discontinuous. We discuss the implications of this fact for the dynamics and propose ways to alleviate the situation. As far as we know, discontinuity of the evolution operator has escaped identification in previous studies of the model and it raises questions as to what extent the particular model is appropriate for describing physical systems.

The unstable attractors studied in this paper are saddle periodic orbits, or fixed points of a suitably defined Poincaré map. In particular, they have a one-dimensional local stable manifold and an $(n-2)$-dimensional local unstable manifold. At the same time, there exists an open set of points in the phase space that converges to the attractor. The situation is presented schematically in figure 1. The attractor P is a saddle point and its stable set $W^s(P)$ contains an open set S. Initial states inside S collapse onto the local stable set $W^s_{\mathrm{loc}}(P)$ and converge to P. Since P is a saddle point there is a neighbourhood U of P such that all initial states in $U \setminus W^s_{\mathrm{loc}}(P)$ leave U after some time.

These saddle attractors do not satisfy the traditional definition of an attractor [9] which requires that the attractor is in the interior of its basin, but are consistent with the definition due to Milnor [10] which does not restrict an attractor to be stable. Given a measure μ on the phase space M, Milnor [10] defines

Definition 1 (Milnor attractor). *A closed set A is an attractor if the following two conditions are satisfied:*

(i) The set $\varrho(A)$ of all points $x \in M$ for which $\omega(x) \subset A$ has strictly positive measure. Here $\omega(x)$ is the ω-limit set of x.

(ii) There is no strictly smaller closed set $A' \subset A$ so that $\varrho(A') = \varrho(A)$ up to a set of measure 0.

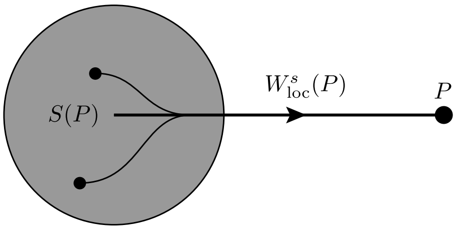


Figure 1. Schematic picture of an unstable attractor. P is a saddle point whose stable set $W^s(P)$ contains an open set S. Initial states from S collapse onto the local stable set $W^s_{\mathrm{loc}}(P)$ and converge to P. Since P is a saddle point, almost all nearby initial states leave P.

This definition implies that a point P (or a periodic orbit) is an attractor, if there is a set of positive measure that converges to P even if points near P move away. Therefore one can define the notion of an unstable attractor. We follow [4]:

Definition 2 (unstable attractor). *A Milnor attractor A is unstable if there is a neighbourhood U of A such that the measure of the set of points that stay in U for all $t \geqslant 0$ is zero.*

In other words, an attractor is unstable if there is a neighbourhood U such that almost all points in U (except a set of measure zero), leave U after some time. Notice that the definition is silent on whether these points eventually come back inside U. Depending on whether they come back or not, Ashwin and Timme [4] define the notions of unstable attractors with positive and zero measure local basin, respectively. Presently, this distinction plays no role.

In the context of this paper the requirement of definitions 1 and 2 for a set to have positive measure is replaced by the stronger requirement that they have non-empty interior.

Unstable attractors exist even in one-dimensional maps $f : [0, 1] \to [0, 1]$. Two such examples, f_1 and f_2, are given in [4] and we reproduce them in figures 2(a) and 2(b), respectively. In both cases the point $x = 1/2$ is an unstable attractor. Notice that although both of these maps are discontinuous, the discontinuity is not essential for the existence of the unstable attractor. Consider for example the continuous map f_3 depicted in figure 2(c) for which $x = 1/2$ still is an unstable attractor. That $x = 1/2$ is an attractor in all cases is explained by the fact that the graph of the one-dimensional map has a 'plateau'. In other words there is a set S of positive measure such that $f_i(S) = 1/2$ for all i. On the other hand, $x = 1/2$ is unstable because the absolute value of the slope of $f_i(x)$ in a neighbourhood of $x = 1/2$ (excluding $x = 1/2$ in the case of f_2) is larger than 1.

A different example is shown in figure 2(d), where the map f_4 is discontinuous at the attractor $x = 1/2$. In this case, points near $x = 1/2$ go away under iterations of f_4, therefore the attractor $x = 1/2$ satisfies definition 2 of an unstable attractor. In the networks studied in this paper we observe an analogous situation. We note that in this case the unstable attractor is *not* linearly unstable, since there is no linear approximation of the dynamics.

Pulse-coupled oscillator systems such as the ones considered in this paper are piecewise affine systems. A lot of work has been done on such systems, and simple particular models include the tent map defined on $\mathbb{T}^1$ [11], or the sawtooth standard map defined on $\mathbb{T}^2$ [12]. Piecewise smooth systems can also be used in order to describe mechanical systems like dry friction [13]. Although the systems that we consider here are piecewise smooth, they are not piecewise invertible. This is due to the fact that the Poincaré map of these systems has 'flat pieces', i.e. there are sets with non-empty interior in the state space that are mapped to the same state. Invertible and even piecewise invertible systems cannot have unstable attractors, although attractors with riddled basins [14–16] or Milnor attractors [17] can occur. Therefore unstable

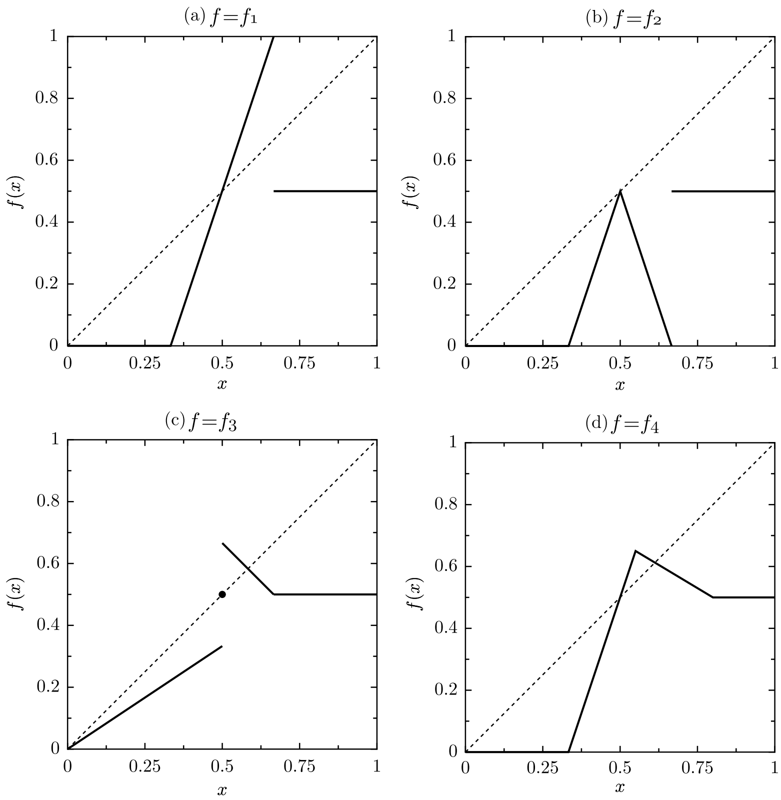


Figure 2. Unstable attractors in one-dimensional maps. Figures (*a*)-(*d*) depict one-dimensional maps $f : [0, 1] \to [0, 1]$ that have an unstable attracting fixed point at $x = 1/2$. The examples in figures (*a*) and (*b*) are from [4].

attractors are characteristic of maps with 'flat pieces'. In [4] it is shown how to construct a semiflow with unstable attractors by perturbing a smooth flow with robust heteroclinic cycles.

The paper is organized as follows. In section 2 the dynamics of the system is defined. In particular, in section 2.1 we define the state space of the system and the evolution operator. In section 2.2 we describe the Mirollo–Strogatz model. Then, in section 2.3 we define a metric for the state space and show that the evolution operator is discontinuous. In section 2.4, we discuss other representations of the dynamics of the system. In section 3, we study the unstable attractors in the system and prove the main theorem on their existence. In particular, in section 3.1 we present some numerical results on a system of three oscillators in order to demonstrate the global behaviour of the system, the occurrence of unstable attractors and the local dynamics around them. In section 3.2, we prove that for networks with $n \geqslant 3$ oscillators there is an open region of parameter values for which an open set of initial states collapses to a fixed point attractor of the return map and in section 3.3, we prove that this attractor is unstable. Finally, in section 4 we explore the role of the discontinuity of the evolution operator and we propose a model with continuous evolution operator.

2. Definition of the dynamics

The system studied in this paper is a delay system [18]. The state space of such systems is an appropriate space $\mathcal{P}_\tau^n$ of functions (see definition 3) defined on the interval $(-\tau, 0]$, where

$\tau > 0$ is the delay of the system, and taking values in an n-dimensional manifold N. The state space thus is infinite dimensional. In our case, points in N represent the phases of the n coupled oscillators, which implies that $N = \mathbb{T}^n$, the n-dimensional torus.

For a given $\phi \in \mathcal{P}_\tau^n$ and for each $t \in (-\tau, 0]$, $\phi(t) \in N$ represents the phases of the oscillators at time t. Using the dynamics of the system, ϕ can be *extended* to a unique function $\phi^+ : (-\tau, +\infty) \to N$, such that $\phi^+(t) = \phi(t)$ for $t \in (-\tau, 0]$ and $\phi^+(t) \in N$ represents the phases of the oscillators at any time $t \geqslant -\tau$. Then the *evolution operator* $\Phi^t : \mathcal{P}_\tau^n \to \mathcal{P}_\tau^n$ is defined by $\Phi^t(\phi)(s) = \phi^t(s) = \phi^+(t+s)$ for any $t \geqslant 0$ and $s \in (-\tau, 0]$. In other words, the evolution operator maps the initial state $\phi = \phi^0$ to the state ϕ^t of the system at time t. The latter is the restriction of ϕ^+ in $(t-\tau, t]$ shifted back to the interval $(-\tau, 0]$.

2.1. Pulse-coupled oscillator networks with delay

We now specialize the above notions of the theory of delay equations to the current setting. In this section we closely follow [4].

Definition 3 (state space, cf [4]). *The state space $\mathcal{P}_\tau^n$ of the system of n pulse-coupled oscillators with delay $\tau > 0$ is the space of phase history functions*

$$\phi : (-\tau, 0] \to \mathbb{T}^n : s \mapsto \phi(s) = (\phi_1(s), \ldots, \phi_n(s)),$$

that satisfy the following conditions:

(i) *Each ϕ_i is upper-semicontinuous, i.e. $\phi_i(s^+) := \lim_{t\to s^+} \phi_i(t) = \phi_i(s)$ and $\phi_i(s^-) := \lim_{t\to s^-} \phi_i(t) \leqslant \phi_i(s)$ for all $s \in (-\tau, 0]$.*

(ii) *Each ϕ_i is only discontinuous at a finite (or empty) set $S_i = \{s_{i,1}, \ldots, s_{i,k_i}\} \subset (-\tau, 0]$ with $k_i \in \mathbb{N}$ and $s_{i,1} > s_{i,2} > \cdots > s_{i,k_i}$.*

(iii) $\mathrm{d}\phi_i(s)/ds = 1$ *for $s \notin S_i$.*

The coupling between the n oscillators is defined using the *pulse response function*.

Definition 4 (pulse response function, cf [4]). *A pulse response function is a map*

$$V : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto V(\theta, \varepsilon), \tag{1}$$

that satisfies the following conditions:

(i) V *is smooth on* $(\mathbb{T} \setminus \{0\}) \times \mathbb{R}_+$.

(ii) $\partial V(\theta, \varepsilon)/\partial\theta > 0$ *on* $(\mathbb{T} \setminus \{0\}) \times (\mathbb{R}_+ \setminus \{0\})$.

(iii) $\partial V(\theta, \varepsilon)/\partial\varepsilon > 0$ *on* $\mathbb{T} \times \mathbb{R}_+$.

(iv) $V(\theta, 0) = 0$ *for all* $\theta \in \mathbb{T}$.

(v) $0 < V(0, \varepsilon) < 1$ *for all* $\varepsilon \in (0, 1)$.

(vi) H*, given by (4), satisfies*

$$H_m(\theta) = H_1 \circ H_{m-1}(\theta) = \overbrace{H_1 \circ \ldots \circ H_1}^{m\text{-times}}(\theta). \tag{2}$$

Notice that in the above definition $\partial V/\partial\theta > 0$, therefore V cannot be smooth everywhere on $\mathbb{T}$. This is reflected in condition (i) of the definition. The pulse response function depends on the parameter $\varepsilon \geqslant 0$, called *coupling strength*. As a shorthand notation we introduce

$$V_m(\theta) = V(\theta, m\hat{\varepsilon}), \qquad \text{for } m = 1, 2, 3, \ldots, \tag{3}$$

where $\hat{\varepsilon} = \varepsilon/(n-1)$. Given a pulse response function V we also define

$$H : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto H(\theta, \varepsilon) = \theta + V(\theta, \varepsilon), \tag{4}$$

and

$$H_m(\theta) = H(\theta, m\hat{\varepsilon}), \qquad \text{for } m = 1, 2, 3, \ldots. \tag{5}$$

Definition 5 (dynamics, cf [4]). *A system of n pulse-coupled oscillators with delay is a quadruple $\mathcal{D} = (n, V, \varepsilon, \tau)$, where V is as in definition 4, $\varepsilon \geqslant 0$ and $\tau \geqslant 0$. Given a system $\mathcal{D}$ and an initial state $\phi \in \mathcal{P}_\tau^n$, we extend ϕ to a function $\phi^+ : (-\tau, +\infty) \to \mathbb{T}^n$ using the following rules:*

(i) $\phi^+(t) = \phi(t)$ *for* $t \in (-\tau, 0]$.
(ii) $\mathrm{d}\phi_i^+(t)/\mathrm{d}t = 1$ *for* $t \geqslant 0$, *if* $\phi_j^+(t-\tau) \neq 0\,(\mathrm{mod}\,\mathbb{Z})$ *for all* $j \neq i$.
(iii) $\phi_i^+(t) = \min\{1, H_m(\phi_i^+(t^-))\}\,(\mathrm{mod}\,\mathbb{Z})$, *if there are* $j_1, \ldots, j_m \neq i$ *such that* $\phi_{j_k}^+(t-\tau) = 0\,(\mathrm{mod}\,\mathbb{Z})$ *for all* $k = 1, \ldots, m$.

The dynamics described in definition 5 can be interpreted in the following way. The phase ϕ_i of each oscillator O_i, $i = 1, \ldots, n$, increases linearly. When the phase reaches the value $1 = 0\,(\mathrm{mod}\,\mathbb{Z})$, then the oscillator O_i *fires* and all the other oscillators O_j, $j \neq i$ receive a *pulse* after a time *delay* τ. In general, an oscillator O_j may receive m simultaneous pulses at time t if m oscillators $O_{i_1}, \ldots, O_{i_m}$ have fired simultaneously at time $t - \tau$. Then the phase of O_j is increased to $H(u_j, m\hat{\varepsilon}) = H_m(u_j)$ where $u_j = \phi_j^+(t^-)$, unless the pulse causes the oscillator to fire and then the phase becomes exactly 1.

The *evolution operator* Φ^t for $t \geqslant 0$ is then defined by

$$\Phi^t : \mathcal{P}_\tau^n \to \mathcal{P}_\tau^n : \phi \mapsto \Phi^t(\phi) = \phi^t = \phi^+|_{(t-\tau,t]} \circ T_t, \tag{6}$$

where T_t is the shift $s \mapsto s + t$ and the *positive semiorbit* of $\phi \in \mathcal{P}_\tau^n$ is given by

$$\mathcal{O}_+(\phi) = \{\Phi^t(\phi) : t \geqslant 0\}. \tag{7}$$

Proposition 1. *The evolution operator Φ^t is well defined.*

Proof. From definition 5 it follows that the extended function ϕ^+ can be determined for all $t \geqslant 0$ and all $\phi \in \mathcal{P}_\tau^n$, given $\mathcal{D}$ and $\phi \in \mathcal{P}_\tau^n$. The only question is whether $\phi^t \in \mathcal{P}_\tau^n$ for all $t \geqslant 0$. First we show that ϕ^t, $t \geqslant 0$ is discontinuous at a finite set. Note that, by definition 3, each component ϕ_i of ϕ has only a finite number k_i of discontinuities in $(-\tau, 0]$. Therefore, $\phi_i(0) < \phi_i(-\tau) + k_i + \tau$, since the phase ϕ_i increases linearly (outside discontinuities) and each discontinuity induces an increase of ϕ_i that is less than 1. This implies that $\phi_i(s) = 0\,(\mathrm{mod}\,\mathbb{Z})$ in a finite set $\{s_{i,1}, \ldots, s_{i,\ell_i}\} \subset (-\tau, 0]$ with ℓ_i elements. Then, the number of discontinuities of ϕ_j^+, in $(0, \tau]$ (and hence of ϕ_j^τ) also is finite for all $j = 1, \ldots, n$. This follows from the fact that the number of discontinuities of ϕ_j^+ in $(0, \tau]$ is less than or equal to the number of firings of all the ϕ_i in $(-\tau, 0]$ for $i = 1, \ldots, n$ and $i \neq j$, therefore it is less than or equal to $\sum_{i=1}^n \ell_i$. This shows that advancing time by τ the number of discontinuities remains finite. It follows by induction that the number of discontinuities of ϕ_i^+ in any interval $((m-1)\tau, m\tau]$, $m \in \mathbb{N}$ is finite for all $i = 1, \ldots, n$. Thus, the number of discontinuities of ϕ_i^+ in any interval $(t - \tau, t]$, $t \geqslant 0$ (and hence of ϕ_i^t) is finite for all $i = 1, \ldots, n$. The facts that ϕ^t is upper-semicontinuous and $\mathrm{d}\phi_i^t/\mathrm{d}s = 1$ (outside discontinuities) are a direct consequence of properties (iii) and (ii), respectively, of definition 5. □

For a given system $\mathcal{D} = (n, V, \varepsilon, \tau)$, the *accessible state space* is $\mathcal{P}_\mathcal{D} = \Phi^\tau(\mathcal{P}_\tau^n)$. In other words, $\phi \in \mathcal{P}_\mathcal{D}$ if there is a state $\psi \in \mathcal{P}_\tau^n$ such that $\Phi^\tau(\psi) = \phi$, i.e. $\mathcal{P}_\mathcal{D}$ includes only those states that are dynamically accessible. From now on, we restrict our attention to $\mathcal{P}_\mathcal{D}$.

