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Uncovering Fractional Monodromy

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Abstract: The uncovering of the role of monodromy in integrable Hamiltonian fibrations has been one of the major advances in the study of integrable Hamiltonian systems in the past few decades: on one hand monodromy turned out to be the most fundamental obstruction to the existence of global action-angle coordinates while, on the other hand, it provided the correct classical analogue for the interpretation of the structure of quantum joint spectra. Fractional monodromy is a generalization of the concept of monodromy: instead of restricting our attention to the toric part of the fibration we extend our scope to also consider singular fibres. In this paper we analyze fractional monodromy for n_1:$(-n_2)$ resonant Hamiltonian systems with n_1, n_2 coprime natural numbers. We consider, in particular, systems that for $n_1, n_2 > 1$ contain one-parameter families of singular fibres which are 'curled tori'. We simplify the geometry of the fibration by passing to an appropriate branched covering. In the branched covering the curled tori and their neighborhood become untwisted thus simplifying the geometry of the fibration: we essentially obtain the same type of generalized monodromy independently of n_1, n_2. Fractional monodromy is then recovered by pushing the results obtained in the branched covering back to the original system.

1. Introduction

In this paper we consider Liouville integrable Hamiltonian systems defined by a smooth Hamiltonian function H on the phase space $\mathbf{R}^4$ with canonical coordinates (q_1, q_2, p_1, p_2). The second, also smooth, integral of motion J is assumed to generate an $\mathbf{S}^1$ symmetry action. The *integral map*

$$F : \mathbf{R}^4 \to \mathbf{R}^2 : p \mapsto F(p) = (J(p), H(p)) \tag{1}$$

encodes the dynamics and the geometry of our systems. F defines a fibration of $\mathbf{R}^4$ which we call *integrable Hamiltonian fibration*.

The complete understanding of the integrable Hamiltonian fibration is one of the central problems in the modern theory of Hamiltonian systems [3,11,16]. The most fundamental fact in this direction is the *Arnol'd-Liouville theorem* [1,2]. Recall that a value $f \in \mathbf{R}^2$ of F is *regular* if DF_p is submersive for all

$$p \in F^{-1}(f) = \{p \in \mathbf{R}^4 \ : \ F(p) = f\},$$

and f is *critical* if this condition fails at some $p \in F^{-1}(f)$. We denote by R the set of regular values of F. The Arnol'd-Liouville theorem tells us that if $f \in \mathsf{R}$ and if the fibre $F^{-1}(f)$ is compact then it is a smooth two dimensional torus $\mathbf{T}^2$ or the disjoint union of such tori. In fact, $F_{\mathsf{R}} = F|F^{-1}(\mathsf{R})$ defines a $\mathbf{T}^2$ fibre bundle. If, on the other hand, f is a *critical value* of F then the fibre $F^{-1}(f)$ has at least one connected component that is not a smooth $\mathbf{T}^2$. This connected component is either a lower dimensional torus ($\mathbf{S}^1$ or point) or a singular set. If F is polynomial this singular set is a singular algebraic variety. Furthermore, the Arnol'd-Liouville theorem tells us that R is open in $\mathbf{R}^2$ and that if $f \in \mathsf{R}$ then there is an open neighborhood $U \subseteq \mathsf{R}$ of f where *local* action-angle coordinates can be defined.

1.1. Standard monodromy. After this basic classification of the fibres of F we can ask what is the *global geometry* of the integrable Hamiltonian fibration. The monodromy of the regular part F_{R} of this fibration gives the first insights and a partial answer to our question. If the monodromy is non-trivial then the bundle F_{R} is non-trivial.[1] Duistermaat [16] was the first to highlight the importance of monodromy to integrable Hamiltonian systems from the point of view of the existence of *global* smooth action-angle coordinates: if the monodromy is non-trivial then such coordinates do not exist.

Let us briefly describe the concept of monodromy as it applies to our context. Consider a regular value $f \in \mathsf{R}$ of F and a piecewise smooth closed path $\Gamma : [0, 1] \to \mathsf{R}$ in the set of regular values of F that starts and ends at $f = \Gamma(0) = \Gamma(1)$. Assume that the fibre $F^{-1}(f)$ is connected, so it is a smooth $\mathbf{T}^2$. The fact that image$(\Gamma) \subset \mathsf{R}$ implies that the number of connected components of $F^{-1}(\Gamma(t))$ remains constant for $t \in [0, 1]$ so all fibres along Γ are $\mathbf{T}^2$. Let (a, b) be a basis of the homology group $H_1(F^{-1}(f)) \simeq \mathbf{Z}^2$. Since the first homology group of regular fibres is a discrete set there is a unique notion of parallel transport of homology cycles along Γ which defines an automorphism of $H_1(F^{-1}(f))$. This automorphism is the same for all paths in the homotopy class $[\Gamma]$ of Γ. Thus one defines on the first fundamental group $\pi_1(\mathsf{R}; f)$ the monodromy map