2.2. The Mirollo–Strogatz model

A pulse response function V that satisfies all the requirements of definition 4 is provided by the Mirollo–Strogatz model [5] where the pulse response function is

$$V_{\mathrm{MS}}(\theta, \varepsilon) = f^{-1}(f(\theta) + \varepsilon) - \theta, \tag{8}$$

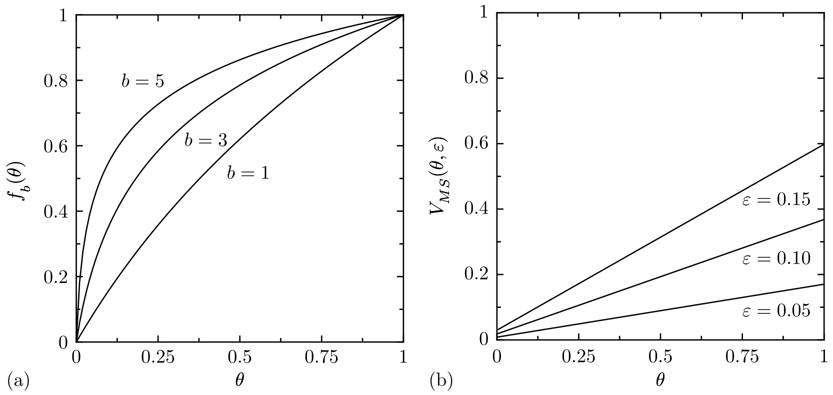


Figure 3. (*a*) Graph of f_b (9) as a function of θ for different values of b. (*b*) Graph of V_{MS} (10) as a function of θ for $f = f_b$, $b = 3$ and different values of ε.

and f is a function which is concave down ($f'' < 0$) and monotonically increasing ($f' > 0$). Moreover, $f(0) = 0$ and $f(1) = 1$. A concrete example is given by

$$f_b(\theta) = \frac{1}{b}\ln(1 + (\mathrm{e}^b - 1)\theta). \tag{9}$$

We present a sketch of the function f_b for various values of b in figure 3(*a*). For any given positive value of ε, the pulse response function $V_{\mathrm{MS}}(\theta, \varepsilon)$ for $f = f_b$ as in (9) is affine:

$$V_{\mathrm{MS}}(\theta, \varepsilon) = \upsilon_\varepsilon + K_\varepsilon\theta, \tag{10}$$

where $\upsilon_\varepsilon = (\mathrm{e}^{b\varepsilon} - 1)/(\mathrm{e}^b - 1)$ and $K_\varepsilon = \mathrm{e}^{b\varepsilon} - 1$. The graph of V_{MS} (10) is depicted in figure 3(*b*) for different values of ε.

In the numerical computations in this paper, we use the Mirollo–Strogatz model with f_b as in (9) with fixed $b = 3$. After fixing b, the parameter space of the system is $\{(\varepsilon, \tau) : \varepsilon > 0, \tau > 0\} = \mathbb{R}^2_+$ where we recall that τ is the *delay* and ε is the *coupling strength*.

The qualitative results of our analysis depend only on the properties of the pulse response function V given in definition 4 and not on the specific choice of the Mirollo–Strogatz model (8) nor on the choice $f = f_b$ (9).

2.3. Metric

We introduce a metric d on $\mathcal{P}_\mathcal{D}$ followed by a brief study on the continuity of the evolution operator Φ^t with respect to d. Recall that given a phase history function $\phi \in \mathcal{P}^n_\tau$, we can define the extended phase history function ϕ^+.

We define a *lift* [19] of an extended phase history function ϕ^+ as any function $L_\phi : (-\tau, +\infty) \to \mathbb{R}^n$ such that

(i) $L_\phi(s) \bmod \mathbb{Z}) = \phi^+(s)$ and
(ii) for any $s \in (-\tau, +\infty)$ and for $i = 1, \ldots, n$,

$$(L_\phi)_i(s) - (L_\phi)_i(s^-) = \phi^+_i(s) - \phi^+_i(s^-).$$

It follows from these properties that if $L^{(1)}_\phi$ and $L^{(2)}_\phi$ are two lifts of the same extended phase history function ϕ^+ then they differ by a constant integer vector, i.e. $L^{(1)}_\phi(s) - L^{(2)}_\phi(s) = k \in \mathbb{Z}^n$, for all $s \in (-\tau, \infty)$.

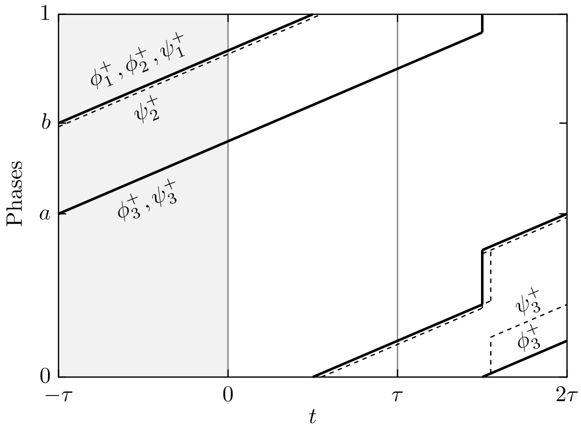


Figure 4. An initial state $\phi \in \mathcal{P}_\mathcal{D}$ for which the map Φ^τ is discontinuous. The graphs of ϕ_j^+, $j = 1, 2, 3$ are represented by the solid lines. The graphs of ψ_j^+, $j = 1, 2, 3$ where ψ^+ is the extended phase history function that corresponds to the initial state ψ are represented by the dashed lines. Recall that ϕ (respectively, ψ) is the restriction of ϕ^+ (respectively, ψ^+) to $(-\tau, 0]$ (represented by the grey region). The two states diverge abruptly after $t = 3\tau/2$. On the vertical axis, $a = \vartheta - \frac{5}{2}\tau$ and $b = 1 - \frac{3}{2}\tau$.

Definition 6 (metric on $\mathcal{P}_\mathcal{D}$). *The metric $d : \mathcal{P}_\mathcal{D} \times \mathcal{P}_\mathcal{D} \to \mathbb{R}$ is given by*

$$\mathrm{d}(\phi, \psi) = \min_{k \in \mathbb{Z}^n} \sum_{i=1}^{n} \int_{-\tau}^{\tau} |(L_\phi)_i(s) - (L_\psi)_i(s) - k_i| \mathrm{d}s, \tag{11}$$

where L_ϕ and L_ψ are arbitrary lifts of ϕ and ψ, respectively.

Remark 1. Because of the delay τ, the distance in $\mathbb{T}^n$ between the points $\phi(0)$ and $\psi(0)$ is not a suitable metric for this system. Instead, it is important to take into account the values of ϕ and ψ at least in the interval $(-\tau, 0]$. Nevertheless, if the integral in (11) runs from $-\tau$ to 0 thus defining a metric d', there are several states at which the evolution operator is discontinuous with respect to d'. For the chosen metric d in (11) the only states for which the evolution operator is discontinuous are those that are related to the *overfiring* effect which is discussed later in this section.

2.3.1. Discontinuity of the evolution operator. In general, the evolution operator $\Phi^t : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{D}$ is not continuous for all $t \geqslant 0$. We demonstrate this by a simple example. Consider a system of $n = 3$ oscillators and the initial state ϕ given by

$$\phi_1(s) = \phi_2(s) = 1 - \tfrac{1}{2}\tau + s,$$
$$\phi_3(s) = \vartheta - \tfrac{3}{2}\tau + s,$$

for $s \in (-\tau, 0]$, where $\vartheta \in (0, 1)$ is close enough to 1, so that $H_1(\vartheta) > 1$ which also implies that $H_2(\vartheta) > 1$. Following the rules of definition 5 we extend ϕ to a function ϕ^+ defined on $(-\tau, 2\tau]$. The graphs of ϕ_1^+, ϕ_2^+ and ϕ_3^+ are depicted in figure 4 with the solid lines. Recall that $\phi^+|_{(-\tau,0]} = \phi$. The most important thing to note is that the oscillator O_3 receives two simultaneous pulses at $t_f = 3\tau/2$ while $\phi_3^+(t_f^-) = \vartheta$. Therefore, $\phi_3^+(t_f) = \min\{1, H_2(\vartheta)\} \pmod{\mathbb{Z}} = 0$.

Then, consider an initial state ψ given by $\psi_1(s) = \phi_1(s)$, $\psi_2(s) = \phi_2(s) - \epsilon$ and $\psi_3(s) = \phi_3(s)$ for $s \in (-\tau, 0]$, where $\epsilon > 0$ is small. The distance between ϕ and ψ is

$$d(\phi, \psi) = 2\tau\epsilon = O(\epsilon).$$

The graphs of the components ψ_1^+, ψ_2^+ and ψ_3^+ of the extended phase history function ψ^+ are depicted in figure 4 with the dashed lines. The main difference between ϕ and ψ is that while in the former the oscillators O_1 and O_2 are synchronized, in the latter they are not. For this reason, the oscillator O_3 receives a single pulse at $t = t_f = 3\tau/2$ while $\psi_3^+(t_f^-) = \vartheta$. Then, $\psi_3^+(t_f) = \min\{1, H_1(\vartheta)\} \pmod{\mathbb{Z}} = 0$, since as we showed before $H_1(\vartheta) > 1$. O_3 receives a second pulse at $t = t_f + \epsilon$ while $\psi_3^+(t_f + \epsilon^-) = \epsilon$. Hence its phase becomes $\psi_3^+(t_f + \epsilon) = V_1(\epsilon) = V_1(0) + O(\epsilon)$.

Therefore, for $s \in (3\tau/2 + \epsilon, 2\tau]$, we have that $\psi_3^+(s) - \phi_3^+(s) = V_1(0) + O(\epsilon)$ where $V_1(0) > 0$ does not depend on ϵ. Then, it is easy to see that

$$\mathrm{d}(\Phi^\tau(\psi), \Phi^\tau(\phi)) = \tfrac{1}{2}\tau V_1(0) + O(\epsilon).$$

This shows that the evolution operator Φ^τ is discontinuous at ϕ with respect to the metric d.

We conjecture that the discontinuity of the evolution operator is independent of the choice of a 'reasonable' metric but depends only on the dynamics of the system and, in particular, on the fact that in the example above O_3 fires by receiving two simultaneous pulses but could have fired after receiving a single pulse. Also this discontinuity should not be confused with the fact that the phases of the oscillators are discontinuous functions of time.

Motivated by the previous discussion we introduce the following definition.

Definition 7. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ and $\theta \in \mathbb{T}$, we define $\nu(\theta)$ as the minimum positive integer for which $H_{\nu(\theta)}(\theta) := H(\theta, \frac{\nu(\theta)}{n-1}\varepsilon) \geqslant 1$.*

In other words, $\nu(\theta)$ is the minimum number of pulses that will make an oscillator with phase θ fire. Consider an oscillator whose phase at time t^- is θ and fires after receiving m pulses at t. We say that the oscillator *overfires by $m - \nu(\theta)$ pulses* at t if $\nu(\theta) < m$, i.e. if the oscillator fires after receiving more simultaneous pulses than the strictly necessary number $\nu(\theta)$.

Restating remark 2.3.1 in these terms, we can say that the map $\Phi^\tau : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{D}$ is discontinuous at $\phi \in \mathcal{P}_\mathcal{D}$ because the oscillator O_3 overfires by 1 pulse at $t = t_f = 3\tau/2$. In section 4 we discuss possible ways to modify the system so that the evolution operator becomes continuous.

The existence of discontinuous evolution has been observed in [3, 20] where initial states that give such evolution are characterized as *superunstable*.

2.4. *Other representations of the dynamics*

It is often useful in what follows to use alternative representations of the dynamics. In this section we introduce, following [4], the *past firings* and the *event* representation.

2.4.1. The past firings representation. It follows from definition 5 that the evolution of an initial state $\phi \in \mathcal{P}_\mathcal{D}$ depends only on the values $\phi_i(0)$ and the firing sets $\Sigma_i(\phi)$ that are defined as follows:

Definition 8. *Given a phase history function $\phi \in \mathcal{P}_\mathcal{D}$, the firing sets $\Sigma_i(\phi) \subset (-\tau, 0]$, $i = 1, \ldots, n$ are the sets of solutions of the equation $\phi_i(s) = 0$ for $s \in (-\tau, 0]$. The total firing set is the disjoint union*

$$\Sigma(\phi) = \bigsqcup_{i=1}^{n} \Sigma_i(\phi) = \{(i, \sigma) : \sigma \in \Sigma_i(\phi), i = 1, \ldots, n\}.$$

Therefore, if we are interested only in the future evolution of the system we can consider the following equivalence relation in $\mathcal{P}_{\mathcal{D}}$.

Definition 9. *Two phase history functions ϕ_1, ϕ_2 in $\mathcal{P}_{\mathcal{D}}$ are equivalent, denoted by $\phi_1 \sim \phi_2$, if $\phi_1(0) = \phi_2(0)$ and $\Sigma(\phi_1) = \Sigma(\phi_2)$. Let $\mathbb{P}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}}/\sim$ the quotient set of equivalence classes and denote by $[\phi] \in \mathbb{P}_{\mathcal{D}}$ the equivalence class of $\phi \in \mathcal{P}_{\mathcal{D}}$.*

Points $[\phi] \in \mathbb{P}_{\mathcal{D}}$ are completely determined by the values of the phases $\phi_i(0)$ and the firing sets $\Sigma(\phi)$ (which may be empty). We denote the elements of $\Sigma_i(\phi)$ by $\sigma_{i,1} > \sigma_{i,2} > \cdots > \sigma_{i,k_i}$ where k_i is the cardinality of $\Sigma_i(\phi)$. Remark that by definition, $\phi_i(0) \geqslant \sigma_{i,1}$, and $\phi_i(0) = 0$ if and only if $\sigma_{i,1} = 0$.

It is possible to give an equivalent description of the dynamics described by definition 5, using only the variables $\phi_i(0)$ and $\sigma_{i,j}$. For such a definition see [4]. Also notice the following proposition.

Proposition 2. *If $\phi_1 \sim \phi_2$ then*

(i) $\Phi^t(\phi_1) \sim \Phi^t(\phi_2)$ *for* $t \geqslant 0$ *and*
(ii) $\Phi^t(\phi_1) = \Phi^t(\phi_2)$ *for* $t \geqslant \tau$.

2.4.2. Poincaré map. Given a network of n oscillators with dynamics defined by the pulse response function V, the pulse strength ε and the delay τ, we can simplify the study of the system $\mathcal{D} = (n, V, \varepsilon, \tau)$ by considering intersections of the positive semiorbits $\mathcal{O}_+(\phi)$ with the set

$$\boldsymbol{P} = \{\phi \in \mathcal{P}_{\mathcal{D}} \,:\, \phi_n(0) = 0\}. \tag{12}$$

The set $\boldsymbol{P}$ is called a (Poincaré) *surface of section* [21, 22]. We make $\boldsymbol{P}$ a metric space by restricting the metric d, see (11).

The evolution operator Φ, see (6), defines a map $R : \boldsymbol{P} \to \boldsymbol{P}$ in the following way. Consider any $\phi \in \boldsymbol{P}$, i.e. such that $\phi_n(0) = 0$. Since the phases of the oscillators are always increasing there is a minimum time $t(\phi)$ such that the phase of O_n becomes 0 again, i.e. such that $\Phi^{t(\phi)}(\phi)_n(0) = 0$. We define

$$R(\phi) = \Phi^{t(\phi)}(\phi). \tag{13}$$

The map R is called a *Poincaré map* or *return map*. Furthermore, we can define the quotient map

$$R_\sim : \boldsymbol{P}/\sim \to \boldsymbol{P}/\sim : [\phi] \mapsto [R(\phi)],$$

of the return map R, where $\sim$ is the equivalence relation given by definition 9. By proposition 2 the map $R_\sim$ is well defined.

2.4.3. The event representation. Given a phase history function ϕ, the firing sets $\Sigma_i(\phi) = \{\sigma_{i,1}, \ldots, \sigma_{i,k_i}\}$ describe at which moments in the interval $(-\tau, 0]$ the oscillator O_i fires. Hence, they also describe at which instants in the interval $(0, \tau]$ the oscillators O_ℓ, $\ell \neq i$ would receive a pulse from O_i, making ϕ_ℓ^+, $\ell \neq i$ discontinuous at $\sigma_{i,j} + \tau \in (0, \tau]$, for $j = 1, \ldots, k_i$. Also, notice that if the phase of the oscillator O_i at time 0 is $\phi_i(0)$ then the oscillator will fire after time $1 - \phi_i(0)$, unless it receives a pulse before it fires. From this point of view, the numbers $\sigma_{i,j}$ and $\phi_i(0)$ where $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$ give information about events that are going to happen in the system.

We can take advantage of this point of view in order to construct a representation of the dynamics that is well suited for implementation in a computer program. This *event representation* is a symbolic description of the dynamics in which the state of the system is represented by a sequence of events (pulse receptions and firings) that are going to happen. Each event E in the sequence is characterized by a triplet $[K(E), O(E), T(E)]$. Here $K(E)$ denotes the type of the event F or mP where F stands for firing event and mP (m a natural number) stands for the simultaneous reception of m pulses. $O(E) \in \{1, \dots, n\}$ denotes the index of the oscillator associated with the event, i.e. the number of the oscillator which is going to fire or receive pulses. Finally, $T(E) \in [0, 1]$ denotes after how much time the event is taking place. For example, the event denoted by $[F, 2, 0.4]$ signifies that the oscillator O_2 will fire after time 0.4 (and this means that its current phase is $1 - 0.4 = 0.6$), while the event denoted by $[P, 1, 0.3]$ signifies that O_1 will receive a pulse after time 0.3. We use the shorthand notation $[F, (i_1, \dots, i_k), t]$ and $[mP, (i_1, \dots, i_k), t]$ to indicate that the oscillators $O_{i_1}, \dots, O_{i_k}$ fire or receive m pulses, respectively, after time t.

Given a particular initial state $\phi \in \mathcal{P}_\mathcal{D}$, such that its equivalence class $[\phi] \in \mathbb{P}_\mathcal{D}$ is characterized by phases $\phi_i(0)$ and firing times $\sigma_{i,j}$ for $i = 1, \dots, n$ and $j = 1, \dots, k_i$, consider the space $\mathcal{A}$ of event sequences $\{E_1, E_2, \dots, E_k\}$ of finite (but not fixed) length and the map

$$\mathcal{E} : \mathbb{P}_\mathcal{D} \to \mathcal{A} : [\phi] \to \mathcal{E}([\phi]), \tag{14}$$

which maps $[\phi]$ to the event sequence $\mathcal{E}([\phi])$ constructed in the following way. First, consider the set Y consisting of the following events:

(i) $[F, i, 1 - \phi_i(0)]$ for $i = 1, \dots, n$ and
(ii) $[P, \ell, \tau + \sigma_{i,j}]$ for $\ell = 1, \dots, n$ with $\ell \neq i$.

Then, impose time ordering on Y (i.e. order the events so that events that occur earlier appear first) and in the case that there are $m > 1$ identical events $[P, i, t]$ collect them together to $[mP, i, t]$ to obtain $\mathcal{E}([\phi])$. It follows that $\mathcal{E}$ is injective and the inverse map $\mathcal{E}^{-1} : \mathcal{E}(\mathbb{P}_\mathcal{D}) \subset \mathcal{A} \to \mathbb{P}_\mathcal{D}$ is well defined.

Next, define the map

$$\Phi_\mathcal{A} : \mathcal{E}(\mathbb{P}_\mathcal{D}) \to \mathcal{E}(\mathbb{P}_\mathcal{D}) \tag{15}$$

using the following algorithm:

(i) For $Z \in \mathcal{E}(\mathbb{P}_\mathcal{D})$, consider the first event E_1 of Z and let $\mathfrak{t} = T(E_1)$. If $T_1 \neq 0$ then set $T(E)$ to $T(E) - \mathfrak{t}$ for all $E \in Z$.
(ii) Take the sequence Z_0 of events $E \in Z$ with $T(E) = 0$ and define $Z_+ = Z \setminus Z_0$. For each event $E \in Z_0$ do the following:
  (a) If $K(E) = F$, then
    1. append to Z_+ the event $[F, O(E), 1]$;
    2. append to Z_+ the events $[P, \ell, \tau]$ for all $\ell \in \{1, \dots, n\}$ with $\ell \neq O(E)$.
  (b) If $K(E) = mP$, then
    1. find the (unique) event $E' \in Z_+$ with $K(E') = F$ and $O(E') = O(E)$;
    2. set $T(E')$ to $\max\{T(E') - V(1 - T(E'), m\hat{\varepsilon}), 0\}$.
(iii) Impose time ordering on Z_+ and collect together identical pulse events.
(iv) Set $\Phi_\mathcal{A}(Z) = Z_+$.

Proposition 3 follows from the definition of $\Phi_\mathcal{A}$.

Proposition 3. *(i) The map $\Phi_\mathcal{A} : \mathbb{P}_\mathcal{D} \to \mathcal{E}(\mathbb{P}_\mathcal{D})$ is well defined.*
(ii) $[\Phi^\mathfrak{t}(\phi)] = \mathcal{E}^{-1}(\Phi_\mathcal{A}(Z))$ where $Z = \mathcal{E}([\phi])$ and $\mathfrak{t}$ is determined at the first step of the algorithm.

(iii) *Consider an initial state* $\phi \in \mathcal{P}_{\mathcal{D}}$ *and the corresponding event sequence* $\mathcal{E}([\phi])$*. If we apply* $\Phi_{\mathcal{A}}$*, m times to* $\mathcal{E}([\phi])$ *and the time that elapses at the* j*th* ($j = 1, \ldots, m$) *application is* $\mathfrak{t}_j$ *with* $\mathfrak{t} = \sum_j \mathfrak{t}_j$*, then it is possible to reconstruct* ϕ^+ *on the interval* $[0, \mathfrak{t}]$.

The last part of proposition 3 implies that if $\mathfrak{t} \geqslant \tau$ then it is possible to obtain from the sequence $\{Z, \Phi_{\mathcal{A}}(Z), \Phi^2_{\mathcal{A}}(Z), \ldots, \Phi^m_{\mathcal{A}}(Z)\}$, where $Z = \mathcal{E}([\phi])$, not only the equivalence class $[\Phi^t(\phi)]$ but also the phase history function $\Phi^t(\phi) = \phi^+|_{(t-\tau,t]} \circ T_t$ for any time $t \in [\tau, \mathfrak{t}]$.

3. The unstable attractor

First, we introduce the functions

$$g_1(\tau) = H_1(1 + \tau + H_{n-2}(\tau) - H_{n-1}(2\tau)), \tag{16a}$$

$$g_2(\tau) = H_{n-1}(2\tau), \tag{16b}$$

$$g_3(\tau) = H_1(\tau + H_{n-2}(\tau)). \tag{16c}$$

Notice that in our notation we suppress the dependence of g_i, for $i = 1, 2, 3$, on the coupling strength ε and the pulse response function V.

Now, we can restate the main result of this paper (theorem 1) in more detail using the terminology introduced in section 2.

Theorem 2. *The Poincaré map* $R : \boldsymbol{P} \to \boldsymbol{P}$ *of a system* $\mathcal{D} = (n, V, \varepsilon, \tau)$ *(see definition 5) such that*

(i) $n \geqslant 3$,
(ii) V *is given by the Mirollo–Strogatz model (section 2.2),*
(iii) $g_1(\tau) < 1$,
(iv) $g_3(\tau) < 1$,
(v) $\nu(T(\tau)) = n - 1$, *where* $T(\tau) = 1 + 2\tau - H_1(\tau + H_{n-2}(\tau))$,

has a linearly unstable attractor $\phi^P \in \boldsymbol{P}$*. For any fixed values* $b > 0$ *and* $n \geqslant 3$ *the conditions (i)–(v) define an open non-empty set in the parameter space* (ε, τ).

In section 3.1 we illustrate numerically the theorem in the case of $n = 3$ oscillators. The proof of the theorem is then given in section 3.2 where we prove the existence of the attractor and in section 3.3 where we prove its instability.

3.1. Numerical evidence of the unstable attractor in networks with $n = 3$ *oscillators*

We consider a network of $n = 3$ oscillators with the dynamics described in section 2.1 and the Mirollo–Strogatz pulse response function V_{MS}, see (8). We set $b = 3$ and restrict our attention to the parameter region $0 < \varepsilon < 0.3$ and $0 < \tau < 0.3$. The purpose of the numerical computations in this section is to give a global picture of the different attractors and their basins for different values of parameters, and investigate the question of the existence of unstable attractors.

Recall that the permissible state space $\mathcal{P}_{\mathcal{D}}$ of the system is infinite dimensional. We restrict our attention to the Poincaré section (section 2.4.2) given by $\boldsymbol{P} = \{\phi \in \mathcal{P}_{\mathcal{D}} : \phi_3(0) = 0\}$ and we study the Poincaré map R defined by the intersections of the positive semiflow with $\boldsymbol{P}$. Moreover, we consider phase history functions up to the equivalence defined in section 2.4.1; in other words we consider $R_\sim : \boldsymbol{P}/\sim \to \boldsymbol{P}/\sim$. In order to make the task of the numerical investigation tractable we consider only initial states that contain the minimal number of the past firings in the time interval $(-\tau, 0]$.

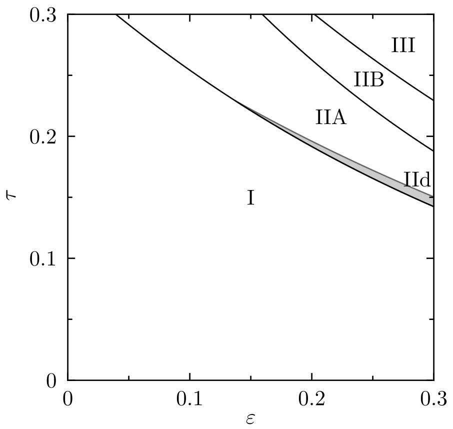


Figure 5. Regions I, IIA, IIB and III in the parameter space for the system of three oscillators with the Mirollo–Strogatz model with $f = f_b$, $b = 3$. Region IId is represented by the thin light grey strip inside region IIA. In regions I and III an open set is mapped to a stable fixed point attractor. In regions IIA and IIB an open set is mapped in a finite number of iterations to an unstable fixed point attractor.

We denote $\phi_1(0) = \theta_1$ and $\phi_2(0) = \theta_2$. Then, if $\theta_1, \theta_2 \geqslant \tau$, the initial states in the event representation (section 2.4.3) have the general form

$$\mathcal{E}([\phi]) = [P, (1, 2), \tau], [F, 1, 1 - \theta_1], [F, 2, 1 - \theta_2], [F, 3, 1]$$

up to time ordering. If $\theta_1 < \tau$ then O_1 has fired in the interval $[-\theta_1, 0)$. We make the extra assumption that in this case O_1 has fired exactly at time $-\theta_1$. Therefore, we add to $\mathcal{E}([\phi])$ the events $[P, (2, 3), \tau - \theta_1]$. Similarly, if $\theta_2 < \tau$ we add to $\mathcal{E}([\phi])$ the events $[P, (1, 3), \tau - \theta_2]$. In each case we time-order $\mathcal{E}([\phi])$.

We denote this particular set of initial states by $\mathcal{S}$ and we parametrize it by $\theta_1, \theta_2 \in [0, 1]^2$. In terms of the past firings representation (section 2.4.1) the initial state is given by $\theta_3 = 0$, $\sigma_{3,1} = 0$, $\sigma_{i,1} = (-\theta_i$ if $\theta_i < \tau$; undefined if $\theta_i \geqslant \tau)$ for $i = 1, 2$.

The square $[0, 1]^2$ is scanned with a step size of 10^{-3} in the phases θ_1 and θ_2 and for each point $(\theta_1, \theta_2) \in [0, 1]^2$ we identify the attractor to which the corresponding orbit converges. Notice, that in general, for a point $p \in \mathcal{S}$ represented by the phases (θ_1, θ_2) its image $R(p)$ might not belong in $\mathcal{S}$ and therefore the Poincaré map $R : \boldsymbol{P} \to \boldsymbol{P}$ cannot be restricted to $\mathcal{S}$. It also implies that one should be careful when interpreting pictures of the dynamics on $[0, 1]^2$ since these are just projections from $\boldsymbol{P}$ to $\mathcal{S}$.

In the numerical simulations, we initially observe four different types of qualitative behaviour of the system in different parameter regions which are labelled as I, IIA, IIB and III in figure 5. In all the cases we observe the existence of a set S of non-empty interior, which is mapped in one iteration to a single point $R(S)$ on the line $\theta_1 = \theta_2$. This means that points in S become synchronized in one iteration of the Poincaré map R. Regions I, II (which consists of subregions IIA, IIB) and III in figure 5 are distinguished based on whether $R(S)$ belongs in S or not. Also, in all cases there exists a point P with $\theta_1 = \theta_2 = H_{n-2}(\tau)$ which is a saddle fixed point of R. This corresponds to a periodic orbit $\hat{P}$ in the state space $\mathcal{P}_{\mathcal{D}}$.

We take parameter values $\varepsilon = 0.25$ and $\tau = 0.15$ representing region I and the results are shown in figure 6(a). Observe that there is an open set S which is mapped in one iteration to a single point $R(S)$ which lies inside S. This implies that $R(S)$ is a fixed point attractor of R and is obviously stable since neighbouring points are mapped to it. There is also another

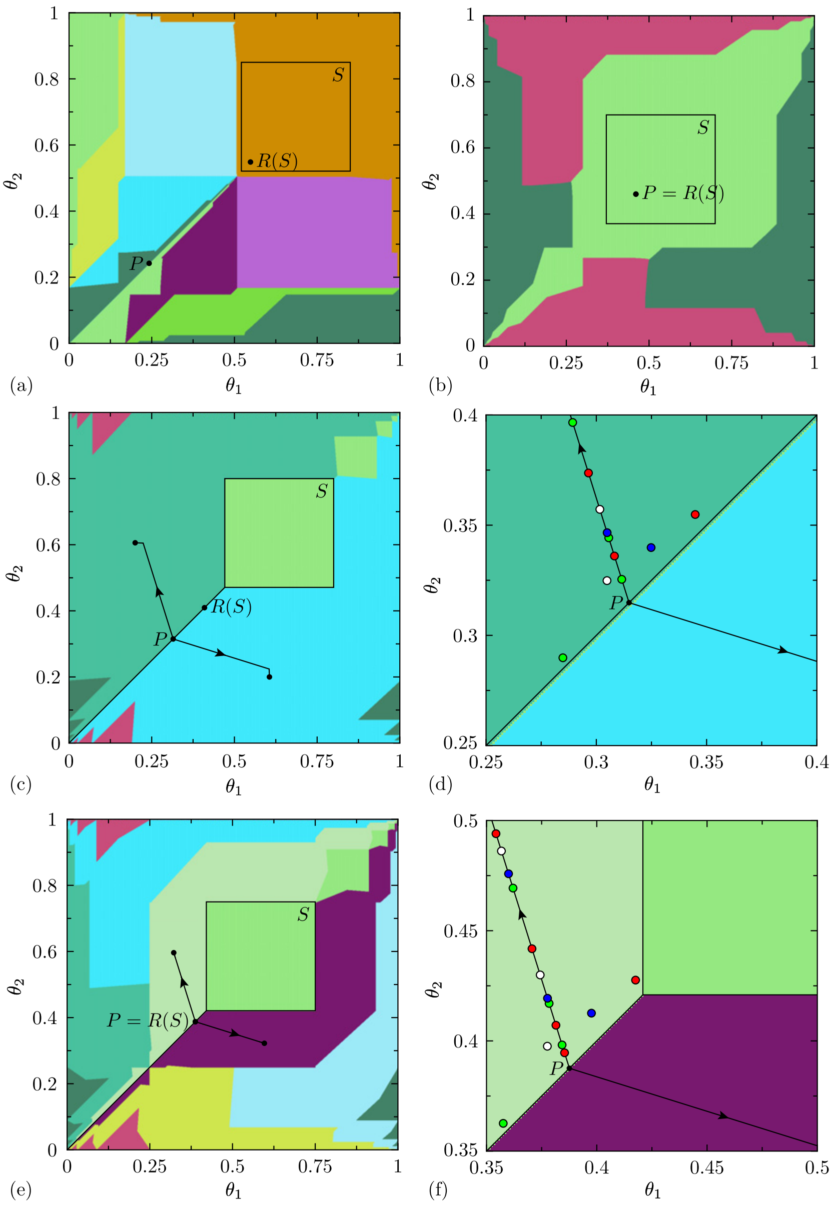


Figure 6. Basins of attraction for the system of three oscillators with the Mirollo–Strogatz model with $f = f_b$, $b = 3.0$ and $\varepsilon = 0.25$. In all cases we identify the square S that is mapped in one iteration to the single point $R(S)$. (*a*) Parameter region I, $\tau = 0.15$. (*b*) Region III, $\tau = 0.30$. (*c*) Region IIA, $\tau = 0.20$. (*d*) A detail of (*c*) in which we show four points close to P that are mapped on the unstable manifold in one iteration. (*e*) Region IIB, $\tau = 0.25$. (*f*) A detail of (*e*) that shows the local dynamics.

fixed point P but only a set of measure zero converges to it. In this case we do not observe any unstable attractors.

In region III we consider parameter values $\varepsilon = 0.25$, $\tau = 0.3$ and the results are shown in figure 6(*b*). Here the region S is mapped in one iteration to the single point $R(S)$ inside S. In this case we again have a stable fixed point attractor.

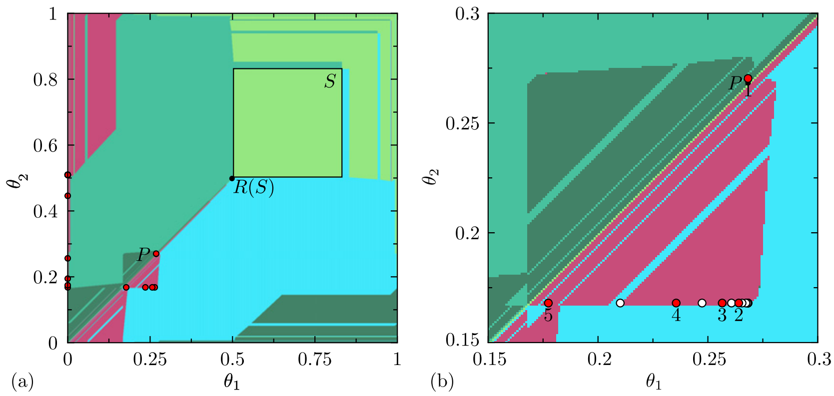


Figure 7. (a) The basins of attraction in region IId for $\varepsilon = 0.25$ and $\tau = 0.168$. (b) Discontinuity of the Poincaré map. Two orbits that begin near P are shown. The first orbit begins at distance 10^{-6} from P and it is marked with white points. The second begins at distance 2×10^{-3} from P and is marked with filled points which are numbered 1, 2, 3 etc. In both cases the value of θ_2 changes abruptly from a value close to $H_1(\tau)$ to a value close to τ.

For region IIA, we consider the parameter values $\varepsilon = 0.25$ and $\tau = 0.20$ (figure 6(c)). The initial states in S are mapped in one iteration to the point $R(S)$ outside S on the diagonal $\theta_1 = \theta_2$. Next, in a *finite number of iterations* the point $R(S)$ is mapped to the fixed point P. Moreover, we numerically observe that points on the diagonal $\theta_1 = \theta_2$ that are close to P are mapped to P in one iteration of R. This means that part of the diagonal $\theta_1 = \theta_2$ belongs to the stable manifold $W^s(P)$ of P.

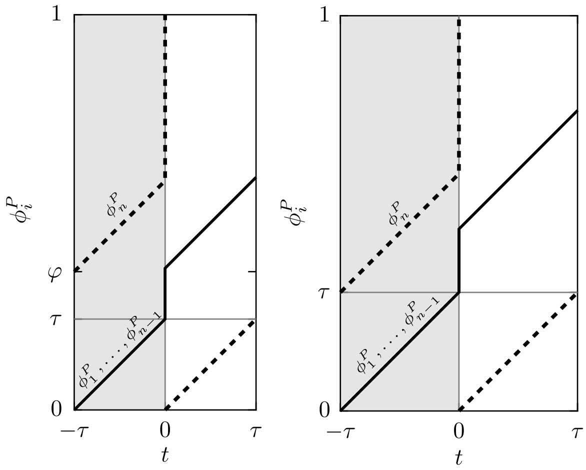
On the other hand, points close to P but not on the diagonal $\theta_1 = \theta_2$ converge to another attractor of the system. This shows that P is an unstable attractor. The exact behaviour of points close to P and outside the diagonal is shown in figure 6(d). In this figure we consider four different points close to P. We observe that in one iteration all points are mapped on the same line $W^u(P)$ which is invariant under R. After the points are mapped on $W^u(P)$ they move away from P and eventually they converge to another attractor. This means that $W^u(P)$ is the unstable invariant manifold of the saddle point P.

We observe exactly the same phenomena in region IIB. We depict region IIB for parameter values $\varepsilon = 0.25$ and $\tau = 0.25$ in figure 6(e). The only difference from the previous case is that now the points $R(S)$ and P coincide. The dynamics near P shown in figure 6(f) qualitatively is the same as in region IIA.

Besides these four regions there exists, inside region IIA, a small region that we denote by IId. We depict the basins of attraction in figure 7(a) choosing $\varepsilon = 0.25$ and $\tau = 0.168$. In this region, the convergence of the set S to the fixed point attractor P occurs exactly as in region IIA. The difference is that if we consider any point Q (off the diagonal $\theta_1 = \theta_2$) at an arbitrarily small (positive) distance off P, we observe that the distance of $R(Q)$ to P is bounded away from 0 by a positive constant (figure 7(b)). This means that the Poincaré map R is discontinuous at P. This occurs because in region IId, the oscillator O_3 overfires by 1 pulse along the periodic orbit $\hat{P}$ that corresponds to the fixed point P.