$$\mu : \pi_1(\mathsf{R}; f) \to \mathrm{Aut}(H_1(F^{-1}(f))),$$

that maps each homotopy class of closed paths in R to the corresponding automorphism. The matrix of $\mu([\Gamma])$ is called the monodromy matrix along Γ and in general is an element of $\mathrm{GL}(2, \mathbf{Z})$. An integrable system has non-trivial monodromy if μ is not trivial, that is, if there is a homotopy class $[\Gamma]$ for which $\mu([\Gamma])$ is not the identity. In other words, if the parallel transport of the initial basis (a, b) along the closed path Γ gives a basis $(a', b') \neq (a, b)$ of $H_1(F^{-1}(f))$ then F has non-trivial monodromy.

We emphasize that the idea of characterizing an aspect of the global geometry of the integrable Hamiltonian fibration by the parallel transport of homology cycles along a closed path is, with appropriate modifications, central to the concept of fractional monodromy and to our approach in the present paper.

[1] The inverse of this statement is not true; monodromy is not the only obstruction to the triviality of F_{R}. All such obstructions are classified in [16,26] but we will not discuss these here.

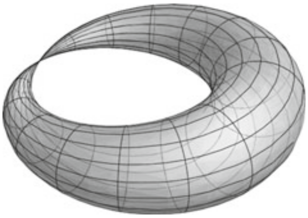

Fig. 1. Single-pinched torus

There are several examples of Hamiltonian systems with non-trivial monodromy and thus no global action coordinates, see [11,13–15,17,23,34,35]. One sufficient condition for a system to have non-trivial monodromy is the existence of *focus-focus* singularities. Consider a critical value c of F such that the fibre $F^{-1}(c)$ is a k-pinched torus, i.e., it contains $k > 0$ focus-focus singularities and no other critical points of F, see Fig. 1. In this case the critical value c is isolated in the image of F, i.e., there is an open neighborhood of c that contains no other critical values of F. Then the *geometric monodromy theorem* [33,36] states that if Γ is a closed path in R that encircles once the critical value c then the monodromy matrix $\mu([\Gamma])$ is conjugate in $\mathrm{GL}(2, \mathbf{Z})$ to

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}).$$

This result highlights the central role that critical fibres play for the global geometry of the fibration and it shows that standard monodromy, although a global aspect of the fibration, has essentially local origins.

1.2. Fractional monodromy. If instead of pinched tori we consider critical fibres $F^{-1}(c)$ with critical points p where $\operatorname{rank} DF_p = 1$ then the situation becomes very different compared to standard monodromy. Such critical fibres are not isolated but generically appear in one-parameter families. When p is a hyperbolic singularity then such families locally separate the phase space. It follows that in this case there is a curve C of critical values in the image of F that locally separates R, i.e., if c is a critical value on C then there is an open neighborhood U of c such that $U \setminus \mathsf{C}$ consists of two disjoint parts of regular values of F separated by C. The geometry of the integrable Hamiltonian fibration in the neighborhood of one such critical value has been studied in [3,10,37]. But in order to understand the role of the whole one-parameter family for the geometry of the integrable Hamiltonian fibration we must consider not only the neighborhood of one critical value but also how the complete family is embedded in phase space. In other words, and unlike the case of pinched tori where we need only local assumptions in order to make statements about the global geometry, here global assumptions about the one-parameter family become essential [5].

The first system studied from this point of view is an 1:(−2) resonant system [19,29,30]. In this system the Hamiltonian H is chosen so that it has the integral

$$J = \frac{1}{2}(p_1^2 + q_1^2) - (p_2^2 + q_2^2),$$

and so that the integral map $F = (J, H)$ has an one-parameter family C_- of critical values (Fig. 2(a)) where each fibre $F^{-1}(c)$, $c \in \mathsf{C}_-$ is a *curled torus*, see Fig. 2(b).[2] The

[2] Dynamically, a curled torus is a hyperbolic periodic orbit with reflection, together with its coinciding stable and unstable manifolds. Geometrically, the curled torus can be seen as a cylinder over a Fig. 8 such that the bottom and top sides of the cylinder are glued together by giving a half-twist to one of them.