3.2. Existence of a fixed point attractor

In section 3.1 we numerically observed the existence of an unstable attractor P in the case of $n = 3$ oscillators. Here, we begin the proof of theorem 2 by establishing the existence of a



(a) Regions I,II (b) Regions I,II

Figure 8. Graphs of the components of the phase history function ϕ^P extended to the interval $(-\tau, \tau]$. The solid line represents the phases ϕ_i^P, $i = 1, \ldots, n-1$ and the dashed line represents ϕ_n^P. (*a*) The case $H_1(\tau + H_{n-2}(\tau)) < 1$. Here $\varphi = \tau + 1 - H_1(\tau + H_{n-2}(\tau))$. (*b*) The case $H_1(\tau + H_{n-2}(\tau)) \geqslant 1$.

fixed point attractor ϕ^P on the surface of section $\boldsymbol{P}$ for networks with $n \geqslant 3$ oscillators. The phase history function ϕ^P corresponds to the point P in the previous section and it is defined by (see also figure 8)

(i) For (ε, τ) such that $H_1(\tau + H_{n-2}(\tau)) < 1$,

$$\phi_i^P(s) = \begin{cases} \tau + s, & \text{for } i = 1, \ldots, n-1, \\ 2\tau + 1 - H_1(\tau + H_{n-2}(\tau)) + s, & \text{for } i = n, \end{cases} \quad \text{for } s \in (-\tau, 0), \tag{17a}$$

and

$$\phi_i^P(0) = \begin{cases} H_{n-2}(\tau), & \text{for } i = 1, \ldots, n-1, \\ 0, & \text{for } i = n. \end{cases} \tag{17b}$$

(ii) For (ε, τ) such that $H_1(\tau + H_{n-2}(\tau)) \geqslant 1$,

$$\phi_i^P(s) = \begin{cases} \tau + s, & \text{for } i = 1, \ldots, n-1, \\ 2\tau + s, & \text{for } i = n, \end{cases} \quad \text{for } s \in (-\tau, 0), \tag{18a}$$

and

$$\phi_i^P(0) = \begin{cases} H_{n-2}(\tau), & \text{for } i = 1, \ldots, n-1, \\ 0, & \text{for } i = n. \end{cases} \tag{18b}$$

Then, the following lemma gives the conditions under which ϕ^P is a fixed point attractor.

Lemma 4. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ such that*

(i) $n \geqslant 3$,
(ii) $H_{n-1}(2\tau) - H_{n-2}(\tau) - \tau$ *is a strictly increasing function in* τ *for any fixed* $\varepsilon > 0$,
(iii) $g_1(\tau) < 1$,
(iv) $g_3(\tau) < 1$,

there is an open set of initial states $\hat{S} \subset \boldsymbol{P}$ that converges to the fixed point attractor $\phi^P \notin \hat{S}$ in finite iterations of the Poincaré map $R : \boldsymbol{P} \to \boldsymbol{P}$.

Note that the Mirollo–Strogatz pulse response function V_{MS} (10) satisfies the requirement (ii) of lemma 4. Therefore, lemma 4 implies the existence of a fixed point attractor of the Poincaré map R when V is given by the Mirollo–Strogatz model, hence it can be considered as the first part of theorem 2. The second part of theorem 2 is the proof of the instability of ϕ^P which is given in section 3.3.

The set $\hat{S}$ is determined in the course of the proof. The connection between lemma 4 and the numerical observations of section 3.1 is that the periodic attractor ϕ^P corresponds to the fixed point P, $\hat{S}$ is related to the set S that collapses to the single point $R(S)$ and the parameter region for which conditions (iii) and (iv) of lemma 4 hold, corresponds to region II.

The plan of the proof is as follows. First, we identify a set S on the surface of section $\boldsymbol{P}$ that is mapped to a single point $\phi^S \in \boldsymbol{P}$ in one iteration of the Poincaré map R and we find a non-empty set $\hat{S} \subset S$ that is open in $\boldsymbol{P}$. Then, we find criteria on the parameters (ε, τ) and the pulse response function such that ϕ^S is inside or outside $\hat{S}$. In this way the parameter space is separated into regions I, II and III such that in regions I and III, $\phi^S \in \hat{S}$ while in region II, $\phi^S \notin \hat{S}$. Region II is exactly the region of the parameter space in which the requirements (iii) and (iv) of lemma 4 are satisfied. The third step is to show that $\phi^P \in \boldsymbol{P}$ is a fixed point of R. Finally, we show that in the regions II and III, ϕ^S (hence $\hat{S}$) is mapped to ϕ^P in a finite number of iterations of R. Then, lemma 4 follows directly from these facts.

3.2.1. Collapse of $S \subset \boldsymbol{P}$ onto a single point. In this section, we identify an open set $S \in \boldsymbol{P}$ which is mapped to a single point $\phi^S = R(S)$ in one iteration of the return map R.

Proposition 5. *Consider a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ such that*

(i) $n \geqslant 3$,
(ii) $\tau + H_{n-2}(\tau) < 1$,
(iii) $H_{n-1}(2\tau) - H_{n-2}(\tau) - \tau$ is a strictly increasing function in τ for any fixed $\varepsilon > 0$,

and consider the set

$$S = \{\phi \in \boldsymbol{P} : \phi_i(0) = \theta_i,\ \Sigma_i(\phi) = \emptyset \text{ for } i = 1, \ldots, n-1, \text{ and}$$
$$\phi_n(0) = 0,\ \Sigma_n(\phi) = \{0\}\}, \tag{19}$$

where θ_i satisfies the relations $\tau < \theta_i < 1 - \tau$ and $H_1(\theta_i + \tau) > 1$ for $i = 1, \ldots, n-1$. The set S is mapped in one iteration of R onto the single phase history function $\phi^S = R(S) \in \boldsymbol{P}$ given by

(i) If $H_{n-1}(2\tau) < 1 - \tau$, then

$$\phi_i^S(s) = \begin{cases} H_{n-2}(\tau) + 1 - H_{n-1}(2\tau) + s, & \text{for } i = 1, \ldots, n-1, \\ 1 + s, & \text{for } i = n, \end{cases} \quad \text{for } s \in (-\tau, 0]. \tag{20}$$

(ii) If $1 - \tau < H_{n-1}(2\tau) < 1$, then

$$\phi_i^S(s) = \begin{cases} \tau + 1 - H_{n-1}(2\tau) + s, & \text{for } i = 1, \ldots, n-1, \\ 2\tau + 1 - H_{n-1}(2\tau) + s, & \text{for } i = n, \end{cases} \tag{21a}$$

for $s \in (-\tau, H_{n-1}(2\tau) - 1)$, and

$$\phi_i^S(s) = \begin{cases} H_{n-2}(\tau) + 1 - H_{n-1}(2\tau) + s, & \text{for } i = 1, \ldots, n-1, \\ 1 + s, & \text{for } i = n, \end{cases} \tag{21b}$$

for $s \in [H_{n-1}(2\tau) - 1, 0]$.

(iii) If $1 \leqslant H_{n-1}(2\tau)$*, then* $\phi^S = \phi^P$*, i.e.*

$$\phi_i^S(s) = \begin{cases} \tau + s, & \text{for } i = 1, \dots, n-1, \\ 2\tau + s, & \text{for } i = n, \end{cases} \qquad \text{for } s \in (-\tau, 0), \tag{22a}$$

and

$$\phi_i^S(0) = \begin{cases} H_{n-2}(\tau), & \text{for } i = 1, \dots, n-1, \\ 1, & \text{for } i = n. \end{cases} \tag{22b}$$

Proof. The evolution of the initial states in S can be represented in the event sequence representation as follows:

$[P, (1, \dots, n-1), \tau], [F, 1, 1-\theta_1], \dots, [F, n-1, 1-\theta_{n-1}], [F, n, 1]$

$\xrightarrow{1} [P, (1, \dots, n-1), 0], [F, 1, 1-\theta_1-\tau], \dots, [F, n-1, 1-\theta_{n-1}-\tau], [F, n, 1-\tau]$

$\xrightarrow{2} [F, (1, \dots, n-1), 0], [F, 3, 1-\tau]$

$\xrightarrow{3} [(n-2)P, (1, \dots, n-1), \tau], [(n-1)P, n, \tau], [F, n, 1-\tau], [F, (1, \dots, n-1), 1]$

$\xrightarrow{4} [(n-2)P, (1, \dots, n-1), 0], [(n-1)P, n, 0], [F, n, 1-2\tau], [F, (1, \dots, n-1), 1-\tau].$

In transition 2 we used the fact that $H_1(\theta_i + \tau) > 1$ which implies that all the oscillators $O_1, \dots, O_{n-1}$ fire after receiving the pulse from O_n. Notice that after this step the oscillators $O_1, \dots, O_{n-1}$ are synchronized. In transition 3 we used the fact that $\tau < 1/2$ so that $\tau < 1-\tau$ in order to get the correct time ordering. At this point we have to distinguish two cases.

If $H_{n-1}(2\tau) < 1$ the evolution is

$\xrightarrow{5} [F, n, 1-H_{n-1}(2\tau)], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)]$

$\xrightarrow{6} [F, n, 0], [F, (1, \dots, n-1), H_{n-1}(2\tau) - H_{n-2}(\tau)]$

$\xrightarrow{7} [P, (1, \dots, n-1), \tau], [F, (1, \dots, n-1), H_{n-1}(2\tau) - H_{n-2}(\tau)], [F, n, 1].$

In transition 5 we used the fact that $H_{n-1}(2\tau) < 1$, to obtain that O_n does not fire after receiving the $n-1$ pulses. In transitions 5 and 7 we used (A.4) in order to obtain the correct time ordering. The last event sequence corresponds to a phase history function $\phi^S \in \boldsymbol{P}$. Reconstructing ϕ^S from the successive event sequences we obtain (20) and (21). In both cases, all the oscillators receive pulses at $t = H_{n-1}(2\tau) - 1$. The difference is that in the latter case $-\tau < H_{n-1}(2\tau) - 1 < 0$, therefore ϕ^S must have discontinuities at $t = H_{n-1}(2\tau) - 1$, while in the former case $H_{n-1}(2\tau) - 1 \leqslant -\tau$, therefore ϕ^S has no discontinuities in $(-\tau, 0]$.

In the case $H_{n-1}(2\tau) \geqslant 1$ we obtain

$\xrightarrow{5} [F, n, 0], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)]$

$\xrightarrow{6} [P, (1, \dots, n-1), \tau], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)], [F, n, 1].$

In this case, we used in transition 5 the fact that $H_{n-1}(2\tau) \geqslant 1$, to obtain that O_n fires. In the same transition we used the fact that $H_{n-2}(\tau) < 1-\tau < 1$ in order to show that the oscillators O_i $(i = 1, \dots, n-1)$ do not fire. Finally, in transition 6 we used that $\tau < 1-H_{n-2}(\tau)$, in order to obtain the correct time ordering. The last event sequence corresponds to a phase history function $\phi^S \in \boldsymbol{P}$. Reconstructing ϕ^S from the successive event sequences we obtain (22). $\square$

Proposition 6. *S contains the non-empty open set*

$$\hat{S} = \{\phi \in S : \phi_n(-\tau) > 0\}. \tag{23}$$

Proof. See appendix A.2. $\square$

3.2.2. Parameter regions. In this section we show that given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ there is a neighbourhood in the parameter space with non-empty interior for which $\phi^S \notin \hat{S}$.

Then we immediately read from lemma 5 that $\phi^S = \phi^P$ if and only if $g_2(\tau) = H_{n-1}(2\tau) \geqslant 1$, see (16*b*). Also notice that ϕ^S is inside $\hat{S}$ if the phases $\phi_i^S(0)$, $i = 1, \ldots, n-1$ satisfy

$$H_1(\phi_i^S(0) + \tau) > 1. \tag{24}$$

Applying inequality (24) in the case that $g_2(\tau) < 1$ we get that $\phi^S \in \hat{S}$ if and only if $H_1(1 + \tau + H_{n-2}(\tau) - H_{n-1}(2\tau)) = g_1(\tau) > 1$, see (16*a*). Applying the same inequality in the case $g_2(\tau) > 1$ we obtain that $\phi^S \in \hat{S}$ if and only if $H_1(\tau + H_{n-2}(\tau)) = g_3(\tau) > 1$, see (16*c*).

Combining these together we have the following cases:

(i) Case I: If $g_2(\tau) < 1$ and $g_1(\tau) > 1$, then $\phi^S \in \hat{S}$ and $\phi^S \neq \phi^P$.
(ii) Case IIA: If $g_2(\tau) < 1$ and $g_1(\tau) < 1$, then $\phi^S \notin \hat{S}$ and $\phi^S \neq \phi^P$.
(iii) Case IIB: If $g_2(\tau) > 1$ and $g_3(\tau) < 1$, then $\phi^S \notin \hat{S}$ and $\phi^S = \phi^P$.
(iv) Case III: If $g_2(\tau) > 1$ and $g_3(\tau) > 1$, then $\phi^S \in \hat{S}$ and $\phi^S = \phi^P$.

Notice that if ϕ^S is inside $\hat{S}$ then it is a fixed point of R and its basin of attraction contains $\hat{S}$. Therefore, in order for $\hat{S}$ to converge to an unstable attractor the system must realize at least one of the cases IIA or IIB. If either of these cases are realized, then we show that the unstable attractor is ϕ^P (17). In the rest of this section we determine the conditions under which the cases IIA and IIB can appear in the system.

Proposition 7. *In a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ such that*

(i) $n \geqslant 3$,
(ii) $H_{n-1}(2\tau) - H_{n-2}(\tau) - \tau$ is a strictly increasing function in τ for any fixed $\varepsilon > 0$,

the cases IIA and IIB can be realized if and only if $H_{n-1}(0) = H(0, \varepsilon) < 1$.

Note that the relation $H(0, \varepsilon) < 1$ in the case of the Mirollo–Strogatz model is equivalent to $\varepsilon < 1$.

Proof. The functions g_2 and g_3 are strictly increasing. Moreover, if we denote derivatives by $'$ we have

$$g_2'(\tau) = 2 + 2V_{n-1}'(2\tau) > 2,$$
$$g_3'(\tau) = 2 + V_{n-2}'(\tau) + V_1'(2\tau + V_{n-2}(\tau))(2 + V_{n-1}'(\tau)) > 2.$$

Notice also that

$$g_2(0) = H_{n-1}(0) = H_1(H_{n-2}(0)) = g_3(0) < 1. \tag{25}$$

These facts imply that given a pulse response function V and a coupling strength ε such that $H_{n-1}(0) < 1$, there are unique $\tau_2, \tau_3 \in (0, 1/2)$ (that depend on V and ε) such that

$$g_2(\tau_2) = 1 \qquad \text{and} \qquad g_3(\tau_3) = 1.$$

From (A.5) we conclude that $\tau_2 < \tau_3$. Notice then that

$$\begin{aligned} g_1(\tau_2) &= H_1(1 + \tau_2 + H_{n-2}(\tau_2) - H_{n-1}(2\tau_2)), \\ &= H_1(1 + \tau_2 + H_{n-2}(\tau_2) - g_2(\tau_2)), \\ &= H_1(\tau_2 + H_{n-2}(\tau_2)) = g_3(\tau_2) < 1. \end{aligned} \tag{26}$$

Since $g_1(0) > 1$ (A.6), $g_1(\tau_2) < 1$ and g_1 is strictly decreasing (because $H_{n-1}(2\tau) - H_{n-2}(\tau) - \tau$ is strictly increasing) we conclude that there is a unique $\tau_1 < \tau_2$ such that $g_1(\tau_1) = 1$.

Therefore we can characterize cases I, IIA, IIB and III in the following way:

(i) Case I: $0 < \tau < \tau_1$, because then $g_1(\tau) > 1$, $g_2(\tau) < 1$ and $g_3(\tau) < 1$.
(ii) Case IIA: $\tau_1 < \tau < \tau_2$, because then $g_1(\tau) < 1$, $g_2(\tau) < 1$ and $g_3(\tau) < 1$.
(iii) Case IIB: $\tau_2 < \tau < \tau_3$, because then $g_1(\tau) < 1$, $g_2(\tau) > 1$ and $g_3(\tau) < 1$.
(iv) Case III: $\tau_3 < \tau < 1/2$, because then $g_1(\tau) < 1$, $g_2(\tau) > 1$ and $g_3(\tau) > 1$.

This shows that if $H_{n-1}(0) < 1$ then the cases IIA and IIB can be realized.

For the opposite, notice that if $H_{n-1}(0) > 1$ then $g_2(\tau) = g_3(\tau) > 1$ for all $\tau \geqslant 0$. This implies that only case III can be realized in this situation. □

From now on, we restrict our attention to parameters for which the cases IIA and IIB can be realized.

Definition 10. *For a given system $\mathcal{D} = (n, V, \varepsilon, \tau)$ such that*

(i) $n \geqslant 3$,
(ii) $H_{n-1}(2\tau) - H_{n-2}(\tau) - \tau$ is a strictly increasing function in τ for any fixed $\varepsilon > 0$,

the permissible parameter region is

$$\mathcal{M}_V = \{(\varepsilon, \tau) \in \mathbb{R}^2_+ \,:\, 0 < \tau < \tau_4,\ H_{n-1}(0) = H(0, \varepsilon) < 1\} \tag{27}$$

where $\tau_4 \in (0, 1/2)$ is the unique solution of $\tau + H_{n-2}(\tau) = 1$.

Notice that $g_3(\tau) = H_1(\tau + H_{n-2}(\tau)) > \tau + H_{n-2}(\tau)$ and $(\tau + H_{n-2}(\tau))' = 2 + V'_{n-2}(\tau) > 2$, therefore $\tau_3 < \tau_4 < 1/2$.

Definition 11. *For a given system $\mathcal{D} = (n, V, \varepsilon, \tau)$ such that*

(i) $n \geqslant 3$,
(ii) $H_{n-1}(2\tau) - H_{n-2}(\tau) - \tau$ is a strictly increasing function in τ for any fixed $\varepsilon > 0$,

we define the following regions in $\mathcal{M}_V$:

(i) Region I: $\{(\varepsilon, \tau) \in \mathcal{M}_V \,:\, 0 < \tau < \tau_1\}$.
(ii) Region IIA: $\{(\varepsilon, \tau) \in \mathcal{M}_V \,:\, \tau_1 < \tau < \tau_2\}$.
(iii) Region IIB: $\{(\varepsilon, \tau) \in \mathcal{M}_V \,:\, \tau_2 < \tau < \tau_3\}$.
(iv) Region III: $\{(\varepsilon, \tau) \in \mathcal{M}_V \,:\, \tau_3 < \tau < \tau_4\}$.

We also call the union of regions IIA and IIB, region II.

A picture of regions I, IIA, IIB, III for the Mirollo–Strogatz model with $n = 3$ has already been given in figure 5.

3.2.3. ϕ^P is a fixed point of R

Proposition 8. *Given a pulse response function $V \in \mathcal{G}$, ϕ^P ((17),(18)) is a fixed point of R for $(\varepsilon, \tau) \in \mathcal{M}_V$.*

Proof. P evolves in the event sequence representation as follows:

$$[P, (1, \ldots, n-1), \tau], [F, (1, \ldots, n-1), 1 - H_{n-2}(\tau)], [F, n, 1]$$
$$\xrightarrow{1} [P, (1, \ldots, n-1), 0], [F, (1, \ldots, n-1), 1 - \tau - H_{n-2}(\tau)], [F, n, 1 - \tau].$$

In transition 2 we used the fact that for $\tau < \tau_4(V)$ we have that $\tau + H_{n-2}(\tau) < 1$. If $g_3(\tau) = H_1(\tau + H_{n-2}(\tau)) \geqslant 1$ (region III), then the oscillators O_i, $i = 1, \dots, n-1$ fire after receiving the pulse from O_n, so we get

$$\xrightarrow{2}[F, (1, \dots, n-1), 0], [F, n, 1-\tau]$$
$$\xrightarrow{3}[(n-2)P, (1, \dots, n-1), \tau], [(n-1)P, n, \tau], [F, n, 1-\tau], [F, (1, \dots, n-1), 1]$$
$$\xrightarrow{4}[(n-2)P, (1, \dots, n-1), 0], [(n-1)P, n, 0], [F, n, 1-2\tau], [F, (1, \dots, n-1), 1-\tau]$$
$$\xrightarrow{5}[F, n, 0], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)]$$
$$\xrightarrow{6}[P, (1, \dots, n-1), \tau], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)], [F, n, 1],$$

where as in transition 5 we used the fact that when $g_3(\tau) \geqslant 1$ we also have $g_2(\tau) = H_{n-1}(2\tau) > 1$. We observe that we return to the initial state, i.e. ϕ^P is a fixed point of R. If $g_3(\tau) < 1$ (regions I and II), the evolution is

$$\xrightarrow{2}[F, (1, \dots, n-1), 1-H_1(\tau + H_{n-2}(\tau))], [F, n, 1-\tau]$$
$$\xrightarrow{3}[F, (1, \dots, n-1), 0], [F, n, H_1(\tau + H_{n-2}(\tau)) - \tau]$$
$$\xrightarrow{4}[(n-2)P, (1, \dots, n-1), \tau], [(n-1)P, n, \tau], [F, n, H_1(\tau + H_{n-2}(\tau)) - \tau], \\ [F, (1, \dots, n-1), 1]$$
$$\xrightarrow{5}[(n-2)P, (1, \dots, n-1), 0], [(n-1)P, n, 0], [F, n, H_1(\tau + H_{n-2}(\tau)) - 2\tau], \\ [F, (1, \dots, n-1), 1-\tau]$$
$$\xrightarrow{6}[F, n, 0], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)]$$
$$\xrightarrow{7}[P, (1, \dots, n-1), \tau], [F, (1, \dots, n-1), 1-H_{n-2}(\tau)], [F, n, 1].$$

The only transition in question is 6, which is valid only if

$$H_{n-1}(T(\tau)) \geqslant 1,$$

where

$$\begin{aligned} T(\tau) &= 1 + 2\tau - H_1(\tau + H_{n-2}(\tau)), \\ &= 1 + 2\tau - g_3(\tau), \\ &= 1 - V_{n-2}(\tau) - V_1(2\tau + V_{n-2}(\tau)). \end{aligned} \tag{28}$$

Notice that $T(\tau)$ is a strictly decreasing function of τ and this implies that $H_{n-1}(T(\tau))$ is also strictly decreasing. Since $g_3(\tau) < 1$ we have $\tau < \tau_3$. Therefore, in order to prove that $H_{n-1}(T(\tau)) \geqslant 1$ it is enough to show that $H_{n-1}(T(\tau_3)) \geqslant 1$. We find

$$T(\tau_3) = 1 + 2\tau_3 - g_3(\tau_3) = 2\tau_3.$$

and

$$H_{n-1}(T(\tau_3)) = H_{n-1}(2\tau_3) = g_2(\tau_3) > 1. \qquad \square$$

Remark 2. The function $T(\tau)$ (28) is the period of the periodic orbit $\mathcal{O}_+(\phi^P)$ in regions I and II, i.e. $\Phi^{T(\tau)}(\phi^P) = \phi^P$. This can be easily verified by computing how long it takes to go from the initial to the final event sequence in the previous proof. In region III, the period is 2τ.

Remark 3. In regions I and II we used the fact that $H_{n-1}(T(\tau)) \geqslant 1$ to show that O_n fires after receiving $n-1$ pulses. It is possible in general $\nu(T(\tau)) = m < n-1$, i.e. that $H_m(T(\tau)) \geqslant 1$ and $H_{m-1}(T(\tau)) < 1$ for some $m < n-1$, which means that O_n overfires by $n-1-m$ pulses. Recall from section 2.3.1 that in this case the evolution operator Φ^t is discontinuous for some $t > 0$. We come back to this point later.

3.2.4. Finite time convergence. In this section we prove that in regions II and III, the point ϕ^S is mapped to ϕ^P in a finite number of iterations of R.

Proposition 9. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$, ϕ^S (proposition 5) is mapped to ϕ^P (given by (17) and (18)) in finitely many iterations of R, if the system is in regions II or III (definition 11).*

Proof. In the cases IIB and III there is nothing to prove since $\phi^S = \phi^P$. In the case IIA, consider the one-parameter family $[\phi^w] \in \boldsymbol{P}/\sim$ given by

$$\phi_i^w(0) = w, \qquad \Sigma_i(\phi) = \emptyset, \quad \text{for } i = 1, \dots, n-1,$$
$$\phi_n^w(0) = 0, \qquad \Sigma_n(\phi^w) = \{0\},$$

where

$$w \in \mathcal{L} = [W_1, W_2] = [H_{n-2}(\tau), H_{n-2}(\tau) + 1 - H_{n-1}(2\tau)].$$

Clearly, $\phi^P \in [\phi^{W_1}]$ and $\phi^S \in [\phi^{W_2}]$. Then $[\phi^w]$ evolves as

$$\begin{aligned}
&[P, (1, \dots, n-1), \tau], [F, (1, \dots, n-1), 1-w], [F, n, 1]\\
&\xrightarrow{1} [P, (1, \dots, n-1), 0], [F, (1, \dots, n-1), 1-w-\tau], [F, n, 1-\tau]\\
&\xrightarrow{2} [F, (1, \dots, n-1), 1 - H_1(w+\tau)], [F, n, 1-\tau]\\
&\xrightarrow{3} [F, (1, \dots, n-1), 0], [F, n, H_1(w+\tau) - \tau]\\
&\xrightarrow{4} [(n-2), P, (1, \dots, n-1), \tau], [(n-1)P, n, \tau], [F, n, H_1(w+\tau) - \tau],\\
&\qquad [F, (1, \dots, n-1), 1]\\
&\xrightarrow{5} [(n-2)P, (1, \dots, n-1), 0], [(n-1)P, n, 0], [F, n, H_1(w+\tau) - 2\tau],\\
&\qquad [F, (1, \dots, n-1), 1-\tau].
\end{aligned}$$

In transition 2 we used the fact that $H_1(w+\tau) < 1$. Let

$$\mathcal{L}' = \{w \in \mathcal{L} \, : \, H_{n-1}(1 + 2\tau - H_1(w+\tau)) \geqslant 1\}.$$

The interior of $\mathcal{L}'$ is not empty. This follows from the fact that $H_{n-2}(\tau) \in \mathcal{L}'$, since

$$H_{n-1}(1 + 2\tau - H_1(\tau + H_{n-2}(\tau)) = H_{n-1}(T(\tau)) > 1$$

and that the inequality that defines $\mathcal{L}'$ depends smoothly on w. When $w \in \mathcal{L}'$ we obtain the evolution

$$\begin{aligned}
&\xrightarrow{6} [F, n, 0], [F, (1, \dots, n-1), 1 - H_{n-2}(\tau)]\\
&\xrightarrow{7} [P, (1, \dots, n-1), \tau], [F, (1, \dots, n-1), 1 - H_{n-2}(\tau)], [F, n, 1]
\end{aligned}$$

i.e. all phase history functions in the equivalence class $[\phi^w]$ for $w \in \mathcal{L}'$ are mapped to ϕ^P in one iteration of R.

In order to finish the proof we need to show that points in $\mathcal{L}'' = \mathcal{L} \setminus \mathcal{L}'$ enter $\mathcal{L}'$ in a finite number of iterations. When $w \in \mathcal{L}''$ the evolution is

$$\begin{aligned}
&\xrightarrow{6} [F, n, 1 - H_{n-1}(1 + 2\tau - H_1(w+\tau))], [F, (1, \dots, n-1), 1 - H_{n-2}(\tau)]\\
&\xrightarrow{7} [F, n, 0], [F, (1, \dots, n-1), H_{n-1}(1 + 2\tau - H_1(w+\tau)) - H_{n-2}(\tau)].
\end{aligned}$$

This means that when $w \in \mathcal{L}''$, the point $[\phi^w]$ is mapped to $[\phi^{w'}]$ with

$$w' = H_{n-2}(\tau) + 1 - H_{n-1}(1 + 2\tau - H_1(w+\tau)).$$

Consider the function

$$\Delta(w) = w' - w = V_{n-2}(\tau) + V_1(w + \tau) - V_{n-1}(1 - w + \tau - V_1(w + \tau))$$

which is a strictly increasing function of w. For $w = W_2 = \max \mathcal{L} = H_{n-2}(\tau) + 1 - H_{n-1}(2\tau)$ we obtain that

$$\Delta(W_2) = H_{n-1}(2\tau) - H_{n-1}(1 + 2\tau - H_1(W_2 + \tau)).$$

Notice that

$$H_1(W_2 + \tau) = H_1(1 + \tau + H_{n-2}(\tau) - H_{n-1}(2\tau)) = g_1(\tau).$$

In the case IIA we have that $g_1(\tau) < 1$, i.e.

$$1 + 2\tau - H_1(W_2 + \tau) > 2\tau,$$

and since H_{n-1} is strictly increasing we obtain

$$H_{n-1}(1 + 2\tau - H_1(W_2 + \tau)) > H_{n-1}(2\tau).$$

Therefore, $\Delta(W_2) < 0$. Combined with the fact that Δ is strictly increasing and that $W_2 = \max \mathcal{L}$ we obtain that

$$\Delta(w) \leqslant \Delta(W_2) < 0, \text{ for all } w \in \mathcal{L}.$$

Moreover, one can easily see that if $w \in \mathcal{L}''$ then $w' > H_{n-2}(\tau)$, i.e. $w' \in \mathcal{L}$. This means that beginning with a $w \in \mathcal{L}''$ we can iterate this procedure to get a sequence $\ell = w \to w' \to w'' \to \ldots$ until a point in the sequence enters $\mathcal{L}'$. That this will happen after a finite number of steps is ensured by the fact that ℓ is a decreasing sequence (since $w' - w = \Delta(w) < 0$) and the fact that for smaller w the absolute difference $|\Delta(w)| = -\Delta(w)$ between w' and w actually becomes larger (since Δ is a strictly increasing function of w). The first point in the sequence ℓ that enters $\mathcal{L}'$ is then mapped in one iteration to ϕ^P. □

3.3. Instability of the attractor

In the previous section we proved that ϕ^P is a fixed point attractor of R for $(\varepsilon, \tau) \in \mathcal{M}_V$ which implies that it corresponds to a periodic attractor for the evolution operator Φ^t with period T (28). In this section we prove that ϕ^P is unstable in the parameter region II as long as $\nu(T(\tau)) = n - 1$ where $T(\tau)$ is the period of $\mathcal{O}_+(\phi^P)$. In other words, since $\phi_n^P(0^-) = T(\tau)$, we prove that ϕ^P is unstable when the oscillator O_n does not overfire along the periodic positive semiorbit $\mathcal{O}_+(\phi^P)$. We state and prove the instability of ϕ^P in the case that the pulse response function is given by the Mirollo–Strogatz model, i.e. $V = V_{\mathrm{MS}}$ (10). The theorem can be stated and proved in a similar way for the general case but the computations are much more involved.

Lemma 10. *Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ such that*

(i) $n \geqslant 3$,
(ii) V is given by the Mirollo–Strogatz model,
(iii) $g_1(\tau) = H_1(1 + \tau + H_{n-2}(\tau) - H_{n-1}(2\tau)) < 1$,
(iv) $g_3(\tau) = H_1(\tau + H_{n-2}(\tau)) < 1$,
(a) $\nu(T(\tau)) = n - 1$,

the fixed point attractor ϕ^P given by lemma 4 is linearly unstable. For any fixed values $b > 0$ and $n \geqslant 3$ the conditions (i)–(v) define an open non-empty set in the parameter space (ε, τ).

This is the second part of theorem 2, therefore it concludes the proof of the existence of an unstable fixed point attractor.

3.3.1. Proof of the linear instability. In order to prove that ϕ^P is linearly unstable under the conditions given in lemma 10 we identify first an open neighbourhood of ϕ^P in $\boldsymbol{P}$.

Proposition 11. *In region II there is $\rho > 0$ such that if $\mathrm{d}(\phi^P, \phi) = \eta < \rho$ for some $\phi \in \boldsymbol{P}$ then $|\phi_i(0) - H_{n-2}(\tau)| < \eta_2 = O(\eta)$ and $\Sigma_i(\phi) = \emptyset$ for $i = 1, \ldots, n-1$, while $\phi_n(0) = 0$ and $\Sigma_n(\phi) = \{0\}$. Moreover, if O_n overfires at 0 (i.e. $H_m(T(\tau)) > 1$ for some $m < n-1$) then O_n receives the last $n - m + 1$ pulses simultaneously.*

Proof. The proof is given in appendix A.3. □

In the rest of this section we study the dynamics of R in a small neighbourhood of ϕ^P. Consider any permutation λ of $n-1$ symbols. Then define the set U_λ by

$$U_\lambda = \{\phi \in B_\rho(\phi^P) : \phi_{\lambda(1)}(0) \geqslant \phi_{\lambda(2)}(0) \geqslant \cdots \geqslant \phi_{\lambda(n-1)}(0)\}. \tag{29}$$

We denote $\phi_{\lambda(i)}(0) = H_{n-2}(\tau) + \eta_i$ and $R(\phi)_{\lambda(i)}(0) = H_{n-2}(\tau) + \eta_i'$ for $i = 1, \ldots, n-1$ where η_i and η_i' are small (not necessarily positive). The oscillator O_n has at $t = 0$ phase $\phi_n(0) = 0$ and therefore gives a pulse to the other oscillators after time τ. Hence, at time $t = \tau$ each oscillator O_i, $i = 1, \ldots, n-1$ receives a pulse and its phase becomes

$$H_1(\tau + H_{n-2}(\tau) + \eta_i) = H_1(\tau + H_{n-2}(\tau)) + A\eta_i.$$

Here we used that fact V is given by the Mirollo–Strogatz model so we have that

$$H_k(\theta + \eta) = H_k(\theta) + A^k\eta,$$

where $A = K_\varepsilon + 1$, and K_ε is defined in (10).

Notice that since $H_1(\tau + H_{n-2}(\tau)) = g_3(\tau) < 1$, there is $\rho_1 > 0$ such that, if $|\eta_i| < \rho_1$ then $H_1(\tau + H_{n-2}(\tau)) + A\eta_i < 1$. This implies that in this case the oscillators O_i do not fire after receiving the pulse at $t = \tau$. Each oscillator $O_{\lambda(i)}$, $i = 1, \ldots, n-1$ then fires at time

$$t_i = 1 + \tau - H_1(\tau + H_{n-2}(\tau)) - A\eta_i = T - \tau - A\eta_i.$$

Given the original assumption $\eta_1 \geqslant \cdots \geqslant \eta_{n-1}$ we deduce that $t_1 \leqslant \cdots \leqslant t_{n-1}$, i.e. the oscillators fire in the order $O_{\lambda(1)}, \ldots, O_{\lambda(n-1)}$. Define

$$\delta_{i,j} = t_j - t_i = A(\eta_i - \eta_j). \tag{30}$$

Then, the oscillator O_n receives pulses from the oscillators $O_{\lambda(i)}$, $i = 1, \ldots, n-1$ at the moments $\tau + t_i$. It is easy to see that after receiving the first $n-2$ pulses (at $t = t_{n-2} + \tau$) the phase of the oscillator O_n becomes

$$\phi_n(t_{n-2} + \tau) = H_{n-2}(T) - A^{n-1}\eta_1 + A^{n-3}\delta_{1,2} + \ldots + A\delta_{n-3,n-2}.$$

Since $H_{n-2}(T) < 1$, there is $\rho_2 > 0$ such that $\phi_n(t_{n-2} + \tau) < 1$ for $|\eta_i| < \rho_2$. This means that in this case O_n does not fire after receiving the first $n-2$ pulses. When O_n receives the last pulse at $t = t_{n-1} + \tau$ coming from $O_{\lambda(n-1)}$ its phase becomes

$$\phi_n(t_{n-1} + \tau) = \min\{1, H_{n-1}(T) - A^n\eta_1 + \sum_{k=1}^{n-2} A^{n-k-1}\delta_{k,k+1}\}.$$

Since $H_{n-1}(T) > 1$ we deduce that there is $\rho_3 > 0$ such that if $|\eta_i| < \rho_3$, then $\phi_n(t_{n-1} + \tau) = 1$. Therefore, O_n fires at $t = t_{n-1} + \tau = T - A\eta_{n-1}$.

Using similar arguments we can show that the phase $R(\phi)_{\lambda(j)}(0)$ of $O_{\lambda(j)}$ at $t = t_{n-1} + \tau$ (on the surface of section) is $H_{n-2}(\tau) + \eta_j'$, where

$$\eta_j' = -A^{n-2}\delta_{1,j} + \sum_{k=1}^{j-1} A^{n-k-2}\delta_{k,k+1} + \sum_{k=j}^{n-2} A^{n-k-1}\delta_{k,k+1}. \tag{31}$$

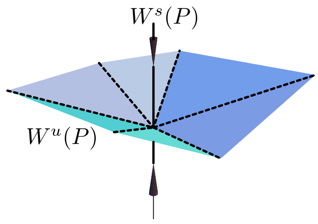


Figure 9. Schematic representation of the local stable and unstable manifolds of ϕ^P for $n = 4$. The stable manifold $W^s(P)$ is one dimensional while the unstable manifold $W^u(P)$ consists of 6 two-dimensional pieces that join in a continuous but not smooth way.