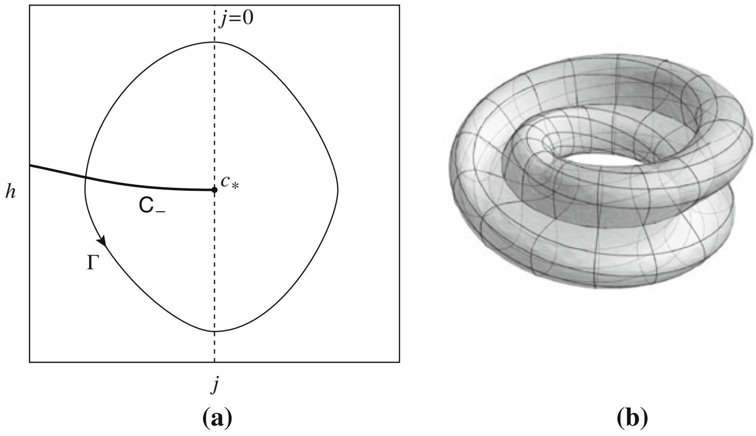

Fig. 2. **(a)** Schematic bifurcation diagram of the 1:(−2) resonant system. **(b)** Curled torus

global assumption that we mentioned before corresponds here to the fact that C_- has an endpoint c_*, see Fig. 2(a), and thus separates R locally but not globally. The consequence of this is that we can take a closed path Γ in the image of F that goes around c_* crossing C_- exactly once. Following the same approach as for standard monodromy [11] we can consider the parallel transport along Γ of a basis of the first homology group $H_1(F^{-1}(j,h))$, where (j,h) is a regular value of $F=(J,H)$. The very surprising fact, first understood intuitively by Zhilinskií, and then proved in [30], is that such parallel transport at the level of homology[3] is actually possible even though Γ goes through a point $c \in \mathsf{C}_-$ that corresponds to a singular fibre of F. More precisely in [30] it is shown that only homology cycles in an index-2 subgroup $\mathsf{H}(j,h)$ of the first homology group $H_1(F^{-1}(j,h))$ can be parallel transported along Γ. Furthermore, for a particular choice of basis (a,b) of $H_1(F^{-1}(j,h))$ the subgroup $\mathsf{H}(j,h)$ is spanned over $\mathbf{Z}$ by $(2a,b)$ and the result of the parallel transport of the latter basis along Γ is $(2a-b,b)$. Expressed in the basis $(2a,b)$ the resulting automorphism of $\mathsf{H}(j,h)$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This property is strongly reminiscent of standard monodromy except for the fact that we have to restrict our attention to the subgroup $\mathsf{H}(j,h)$ of the first homology group $H_1(F^{-1}(j,h))$. The name *fractional monodromy* is due to the fact that the matrix of the resulting automorphism when formally expressed in terms of the basis (a,b) of $H_1(F^{-1}(j,h))$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

We emphasize the formal character of the latter expression, due to the fact that the cycle a cannot be parallel transported along Γ.

[3] The notion of parallel transport of homology cycles used in the proof of standard monodromy has to be generalized in order to take into account that a homology cycle might 'break into two parts' when going through C_-. We discuss in §6 how to define parallel transport in this case.

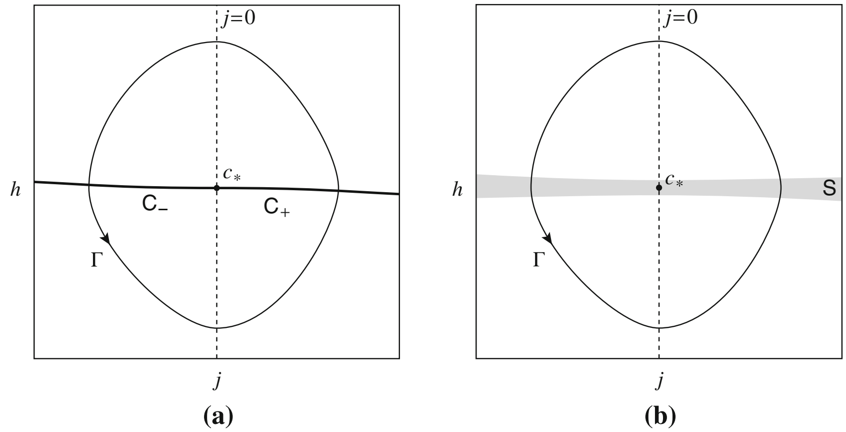

Fig. 3. **(a)** Schematic picture of the bifurcation diagram for n_1:$(-n_2)$ resonant systems as studied in [27,32]. The curves of critical values C_- and C_+ join at the point c_*. **(b)** The situation shown in **(a)** is highly degenerate for $n_1, n_2 \geq 3$ and small perturbations can split each of the curves C_- and C_+ into more than one curve of critical values such that regular values between these new curves lift to the union of disjoint smooth tori in phase space. This new situation is schematically shown in **(b)** where the sets of critical values can be found inside a zone S