In the last expression, each sum is defined to be zero if the starting index is greater than the ending index. Then, we compute

$$\eta'_j - \eta'_{j+1} = A^{n-j-2}(A^j + A - 1)\delta_{j,j+1} = A^{n-j-1}(A^j + A - 1)(\eta_j - \eta_{j+1}).$$

Since $A = K + 1$ and $K > 0$ we have $A > 1$ and $A^{n-j-1}(A^j + A - 1) > 1$. From this we conclude that the phases of the oscillators $O_1, \ldots, O_{n-1}$ have the same ordering in $R(\phi)$ as they had in ϕ, i.e.

$$R(\phi)_{\lambda(1)}(0) \geqslant \cdots \geqslant R(\phi)_{\lambda(n-1)}(0),$$

therefore we can apply the same reasoning to obtain $R^2(\phi)$, $R^3(\phi)$, etc until $R^m(\phi)$ is outside $B_\rho(\phi^P)$ for some $m \geqslant 0$.

Moreover, notice that every application of the Poincaré map R increases the distance between two 'adjacent' phases and from this we can conclude the instability of ϕ^P. In particular, equations (30) and (31) show that the map L from η_i to η'_i is a linear map, whose eigenvalues and eigenvectors we can explicitly compute. The eigenvalues of L are $\lambda_0 = 0$ and $\lambda_j = A^{j-1}(A^{n-1-j} + A - 1) > 1$ for $j = 1, \ldots, n-2$. This shows that L has $n-2$ unstable directions and 1 collapsing (stable) direction. The corresponding eigenvectors are

$$w_0 = \textstyle\sum_{k=1}^{n-1} \boldsymbol{e}_k,$$
$$w_j = \frac{A}{1 - A^{n-1-j}} \textstyle\sum_{k=1}^{n-1-j} \mathbf{e}_k + \sum_{k=n-j}^{n-1} \mathbf{e}_k,$$

where $\boldsymbol{e}_k$, $k = 1, \ldots, n-1$ is the standard basis of $\mathbb{R}^{n-1}$. Moreover, a tedious but straightforward computation shows that if we define $\kappa_1 = 1$, $\kappa_j = A^j(2A-1)(A^j + A - 1)^{-1}(A^{j-1} + A - 1)^{-1}$ for $j = 2, \ldots, n-2$, and $\kappa_{n-1} = A(2A-1)(A-1)^{-1}(A^{n-2} + A - 1)^{-1}$ we have that

$$\sum_{j=1}^{n-1} \kappa_j \eta'_j = 0.$$

This means that all points in U_λ are mapped in exactly one iteration to the set

$$W^u_\lambda = \{\phi : \textstyle\sum_{j=1}^{n-1} \kappa_j(\phi_j(0) - H_{n-2}(\tau)) = 0\}.$$

This set is the $n-2$ dimensional unstable manifold of ϕ^P as it can be shown that it is spanned by the 'unstable' eigenvectors $w_1, \ldots, w_{n-2}$ of L. Recall that this discussion is restricted in the set U_λ, given by (29), where λ is a permutation of $n-1$ symbols. Actually, in each set U_λ there is a respective set W^u_λ. This has already been shown in figure 6 where we depicted the unstable manifolds of ϕ^P for $n = 3$ and we showed that they are one-dimensional sets that join at ϕ^P in a non-smooth way. As a representative of the situation for larger n we depict schematically the case $n = 4$ in figure 9. In this case, there are six permutations of three symbols, hence the unstable manifold W^u consists of six pieces that join along the dashed lines continuously but not smoothly, forming six creases. The vertical line in the same figure corresponds to the stable manifold of ϕ^P in which all the oscillators $O_1, \ldots, O_{n-1}$ are synchronized.

Finally, in order to show that ϕ^P is unstable we need to compute the distance $\mathrm{d}(R(\phi), \phi^P)$. Actually, it is easier to compute instead the distance between the points $\Phi^\tau(R(\phi))$ and $\Phi^\tau(\phi^P)$. Notice that

$$\Phi^\tau(R(\phi))_i(s) = \phi_i(s+\tau) = \begin{cases} H_{n-2}(\tau) + \eta_i' + \tau + s, & \text{for } i = 1, \ldots, n-1 \\ \tau + s, & \text{for } i = n, \end{cases}$$

for $s \in (-\tau, 0)$ and

$$\Phi^\tau(R(\phi))_i(s) = \phi_i(s+\tau) = \begin{cases} H_1(H_{n-2}(\tau) + \tau) + A\eta_i' + s, & \text{for } i = 1, \ldots, n-1 \\ \tau + s, & \text{for } i = n, \end{cases}$$

for $s \in [0, \tau]$. At the same time

$$\Phi^\tau(\phi^P)_i(s) = \phi_i(s+\tau) = \begin{cases} H_{n-2}(\tau) + \tau + s, & \text{for } i = 1, \ldots, n-1 \\ \tau + s, & \text{for } i = n, \end{cases}$$

for $s \in (-\tau, 0)$ and

$$\Phi^\tau(\phi^P)_i(s) = \phi_i(s+\tau) = \begin{cases} H_1(H_{n-2}(\tau) + \tau) + s, & \text{for } i = 1, \ldots, n-1 \\ \tau + s, & \text{for } i = n, \end{cases}$$

for $s \in [0, \tau]$. Therefore

$$\begin{aligned} \mathrm{d}(\Phi^\tau(R(\phi)), \Phi^\tau(\phi^P)) &= \sum_{i=1}^{n} \int_{-\tau}^{\tau} |\phi_i(s) - \phi_i^P(s)| \mathrm{d}s, \\ &= \sum_{i=1}^{n-1} \int_{-\tau}^{0} |\phi_i(s) - \phi_i^P(s)| \mathrm{d}s + \sum_{i=1}^{n-1} \int_{0}^{\tau} |\phi_i(s) - \phi_i^P(s)| \mathrm{d}s \\ &\quad + \int_{-\tau}^{\tau} |\phi_n(s) - \phi_n^P(s)| \mathrm{d}s, \\ &= \sum_{i=1}^{n-1} \tau |\eta_i'| + \sum_{i=1}^{n-1} \tau A |\eta_i'| = \tau(1+A) \sum_{i=1}^{n-1} |\eta_i'|. \end{aligned}$$

The fact that the linear map $L : \eta \mapsto \eta'$ is unstable means that the quantity $\sum_{i=1}^{n-1} |\eta_i'|$ which is the distance in $\mathbb{R}^{n-1}$ from the origin eventually increases. This means that after each step $\mathrm{d}(\Phi^\tau(R(\phi)), \Phi^\tau(\phi^P))$ also increases and that R is unstable.

3.3.2. The parameter region of linear instability is non-empty. The conditions of theorem 2 (and consequently of lemma 10) hold in an open region in the parameter space (ε, τ) that depends on the values of $b > 0$ and $n \geqslant 3$. The following proposition shows that this region is not empty.

Proposition 12. *Consider a pulse response function V, given by the Mirollo–Strogatz model (9) and a fixed $b > 0$ for a network with $n \geqslant 3$ oscillators. Then, the parameter region defined by*

(i) $0 < \varepsilon < 1$,
(ii) $\tau > 0$,
(iii) $g_1(\tau) < 1$,

(iv) $g_3(\tau) < 1$,
(v) $\nu(T(\tau)) = n - 1$, *where* $T(\tau) = 1 + 2\tau - H_1(\tau + H_{n-2}(\tau))$,

is not empty. Moreover, for every $n \geqslant 3$ *there is a range of values of* b *such that for any value of* $\varepsilon \in (0, 1)$ *there is a range of values of* τ *such that the above conditions hold.*

Proof. For given $b > 0$ and $n \geqslant 3$, denote by $\tau_3(\varepsilon)$ the solution of the equation $g_3(\tau) = 1$:

$$\tau_3(\varepsilon) = \frac{-\mathrm{e}^{bn\hat{\varepsilon}} + \mathrm{e}^{\hat{\varepsilon}b+b}}{(\mathrm{e}^{b} - 1)(\mathrm{e}^{2b\hat{\varepsilon}} + \mathrm{e}^{bn\hat{\varepsilon}})}, \tag{32}$$

where $\hat{\varepsilon} = \varepsilon/(n-1)$. It is a decreasing function of ε and $\tau_3(0^+) = 1/2$ while $\tau_3(1^-) = 0$. Moreover, $g_3(\tau) < 1$ for $\tau < \tau_3(\varepsilon)$.

In the same way, define as $\tau_1(\varepsilon)$ the solution of the equation $g_1(\tau) = 1$ which is given by

$$\frac{(-1 + \mathrm{e}^{b\hat{\varepsilon}})(\mathrm{e}^{bn\hat{\varepsilon}} - \mathrm{e}^{\hat{\varepsilon}b+b})}{(\mathrm{e}^{b} - 1)(\mathrm{e}^{2b\hat{\varepsilon}} + \mathrm{e}^{bn\hat{\varepsilon}} - 2\mathrm{e}^{b(n+1)\hat{\varepsilon}})}. \tag{33}$$

It is a decreasing function with $\tau_1(\varepsilon) < \tau_3(\varepsilon)$ for $0 < \varepsilon < 1$, $\tau_1(1^-) = 0$ and $g_1(\tau) < 1$ for $\tau > \tau_1(\varepsilon)$.

Therefore, conditions (i)–(iv) of the lemma are satisfied in the non-empty open region given by

$$\{(\varepsilon, \tau) \subseteq (0, 1) \times (0, 1/2) : \tau_1(\varepsilon) < \tau < \tau_3(\varepsilon)\}. \tag{34}$$

In order to take into account condition (v) of the lemma, notice that $\nu(T(\tau)) = n - 1$ for $\tau \in (h_2(\varepsilon), h_1(\varepsilon))$, where h_1 and h_2 are defined as the solutions of the equations $H_{n-1}(T(\tau)) = 1$ and $H_{n-2}(T(\tau)) = 1$, respectively. These are given by

$$h_1(\varepsilon) = \frac{\mathrm{e}^{b\hat{\varepsilon}} - \mathrm{e}^{bn\hat{\varepsilon}} + \mathrm{e}^{\hat{\varepsilon}b+b} - \mathrm{e}^{b-b(n-2)\hat{\varepsilon}}}{\left(-1 + \mathrm{e}^{b}\right)\left(-2\mathrm{e}^{b\hat{\varepsilon}} + \mathrm{e}^{2b\hat{\varepsilon}} + \mathrm{e}^{bn\hat{\varepsilon}}\right)}, \tag{35}$$

$$h_2(\varepsilon) = -\frac{-\mathrm{e}^{b\hat{\varepsilon}}\left(1 + \mathrm{e}^{b}\right) + \mathrm{e}^{bn\hat{\varepsilon}} + \mathrm{e}^{b-b(n-3)\hat{\varepsilon}}}{\left(-1 + \mathrm{e}^{b}\right)\left(-2\mathrm{e}^{b\hat{\varepsilon}} + \mathrm{e}^{2b\hat{\varepsilon}} + \mathrm{e}^{bn\hat{\varepsilon}}\right)}. \tag{36}$$

They are both decreasing functions with $h_2(\varepsilon) < h_1(\varepsilon)$ for $\varepsilon \in (0, 1)$ and $h_1(1^-) = 0$ while $h_2(1^-) < 0$. Moreover, $h_1(\varepsilon) > \tau_3(\varepsilon)$ for $\varepsilon \in (0, 1)$. This shows that for a given ε the region of validity of the lemma is non-empty if $\max\{h_2(\varepsilon), \tau_1(\varepsilon)\} < \tau_3(\varepsilon)$. Since $\tau_1 < \tau_3$ it is enough to show that there are values of ε such that $h_2 < \tau_3$.

This follows from the fact that $h_2(1^-) < 0 = \tau_3(1^-)$ and that h_2 and τ_3 are smooth decreasing functions of ε. Therefore, there is $\varepsilon_* \in (0, 1)$ such that for $\varepsilon \in (\varepsilon_*, 1)$, it holds that $h_2(\varepsilon) < \tau_3(\varepsilon)$. This shows that for any $n \geqslant 3$ and $b > 0$ there is an open non-empty parameter region in which the conditions of the lemma hold and that is given by

$$\{(\varepsilon, \tau) \subseteq (\varepsilon_*, 1) \times (0, 1/2) : \max\{\tau_1(\varepsilon), h_2(\varepsilon)\} < \tau < \tau_3(\varepsilon)\}. \tag{37}$$

Moreover, one can show that the difference $\tau_3(\varepsilon) - h_2(\varepsilon)$ is a strictly decreasing function of ε, therefore if $\tau_3(0^+) > h_2(0^+)$ we obtain that $\tau_3(\varepsilon) > h_2(\varepsilon)$ for all $\varepsilon \in (0, 1)$, i.e., $\varepsilon_* = 0$. One can verify that for $n = 3$ and $n = 4$, $\tau_3(0^+) > h_2(0^+)$ for all $b > 0$. For $n \geqslant 5$ we obtain that $\tau_3(0^+) > h_2(0^+)$ if and only if $0 < b < \ln \frac{n-2}{n-4}$. Therefore, for all $n \geqslant 3$, there is a range of values $b \in (0, \ln \frac{n-2}{n-4})$ such that for arbitrary $\varepsilon \in (0, 1)$ there is a range of values of τ for which the conditions of theorem 2 hold. $\square$

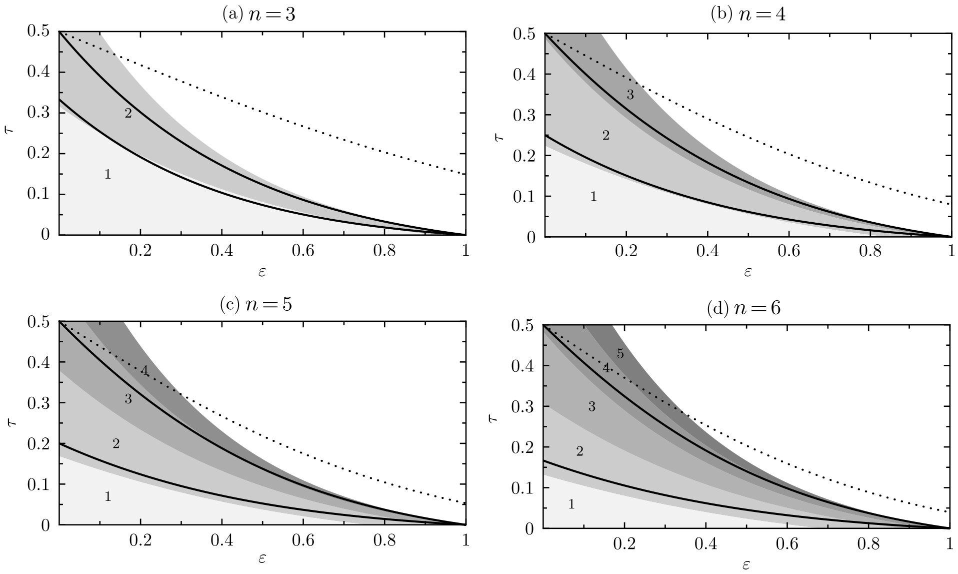


Figure 10. Overfiring regions in the Mirollo–Strogatz model with $b = 3$. The permissible parameter region $\mathcal{M}_V$, see definition 10, lies between the dotted line and the ε-axis. The dotted line represents τ_4 given in definition 10. Region II lies between the two thick black curves. The upper curve is given by τ_1, see (33), and the lower curve by τ_3, see (32). Different shades of grey represent regions with different values of $\nu(T(\tau))$. This value is shown inside each region and ranges from 1 to $n-1$. Notice that the value of $\nu(T(\tau))$ is relevant to our discussion only inside region II since this is the only case in which the attractor ϕ^P is unstable.

4. Discussion

4.1. Applicability of the instability theorem and discontinuity of the evolution operator

We proved theorem 2 in the case that $\nu(T(\tau)) = n - 1$ and we showed that the parameter region (inside region II) for which this condition holds is non-empty for any fixed $b > 0$ and any $n \geqslant 3$. In this section we discuss what happens when $\nu(T(\tau)) < n - 1$. In figure 10, we depict for $b = 3$ the parameter regions in which $\nu(T(\tau))$ has a specific value. The region where $\nu(T(\tau)) = n - 1$ is represented by the darkest shade of grey. Region II lies between the two thick black lines.

We observe that for $n = 3$ the region where $\nu(T(\tau)) = n - 1 = 2$ covers most of region II. For $n = 4$ the situation has changed but still for any ε we can find a small interval of time delays in which $\nu(T(\tau)) = n-1 = 3$. But for $n \geqslant 5$ the situation changes dramatically. Most of the region where $\nu(T(\tau)) = n - 1 = 4$ is now outside region II and for small ε we can find no time delays that belong both in region II and the region where $\nu(T(\tau)) = n - 1$. As the number of oscillators increases the situation becomes progressively worse and for large n there is only a very small intersection (which is not even visible in the figure for $n = 6$) of the region where $\nu(T(\tau)) = n - 1$ and region II for ε very close to 1. Nevertheless, as we proved in proposition 12, such intersection is non-empty for any $n \geqslant 3$.

In figure 10 we also observe that for large n, region II is almost completely covered by regions in which $\nu(T(\tau)) < n - 1$. What can we say about instability in these regions? According to lemma 11, if O_n overfires by $n - 1 - m$ pulses (i.e. if $\nu(T(\tau)) = m$) then the last $n - m$ oscillators to fire should be synchronized. This implies that we can describe phase history functions near ϕ^P by $n - 1$ phases as before, but $n - m$ of them should be equal, so there are only m free variables that characterize phase history functions ϕ near ϕ^P. Therefore, in the case that O_n overfires, we can follow the same arguments as in the previous section to show that ϕ^P is linearly unstable for the neighbourhood of ϕ^P given by proposition 11.

But there is a fundamental problem to this reasoning. ϕ^P is a periodic orbit of Φ. Therefore, we can in principle consider phase history functions not near ϕ^P but near another point along the orbit $\mathcal{O}_+(\phi^P)$. In particular, we can consider phase history functions near $\Phi^\tau(\phi^P)$. Notice, that no oscillator overfires for $\Phi^\tau(\phi^P)$ in the time interval $(-\tau, \tau)$. This implies that no oscillators have to be synchronized in phase history functions ϕ near $\Phi^\tau(\phi^P)$. But if we consider any phase history functions ϕ near ϕ^P with non-synchronized oscillators, then there will be a finite time t (which in this case is $T(\tau)-\tau$, i.e. the time until the oscillator O_n will overfire again) such that the evolution operator Φ^t (and consequently R) are discontinuous for the same reasons as in the example we described in section 2.3.1.

This also explains the discontinuity that we observed numerically in region IId in section 3.1. There, $n = 3$ and $\nu(T(\tau)) = 1$ in region IId, which means that both oscillators O_1 and O_2 should be synchronized for phase history functions ϕ near ϕ^P. Since we considered an initial state with $\theta_1 \neq \theta_2$ we expect the evolution operator to be discontinuous and this is what we observed numerically.

Therefore, the discontinuity of the evolution operator in the specific model studied, presents some issues regarding the instability of certain states because of the overfiring. Notice that this does not mean that all states overfire but nevertheless as pointed out in this section the parameter region in which a specific state (in our case the fixed point ϕ^P) overfires can be much larger than the parameter region where it does not overfire. For this reason we believe that a model with continuous evolution operator would be more appropriate for studying instability in coupled networks. In the following section we propose such a model, which is a modification of the model studied in this paper and for which we conjecture that the evolution operator of the system is continuous.

4.2. A model with continuous evolution operator

We pointed out that the evolution operator of this system is discontinuous. Since this system is modelling physical processes [2, 3, 6], this discontinuity is not merely a curiosity that one can ignore but has to be addressed. In general, in a real physical system (like a neuron network) we cannot expect its components (neurons) to be *perfectly* synchronized as occurs in the present model. This means that we would like to have a model at our disposal that gives the same results for either perfectly synchronized or almost synchronized components. In this respect the present model fails.

A natural question then is, what is the minimal modification to the present model that can make the evolution operator continuous. We propose the following *refractory period model* which is defined exactly as the present model in section 2.1 with the following modification. Consider constants τ_1, τ_2 with $0 < \tau_1 < \tau_2 \ll \tau$. Then redefine the pulse absorption process so that when an oscillator O_i fires at time t then in the time interval $[0, \tau_1]$ any pulse that arrives at the oscillator is ignored, while in the interval $[\tau_1, \tau_2]$ if a pulse arrives at O_i at time t' and its phase is $\theta_i(t'^-) = u_i$ then

$$\theta_i(t') = \theta_i(t) + h(\frac{t' - \tau_1}{\tau_2 - \tau_1})V_1(\theta).$$

Here, $h : [0, 1] \to [0, 1]$ is a strictly increasing smooth function such that $h(0) = 0, h(1) = 1$. If O_i receives a pulse at time $t' > \tau_2$, then it absorbs the pulse completely as in the present model.

In order to see why the evolution operator for the refractory period model is continuous we should revisit section 2.3.1. There, the oscillator O_3 receives two pulses and overfires by one pulse in the phase history function ϕ and then in the nearby phase history function ψ, the same oscillator receives two pulses at different but nearby moments. If we consider the same phase history functions and their evolution in the refractory period model, we observe that when O_3 receives the second pulse, it does not absorb it at all if the time difference between the two pulses is less than τ_1. This means the phases ϕ_3 and ψ_3 are exactly the same. Recall that in section 2.3.1 the discontinuity is caused by the fact that the distance between ϕ_3 and ψ_3 becomes $H_1(0) = O(1)$. Therefore in the refractory period model there is no discontinuity of the evolution operator.

The refractory period model takes into account the fact that neurons have a refractory period in which they cannot absorb any pulses. In this respect our proposal not only makes the evolution operator continuous but also it is more appropriate as a physical model.

Most of the results in this paper concerning unstable attractors in the present model persist for the refractory period model. Moreover, if we consider phase history functions near ϕ^P in the case that O_n overfires, no oscillators have to be synchronized. Then we conjecture that ϕ^P is a linearly unstable attractor for all parameters in region II.

4.3. Conclusions

In this paper we established the existence of unstable attractors in a pulse-coupled oscillator network with delay for arbitrary number of oscillators $n \geqslant 3$. For this purpose we used the mathematical framework given in [4]. The main difference between [4] and this paper is that we introduced a metric in the infinite dimensional state space that allowed us to study issues such as continuity and instability. Moreover, we emphasized the discontinuity of the evolution operator, discussed its effect on the system and proposed a model with continuous evolution operator.

The networks considered in this paper are globally connected and all oscillators are identical. A natural question then would be if unstable attractors persist when changing the connectivity characteristics (topology) of the network and/or when considering non-identical oscillators. It is possible to study networks with different connectivities like random networks [23], small world networks [24] or fractally coupled networks [25, 26]. In random networks, numerical studies [23] show the existence of unstable attractors. Therefore, globally connected networks like the one studied in this paper are not the only ones with unstable attractors. The fact that in the present model the oscillators are identical is essential in our proof of the existence of unstable attractors because it implies that when two oscillators become synchronized they remain synchronized for ever. Therefore, if unstable attractors exist in networks with non-identical oscillators it would be interesting to study what is the exact mechanism that leads to their appearance. It would also be interesting to study the question of existence of unstable attractors in more general coupled map lattices [27, 25, 28].

Another interesting point is the possible existence of heteroclinic connections between unstable attractors [4]. An unstable attractor can represent a task performed by a neuron network. Therefore, the existence of heteroclinic connections means that, in the presence of some external noise, the network can move from one task to another [3] and there are several studies [29–34] which propose that dynamics along heteroclinic orbits are important for information processing in neural systems. In the model studied in this paper there are no heteroclinic connections for $n = 3$ oscillators but we found that there exist such connections for $n \geqslant 4$. We prove the existence of heteroclinic connections in a forthcoming paper [35].

Acknowledgments

The authors would like to thank Tasso Kaper and Marc Timme for useful discussions. KE acknowledges support from the NWO cluster NDNS+.

Appendix A. Technical details

Appendix A.1. Properties of nearby phase history functions

Consider a phase history function $\phi \in \mathcal{P}_{\mathcal{D}}$. Then we show that the characteristics of ϕ determine to a large extent the characteristics of nearby phase history functions $\psi \in \mathcal{P}_{\mathcal{D}}$. In particular, we have the following three propositions. Notice that in the following we make no distinction between a phase history function $\phi : (-\tau, 0] \to \mathbb{T}^n$ and the corresponding extending phase history function $\phi^+ : [-\tau, \infty) \to \mathbb{T}^n$ and we denote both by ϕ.

Proposition 13. *Assume that ϕ_i has no discontinuities in an interval $(s_1, s_2) \subset [-\tau, \tau]$ and that $\phi_i(s) \neq 0 \,(\mathrm{mod}\,\mathbb{Z})$ for all $s \in (s_1, s_2)$. Define $E = \frac{1}{8}M(s_2 - s_1)$ where $M = \min\{V_1(0), 1 - \phi_i(s_2)\}$. Then, if $\psi \in \mathcal{P}_{\mathcal{D}}$ satisfies $d(\phi, \psi) = \eta < E$ we find that $|\phi_i(s) - \psi_i(s)| < \eta_2 = 2\eta/(s_2 - s_1)$ for all $s \in (s_1 + \eta_1, s_2 - \eta_1)$, where $\eta_1 = 2\eta/M$. In particular, ψ_i has no discontinuities in $(s_1 + \eta_1, s_2 - \eta_1)$.*

Proof. Assume, for simplicity, that $s_1 = 0$, $s_2 = S < \tau$ and that $\phi_i(s) = u + s$ with $u > 0$ and $u + S < 1$. Then $M = \min\{V_1(0), 1 - (u + S)\}$.

Suppose that $\psi_i(s)$ has one or more discontinuities in $(\eta_1, S - \eta_1)$ and that one of these discontinuities (caused by $m \geqslant 1$ simultaneous pulses) is at p.

If $\psi_i(p^+) \geqslant \phi_i(p)$, then

$$\mathrm{d}(\phi, \psi) \geqslant \int_p^S |\phi_i(s) - \psi_i(s)|\mathrm{d}s \geqslant (\psi_i(p^+) - \phi_i(p))(S - p) \geqslant (\psi_i(p^+) - \phi_i(p))\eta_1.$$

The second inequality follows from the fact that ϕ_i increases linearly, while ψ_i increases at least linearly. The third inequality follows from $p < S - \eta_1$. Similarly, if $\psi_i(p^-) \leqslant \phi_i(p)$, then

$$\mathrm{d}(\phi, \psi) \geqslant \int_0^p |\phi_i(s) - \psi_i(s)|\mathrm{d}s \geqslant (\phi_i(p) - \psi_i(p^-))p \geqslant (\phi_i(p) - \psi_i(p^-))\eta_1.$$

Again, the second inequality follows from the fact that ϕ_i increases linearly, while ψ_i increases at least linearly. The third inequality follows from $p > \eta_1$. Then, we can distinguish three cases.

If both $\psi_i(p^-)$ and $\psi_i(p^+)$ are greater than $\phi_i(p)$, then $\psi_i(p^+) = \min\{1, \psi_i(p^-) + V_m(\psi_i(p^-))\}$. If $\psi_i(p^+) = 1$ then $\psi_i(p^+) - \phi_i(p) = 1 - \phi_i(p) > 1 - (u + S) \geqslant M$. This means that

$$\mathrm{d}(\phi, \psi) \geqslant M\eta_1 = 2\eta,$$

which is a contradiction. If $\psi_i(p^+) = \psi_i(p^-) + V_m(\psi_i(p^-))$, then $\psi_i(p^+) - \phi_i(p) = V_m(\psi_i(p^-)) + \psi_i(p^-) - \phi_i(p) > V_1(0) \geqslant M$ and we get again a contradiction.

In the second case we assume that both $\psi_i(p^-)$ and $\psi_i(p^+)$ are smaller than $\phi_i(p)$. Then $\phi_i(p) > \psi_i(p^+) = \psi_i(p^-) + V_m(\psi_i(p^-)) > \psi_i(p^-) + V_1(0)$ so we get that $\phi_i(p) - \psi_i(p^-) > V_1(0) \geqslant M$ and

$$\mathrm{d}(\phi, \psi) \geqslant M\eta_1 = 2\eta.$$

In the third case we assume that $\psi_i(p^-) < \phi_i(p)$ and $\psi_i(p^+) > \phi_i(p)$. This implies that $\phi_i(p) - \psi_i(p^-) = \kappa V_m(\psi_i(p^-)) > \kappa V_1(0) \geqslant \kappa M$ for some $\kappa \in (0,1)$ and that $\psi_i(p^+) - \phi_i(p) = \min\{1-\phi_i(p), (1-\kappa)V_1(\psi_i(p^-))\} \geqslant (1-\kappa)M$. Therefore

$$\mathrm{d}(\phi,\psi) \geqslant \kappa M\eta_1 + (1-\kappa)M\eta_1 = 2\eta.$$

In all cases we have reached a contradiction and this implies that ψ_i cannot have any discontinuities in $(\eta_1, S-\eta_1)$.

This also implies that $\psi_i(s) = u' + s$ for $s \in (\eta_1, S-\eta_1)$ and in particular that $\phi_i(s) - \psi_i(s) = u - u'$ is constant in this interval. Then, we obtain

$$\mathrm{d}(\phi,\psi) \geqslant \int_{\eta_1}^{S-\eta_1} |\phi_i(s) - \psi_i(s)|\mathrm{d}s = |u-u'|(S-2\eta_1).$$

Hence, $|u-u'| \leqslant \eta/(S-2\eta_1) \leqslant 2\eta/S$. This concludes the proof of the first statement. □

Proposition 14. *Assume that ϕ_i has a discontinuity at $p \in (-\tau,\tau)$, such that $\phi_i(p^+) = H_m(\phi_i(p^-))$ (i.e. the oscillator O_i receives m simultaneous pulses). Also assume that ϕ_i has no other discontinuities in the open interval $(p-\delta, p+\delta)$ and that $\phi_i(s) \notin \mathbb{Z}$ for all $s \in (p-\delta, p+\delta)$. Then, there is $E' > 0$ such that if $\psi \in \mathcal{P}_\mathcal{D}$ satisfies $d(\phi,\psi) = \eta < E'$ we find that ψ_i receives m pulses in the interval $(p-\eta_1, p+\eta_1)$, where $\eta_1 = 2\eta/M$ and $M = \min\{\phi_i(p+\delta), V_1(0)\}$.*

Proof. Since ϕ_i has no discontinuities in $(p-\delta, p)$ and $(p, p+\delta)$ we can apply the previous result in each one of these intervals. Define $M = \min\{1-\phi_i(p+\delta), V_1(0)\}$. Then for any ψ with $\mathrm{d}(\phi,\psi) = \eta < \frac{1}{8}M\delta$ we conclude that ψ_i has no discontinuities in the intervals $W_1 = (p-\delta+\eta_1, p-\eta_1)$ and $W_2 = (p+\eta_1, p+\delta-\eta_1)$, where $\eta_1 = 2\eta/M$ and $|\phi_i(s) - \psi_i(s)| < \eta_2 = 2\eta/\delta$ in the same intervals. Hence,

$$|\psi_i(p-\eta_1) - \phi_i(p^-) + \eta_1| < \eta_2,$$

and

$$|\psi_i(p+\eta_1) - \phi_i(p^+) - \eta_1| < \eta_2.$$

Combining the two inequalities we obtain

$$2(\eta_1-\eta_2) < \psi_i(p+\eta_1) - \psi_i(p-\eta_1) - V_m(\phi_i(p^-)) < 2(\eta_1+\eta_2).$$

If ψ_i has discontinuities in $(p-\eta_1, p+\eta_1)$ that correspond to reception of κ pulses, then

$$\begin{aligned}\psi_i(p-\eta_1) + V_\kappa(\psi_i(p-\eta_1)) + 2\eta_1 &\leqslant \psi_i(p+\eta_1)\\ &\leqslant \psi_i(p-\eta_1) + 2\eta_1 + V_\kappa(\psi_i(p-\eta_1) + 2\eta_1),\end{aligned}$$

or

$$\begin{aligned}V_\kappa(\psi_i(p-\eta_1)) + 2\eta_1 &\leqslant \psi_i(p+\eta_1) - \psi_i(p-\eta_1),\\ &\leqslant 2\eta_1 + V_\kappa(\psi_i(p-\eta_1)) + V_\kappa'(\psi_i(p-\eta_1))2\eta_1 + O(\eta^2).\end{aligned}$$

From $|\psi_i(p-\eta_1) - \phi_i(p^-) + \eta_1| < \eta_2$, we obtain the estimate

$$V_\kappa(\phi_i(p^-) - (\eta_1+\eta_2)) < V_\kappa(\psi_i(p-\eta_1)) < V_\kappa(\phi_i(p^-) + (\eta_2-\eta_1)),$$

or

$$\begin{aligned}V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\eta_1+\eta_2) + O(\eta^2) &< V_\kappa(\psi_i(p-\eta_1)),\\ &< V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\eta_2-\eta_1) + O(\eta^2).\end{aligned}$$

This implies that

$$
\begin{aligned}
&V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\eta_1+\eta_2) + 2\eta_1 + O(\eta^2)\\
&\qquad \leqslant \psi_i(p+\eta_1) - \psi_i(p-\eta_1)\\
&\qquad \leqslant 2\eta_1 + V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\eta_2-\eta_1) + V_\kappa'(\psi_i(p-\eta_1))2\eta_1 + O(\eta^2),
\end{aligned}
$$

or

$$
\begin{aligned}
-V_\kappa'(\phi_i(p^-))(\eta_1+\eta_2) + 2\eta_1 + O(\eta^2) &\leqslant \psi_i(p+\eta_1) - \psi_i(p-\eta_1) - V_\kappa(\phi_i(p^-))\\
&\leqslant 2\eta_1 + V_\kappa'(\phi_i(p^-))(\eta_2+\eta_1) + O(\eta^2).
\end{aligned}
$$

Combining inequalities we obtain

$$
\begin{aligned}
-\,V_\kappa'(\phi_i(p^-))(\eta_1+\eta_2) - 2\eta_2 + O(\eta^2) &\leqslant V_\kappa(\phi_i(p^-)) - V_m(\phi_i(p^-))\\
&\leqslant V_\kappa'(\phi_i(p^-))(\eta_1+\eta_2) + 2\eta_2 + O(\eta^2)
\end{aligned}
\tag{A.1}
$$

If $\kappa \neq m$ then the difference $|V_\kappa(\phi_i(p^-)) - V_m(\phi_i(p^-))| > |\kappa - m| V_1(0)$ is bounded away from zero. This implies there is some positive $E' < \frac{1}{8}M\delta$ such that the inequality (A.1) does not hold for any $\kappa \neq m$ and $\eta < E'$. Therefore, if $\eta < E'$ we conclude that $\kappa = m$. This concludes the proof of this part. □

Proposition 15. *Assume that ϕ_i has a discontinuity at $p \in (-\tau, \tau)$, such that $\phi_i(p^+) = 1$ (i.e. the oscillator O_i receives $m \geqslant \nu(\phi_i(p^+))$ simultaneous pulses and fires). Also assume that ϕ_i has no other discontinuities in the open interval $(p-\delta, p+\delta)$ and that $\phi_i(s) \notin \mathbb{Z}$ for all $s \in (p-\delta, p+\delta)$. Then, there is $E' > 0$ such that if $\psi \in \mathcal{P}_{\mathcal{D}}$ satisfies $\mathrm{d}(\phi, \psi) = \eta < E'$ we find that ψ_i receives at least $\nu(\phi_i(p^-))$ pulses in the interval $(p-\eta_1, p+\eta_1)$, where $\eta_1 = 2\eta/M$ and $M = \min\{\phi_i(p+\delta), V_1(0)\}$. If ψ_i receives m' pulses with $m' > \nu(\phi_i(p^-))$ then the last $m' - \nu(\phi_i(p^-)) + 1$ pulses are simultaneous.*

Proof. Since ϕ_i has no discontinuities in $(p-\delta, p)$ and $(p, p+\delta)$ we can apply the previous result in each one of these intervals. Define $M = \min\{1 - \phi_i(p+\delta), V_1(0)\}$. Then for any ψ with $\mathrm{d}(\phi, \psi) = \eta < \frac{1}{8}M\delta$ we conclude that ψ_i has no discontinuities in the intervals $W_1 = (p - \delta + \eta_1, p - \eta_1)$ and $W_2 = (p + \eta_1, p + \delta - \eta_1)$, where $\eta_1 = 2\eta/M$ and $|\phi_i(s) - \psi_i(s)| < \eta_2 = 2\eta/\delta$ in the same intervals. Hence,

$$|\psi_i(p-\eta_1) - \phi_i(p^-) + \eta_1| < \eta_2,$$

and

$$|\psi_i(p+\eta_1) - 1 - \eta_1| < \eta_2.$$

Combining the two inequalities we obtain,

$$2(\eta_1 - \eta_2) < \psi_i(p+\eta_1) - \psi_i(p-\eta_1) - (1 - \phi_i(p^-)) < 2(\eta_1 + \eta_2).$$

If ψ_i has discontinuities in $(p-\eta_1, p+\eta_1)$ that correspond to reception of $\kappa < \nu(\phi_i(p))$ pulses, then

$$\psi_i(p+\eta_1) \leqslant \psi_i(p-\eta_1) + 2\eta_1 + V_\kappa(\psi_i(p-\eta_1) + 2\eta_1),$$

or

$$\psi_i(p+\eta_1) \leqslant \psi_i(p-\eta_1) + 2\eta_1 + V_\kappa(\psi_i(p-\eta_1)) + V_\kappa'(\psi_i(p-\eta_1))2\eta_1 + O(\eta^2).$$

As in the previous proposition, since $|\psi_i(p-\eta_1) - \phi_i(p^-) + \eta_1| < \eta_2$, we obtain the estimate

$$
\begin{aligned}
V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\eta_1+\eta_2) + O(\eta^2) &< V_\kappa(\psi_i(p-\eta_1))\\
&< V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\eta_2-\eta_1) + O(\eta^2).
\end{aligned}
$$

Therefore

$$\psi_i(p+\eta_1) \leqslant H_\kappa(\phi_i(p^-)) + \eta_2 + \eta_1 + V'_\kappa(\phi_i(p^-))(\eta_2 - \eta_1) + V'_\kappa(\psi_i(p-\eta_1))2\eta_1 + O(\eta^2).$$

Since $\kappa < \nu(\phi_i(p))$ we deduce that $H_\kappa(\phi_i(p^-))$ therefore taking η small enough we deduce that $\psi_i(p+\eta_1) < 1$ which is a contradiction. This implies that $\kappa \geqslant \nu(\phi_i(p^-))$. Moreover, if $\kappa > \nu(\phi_i(p^-))$ then the last $\kappa - \nu(\phi_i(p^-)) + 1$ pulses must be simultaneous, otherwise $\psi_i(p+\eta_1) > H_1(0)$ which gives a contradiction since $\psi_i(p+\eta_1) = O(\eta)$. □

Appendix A.2. Proof of proposition 6

For $\phi \in \boldsymbol{P}$ we define the open ball of radius ρ around ϕ by

$$B_\rho(\phi) = \{\psi \in \boldsymbol{P} : d(\psi, \phi) < \rho\}.$$

We will prove that for any $\phi \in \hat{S}$ there is $\rho > 0$ such that $B_\rho(\phi) \subset \hat{S}$. This shows that $\hat{S}$ is open.

Since $\phi_n(s) = s$ for $s \in [0, \tau]$ (i.e. no discontinuities and no firings) we use proposition 13 to deduce that there is $\rho > 0$ such that if $\mathrm{d}(\phi, \psi) = \eta < \rho$ then there are $\eta_1, \eta_2 > 0$ such that $|\psi_n(s) - \phi_n(s)| < \eta_2$ for $s \in (\eta_1, \tau - \eta_1)$. Since $\psi_n(0) = 0$ this implies that $\psi_n(s) = s$ for $s \in (0, \tau - \eta_1)$.

Consider now ϕ_i for $i \neq n$. We have that $\phi_i(s) = \theta_i + s$ for $s \in [0, \tau]$. Recall that by definition for $\phi \in S$, $\phi_i(0) = \theta_i$. Then, we deduce that there is $\rho' > 0$ such that if $\mathrm{d}(\phi, \psi) = \eta < \rho'$ then there are $\eta'_1, \eta'_2 > 0$ such that $|\psi_n(s) - \phi_n(s)| < \eta'_2$ for $s \in (\eta'_1, \tau - \eta'_1)$.

We consider the following two cases.

(i) If ϕ_i is continuous at 0 for all $i \neq n$, then there is $\delta > 0$ such that all ϕ_i are continuous in $(-\delta, \tau)$. This implies that there is a $\rho_2 > 0$ such that if $\mathrm{d}(\phi, \psi) = \eta < \rho_2$ then ψ_i has no discontinuities in $(-\delta + \eta_1, \tau - \eta_1)$, where $\eta_1 = O(\eta)$. This implies that if we make ρ_2 small enough, we can ensure that $\eta_1 < \delta/2$. Therefore, no oscillator fires in $(-\tau - \delta + \eta_1, -\eta_1) \supset (-\tau, -\eta_1)$. Moreover, $|\phi_i(s) - \psi_i(s)| < \eta_2$ for $s \in (-\delta + \eta_1, \tau - \eta_1)$ where $\eta_2 = O(\eta)$. Choose again ρ_2 small enough so that $\psi_i(s)$ is bounded away from zero. This implies that $\Sigma_i(\psi) = \emptyset$ for all $i \neq n$ and also that ψ_n has no discontinuities in $(-\delta + \eta_1, \tau - \eta_1)$. Therefore, the oscillator O_n fires only once exactly at 0. This shows that $\Sigma_n(\psi) = \{0\}$. Finally, $|\psi_i(0) - \phi_i(0)| < \eta_2$ therefore by making ρ_2 small enough we can ensure that $\psi_i(0)$ satisfies the defining relations for $\hat{S}$.

(ii) Assume that at least one of the phases ϕ_i, $i \neq n$ is discontinuous at 0. Since $\phi_n(-\tau^+) > 0$, the pulse to ϕ_i cannot come from ϕ_n. In general, assume that there are $1 \leqslant m \leqslant n - 1$ oscillators $O_{j_1}, \ldots, O_{j_m}$ with $j_k \neq n$ for all $k \in \{1, \ldots, m\}$ such that $\phi_{j_k}(-\tau^+) = 0$. Then these oscillators receive $m - 1$ pulses at $t = 0$ while all the other oscillators (including O_n) receive m pulses at $t = 0$. The oscillator O_n fires after receiving these m pulses.
Then there is some $\delta > 0$ such that none of the ϕ_i have any discontinuities in $(-\delta, 0)$. Then we can apply propositions 13–15 in the interval $(-\delta, \delta)$ to deduce that there is $\rho > 0$ such that if $d(\psi, \phi) = \eta < \rho$ then there are $\eta_1, \eta_2 = O(\eta) > 0$ such that
   (a) ψ_{j_k} receive m pulses in $(-\eta_1, \eta_1)$ and they do not fire in the interval $(-\delta + \eta_1, \delta - \eta_1)$,
   (b) $\psi_i, i \neq j_1, \ldots, j_m, n$ receive $m - 1$ pulses in $(-\eta_1, \eta_1)$ and do not fire in the interval $(-\delta + \eta_1, \delta - \eta_1)$,
   (c) ψ_n receives at least 1 pulse in $(-\eta_1, \eta_1)$ and fires exactly once in the interval $(-\delta + \eta_1, \delta - \eta_1)$ and
   (d) all oscillators have no discontinuities in $(\eta_1, \tau - \eta_1)$ and $|\psi_i(s) - \phi_i(s)| < \eta_2$ for all s in this interval.

Moreover, we can choose ρ small enough so that $\eta_1 < \delta/2$. From (d) we conclude that no oscillators fire in $(-\tau + \eta_1, -\eta_1)$.
Notice then that $\psi_n(0) = 0$ and $|\psi_n(s) - s| < \eta_2$ for $s \in (\eta_1, \tau - \eta_1)$. This implies that $\psi_n(s) = s$ in the whole interval $[0, \tau - \eta_1)$ and in particular that ψ_n has no discontinuities in this interval. Therefore no ψ_i, $i \neq n$ fires in $(-\tau, -\eta_1)$. Combining this with (a) and (b) and noticing that $-\delta + \eta_1 < -\eta_1$ we conclude that no ψ_i, $i \neq n$ fires in $(-\tau, 0]$. This means that $\Sigma_i(\psi) = \emptyset$ for $i \neq n$.
Also, from (a)–(c) we conclude that $\psi_{j_1}, \ldots, \psi_{j_m}$ are the oscillators that fire in $(-\tau - \eta_1, -\tau + \eta_1)$ in order to send the corresponding pulses in $(-\eta_1, \eta_1)$. Hence, ψ_n cannot fire in $(-\tau - \eta_1, -\tau + \eta_1)$ and this shows that $\psi_n(-\tau^+) \neq 0$.
Also, all ψ_i are continuous in $(\eta_1, \tau - \eta_1)$ hence ψ_n does not fire in $(-\tau + \eta_1, -\eta_1)$ which means that it does not fire in $(-\tau, -\eta_1)$ and from (c) we know that it fires exactly once in $(-\delta + \eta_1, \delta - \eta_1)$. Combining everything together we conclude that ψ_n fires exactly once at $t = 0$ and $\Sigma_n(\psi) = \{0\}$.
Notice that since no oscillator fires in $(-\tau, 0)$, the oscillators ψ_i have no discontinuities in $[0, \tau)$, therefore $|\psi_i(s) - \phi_i(s)| = |\psi_i(0) - \phi_i(0)| = |\psi_i(0) - \theta_i| < \eta_2$. This shows that $\psi_i(0)$ have values that satisfy $H_1(\psi_i(0) + \tau) > 1$ as long as ρ is chosen to be small enough.

Appendix A.3. Proof of proposition 11

The fixed point ϕ^P (17) is depicted in figure 8(a) for parameters (ε, τ) in region II. Given the form of each ϕ_i^P, $i = 1, \ldots, n$ we apply proposition 13 to conclude that there is a $\rho > 0$ such that if $\mathrm{d}(\phi^P, \phi) = \eta < \rho$, then there are numbers $\eta_1, \eta_2 = O(\eta)$ for which

(i) In $(-\tau + \eta_1, -\eta_1)$, ϕ_i has no discontinuities for any $i \in \{1, \ldots, n\}$, and

$$|\phi_i(s) - \phi_i^P(s)| < \eta_2. \tag{A.2}$$

(ii) In $(\eta_1, \tau - \eta_1)$, ϕ_i has no discontinuities for any $i \in \{1, \ldots, n\}$, and

$$|\phi_i(s) - \phi_i^P(s)| < \eta_2. \tag{A.3}$$

(iii) Each ϕ_i for $i = 1, \ldots, n-1$ receives $n-2$ pulses in $(-\eta_1, \eta_1)$. ϕ_n receives $m \geqslant \nu(T(\tau))$ pulses in the same interval and the last $m - \nu(T(\tau)) + 1$ of them are simultaneous.

Since $\phi_n(0) = 0$, if ϕ_n has any discontinuities in $(0, \eta_1]$, then $\phi_n(\eta_1) - \eta_1 \geqslant V_1(0)$. But we already know from (A.3) that $|\phi_n(\eta_1) - \eta_1| < \eta_2$. Therefore, if we choose ρ small enough so that $\eta_2 < V_1(0)$ we can exclude the possibility that ϕ_n has any discontinuities in $(0, \tau - \eta_1)$. Since $\phi_n(0) = 0$, this implies that $\phi_n(s) = s$ for $s \in [0, \tau - \eta_1)$. In turn, this implies that no ϕ_i, $i \neq n$ fires in $(-\tau, -\eta_1)$. Also, we have established already that no ϕ_i, $i \neq n$ fires in $(-\eta_1, \eta_1)$ therefore $\Sigma_i(\phi) = \emptyset$.

But, each ϕ_i receives $n-2$ pulses in $(-\eta_1, \eta_1)$ while ϕ_n receives at least $\nu(T(\tau)) \geqslant 1$ pulses in $(-\eta_1, 0]$. If $n \geqslant 4$, the only way for all the oscillators (except ϕ_n) to receive $n-2$ pulses is if they receive the pulses from each other and they receive no pulse from ϕ_n. In the case $n = 3$, there is also the possibility that oscillators ϕ_1 and ϕ_2 receive one pulse from ϕ_3 but then ϕ_3 should receive no pulses which is a contradiction. Therefore, for all n, the oscillators ϕ_i, $i \neq n$ receive the $n-2$ pulses from the other oscillators in the same group and receive no pulses from ϕ_n.

This means that ϕ_i fire in $(-\tau - \eta_1, -\tau]$. Also it means that ϕ_n does not fire in $(-\tau - \eta_1, -\tau + \eta_1)$. Putting everything together we conclude that $\Sigma_n(\phi) = \{0\}$. Also these facts imply that in $[0, \tau - \eta_1)$, ϕ_i have no discontinuities. Therefore, $\phi_i(s) = \phi_i(0) + s$ and $|\phi_i(s) - H_{n-2}(\tau) - s| < \eta_2$, from where we get that $|\phi_i(0) - H_{n-2}(\tau)| < \eta_2$.

The oscillator O_n receives $n-1$ pulses in $(-\eta_1, 0]$. If $n-1 > \nu(T(\tau))$ (i.e., ϕ_n^P overfires at 0), then the last $n-\nu(T(\tau))$ must be simultaneous. This concludes the proof.

Appendix A.4. Some useful inequalities

Proposition 16. *For any $V \in \mathcal{F}$, $\tau > 0$ and $\varepsilon > 0$,*

$$H_{n-1}(2\tau) - \tau - H_{n-2}(\tau) > 0. \tag{A.4}$$

Proof. We have

$$\begin{aligned} H_{n-1}(2\tau) - \tau - H_{n-2}(\tau) &= H_1(H_{n-2}(2\tau)) - \tau - H_{n-2}(\tau) \\ &= V_{n-2}(2\tau) - V_{n-2}(\tau) + V_1(2\tau + V_{n-2}(2\tau)). \end{aligned}$$

The last expression is strictly positive, because V_1 is strictly positive and V_{n-2} is strictly increasing. □

Proposition 17. *For any $V \in \mathcal{F}$, $\tau > 0$ and $\varepsilon > 0$,*

$$g_2(\tau) > g_3(\tau), \tag{A.5}$$

where g_2, g_3 are defined in (16).

Proof. We need to prove that

$$H_{n-1}(2\tau) > H_1(\tau + H_{n-2}(\tau)),$$

or equivalently

$$H_1(H_{n-2}(2\tau)) > H_1(\tau + H_{n-2}(\tau)).$$

Since H_1 is strictly increasing the last inequality becomes

$$H_{n-2}(2\tau) > \tau + H_{n-2}(\tau),$$

or $V_{n-2}(2\tau) > V_{n-2}(\tau)$, which is true because V_{n-2} is strictly increasing and $\tau > 0$. □

Proposition 18. *For any $V \in \mathcal{F}$, we have that $H_{n-1}(0) < 1$ if and only if*

$$g_1(0) = H_1(1 + H_{n-2}(0) - H_{n-1}(0)) > 1. \tag{A.6}$$

Proof. We have

$$\begin{aligned} g_1(0) &= H_1(1 + H_{n-2}(0) - H_1(H_{n-2}(0))), \\ &= H_1(1 - V_1(H_{n-2}(0))), \\ &= 1 - V_1(H_{n-2}(0)) + V_1(1 - V_1(H_{n-2}(0))). \end{aligned}$$

Then, since V_1 is strictly increasing we obtain that $g_1(0) > 1$ if and only if

$$1 > H_{n-2}(0) + V_1(H_{n-2}(0)) = H_1(H_{n-2}(0)) = H_{n-1}(0).$$

□

References

[1] Ernst U, Pawelzik K and Geisel T 1995 Synchronization induced by temporal delays in pulse-coupled oscillators *Phys. Rev. Lett.* **74** 1570–3

[2] Timme M, Wolf F and Geisel T 2002 Prevalence of unstable attractors in networks of pulse-coupled oscillators *Phys. Rev. Lett.* **89** 154105

[3] Timme M, Wolf F and Geisel T 2002 Unstable attractors induce perpetual synchronization and desynchronization *Chaos* **13** 377–87

[4] Ashwin P and Timme M 2005 Unstable attractors: existence and robustness in networks of oscillators with delayed pulse coupling *Nonlinearity* **18** 2035–60

[5] Mirollo R and Strogatz S 1990 Synchronization of pulse-coupled biological oscillators *SIAM J. Appl. Math.* **50** 1645–62

[6] Wu W and Chen T 2007 Desynchronization of pulse-coupled oscillators with delayed excitatory coupling *Nonlinearity* **20** 789–808

[7] Pavlidis T 1973 *Biological Oscillators: Their Mathematical Analysis* (New York: Academic)

[8] Winfree A T 1980 *The Geometry of Biological Time* (New York: Springer)

[9] Guckenheimer J and Holmes P 1990 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (New York: Springer)

[10] Milnor J 1985 On the concept of attractor *Commun. Math. Phys.* **99** 177–95

[11] de Melo W and van Strien S 1994 *One-Dimensional Dynamics* (Berlin: Springer)

[12] Ashwin P 1997 Elliptic behaviour in the sawtooth standard map *Phys. Lett.* A **232** 409–16

[13] Leine R I and Nijmeijer H 2004 *Dynamics and Bifurcations of Non-Smooth Mechanical Systems* (Berlin: Springer)

[14] Alexander J C, Kan I, Yorke J A and You Z 1992 Riddled basins *Int. J. Bifurcation Chaos* **2** 795813

[15] Sommerer J C and Ott E A 1993 Physical system with qualitatively uncertain dynamics *Nature* **365** 138–40

[16] Ashwin P and Terry J R 2000 On riddling and weak attractors *Physica* D **142** 87–100

[17] Glendinning P 2001 Milnor attractors and topological attractors of a piecewise linear map *Nonlinearity* **14** 239–57

[18] Diekmann O, van Gills S A, Verduyn Lunel S M and Walther H-O 1995 *Delay Equations: Functional-, Complex-, and Nonlinear Analysis* (Berlin: Springer)

[19] Katok A and Hasselblatt B 1995 *Introduction to the Modern Theory of Dynamical Systems* (Cambridge: Cambridge University Press)

[20] Timme M 2002 Collective dynamics in networks of pulse-coupled oscillators *PhD Thesis* University of Göttingen

[21] Broer H W and Takens F 2008 *Dynamical Systems and Chaos* (Epsilon Uitgaven) (to appear)

[22] Strogatz S H 1994 *Nonlinear Dynamics and Chaos* (Reading, MA: Addison-Wesley)

[23] Zumdieck A, Timme M, Geisel T and Wolf F 2004 Long chaotic transients in complex networks *Phys. Rev. Lett.* **93** 244103

[24] Watts D J and Strogatz S H 1998 Collective dynamics of ‘small-world’ networks *Nature* **393** 440–4

[25] Raghavachari S and Glazier J A 1995 Spatially coherent states in fractally coupled map lattices *Phys. Rev. Lett.* **74** 3297–300

[26] Raghavachari S 1999 Synchrony and chaos in coupled oscillators and neural networks *PhD Thesis* University of Notre Dame, IN

[27] Raghavachari S and Glazier J A 1999 Bursting in neurons with fast and slow inhibition *Preprint* available from http://www.biocomplexity.indiana.edu/cgi-bin/biocomplexity/download.pl?file=051a

[28] Raghavachari S and Glazier J A 1999 Waves in diffusely coupled bursting cells *Phys. Rev. Lett.* **82** 2991–94

[29] Rabinovich M, Volkovskii A, Lecanda P, Huerta R, Abarbanel H D I and Laurent G 2001 Dynamical encoding by networks of competing neuron groups: winnerless competition *Phys. Rev. Lett.* **87** 068102

[30] Huerta R, Nowotny T, Garca-Sanchez M, Abarbanel H D I and Rabinovich M I 2004 Learning classification in the olfactory system of insects *Neural Comput.* **16** 1601–40

[31] Huerta R and Rabinovich M 2004 Reproducible sequence generation in random neural ensembles *Phys. Rev. Lett.* **93** 238104

[32] Hansel D, Mato G and Meunier C 1993 Clustering and slow switching in globally coupled phase oscillators *Phys. Rev.* E **48** 3470–7

[33] Kori H and Kuramoto Y 2001 Slow switching in globally coupled oscillators: robustness and occurrence through delayed coupling *Phys. Rev.* E **63** 046214

[34] Ashwin P and Timme M 2005 When instability makes sense *Nature* **436** 36–7

[35] Broer H W, Efstathiou K and Subramanian E 2007 Heteroclinic connections between unstable attractors submitted for publication