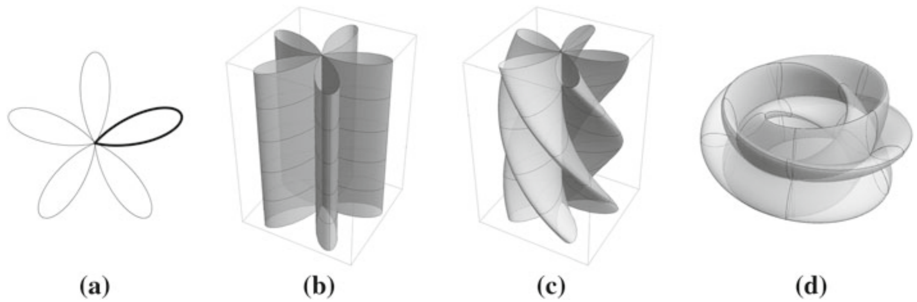

Fig. 4. Representation of a 2:5-curled torus. **(a)** A single 'petal' *(thick curve)* is rotated by angle $\frac{1}{5}(2\pi)$ to form a 'rose' with 5 'petals'. **(b)** A cylinder with finite height is constructed over the 'rose'. **(c)** The upper side of the cylinder is rotated clockwise by $\frac{2}{5}(2\pi)$. **(d)** The upper and lower sides are glued together

Fractional monodromy was soon thereafter studied in $n_1 : (-n_2)$ resonant systems [27,31,32] for n_1, n_2 coprime natural numbers with $n_1 < n_2$. In such systems the Hamiltonian H has, in suitable coordinates, the integral

$$J = \frac{n_1}{2}(p_1^2 + q_1^2) - \frac{n_2}{2}(p_2^2 + q_2^2). \tag{2}$$

The flow of X_J induces in $\mathbf{R}^4$ a resonant n_1:$(-n_2)$ action described in detail in §3.1. For the specific choices of H used in [27,32] the integral map $F = (J, H)$ has one curve of critical values C_- that ends at a point c_* when $n_1 = 1$, or two curves of critical values, C_- and C_+ that join at a point c_* when $n_1 \geq 2$, see Fig. 3. For $f \in \mathsf{C}_-$ the fibre $F^{-1}(j, h)$ is a n_1:n_2-curled torus and for $f \in \mathsf{C}_+$ the fibre $F^{-1}(j, h)$ is a n_2:n_1-curled torus.

A m:n curled torus is most intuitively described in the following way, see Fig. 4. Start with a single 'petal' on a plane, i.e., with a circle which is smooth everywhere except a single singular point. Then construct a 'rose' by joining n copies of the single 'petal' at the singular point, where each copy of the 'petal' is rotated by angle $2\pi/n$ relative to the previous one, see Fig. 4(a). Now consider a (finite height) cylinder over the 'rose', see Fig. 4(b). Glue the upper and lower sides of the cylinder after twisting the upper side by an angle $2\pi(m/n)$ in the clockwise direction, see Fig. 4(c). The resulting gadget, shown in Fig. 4(d), is a m:n-curled torus.

We now come back to probing the global geometry of the integrable Hamiltonian fibration through the parallel transport of homology cycles. Just as we did for 1:(−2) resonant systems we here consider a closed path Γ that goes around c_* and crosses the curves of critical values C_- and C_+ exactly once (the latter only in the case $n_1 \geq 2$), see Fig. 3. Then there is a basis $(n_1 n_2 a, b)$ of an index-$n_1 n_2$ subgroup $\mathsf{H}(j, h)$ of the first homology group $H_1(F^{-1}(j, h))$ which after parallel transport along Γ comes back to $(n_1 n_2 a - b, b)$ [27,32]. Thus in the basis $(n_1 n_2 a, b)$ the monodromy matrix reads

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and formally in the basis (a, b) of $H_1(F^{-1}(j, h))$ the matrix becomes

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{n_1 n_2} & 1 \end{pmatrix}.$$

The situation described so far is highly degenerate when $n_{1,2} \geq 3$. In these cases a generic integrable perturbation of H that commutes with J breaks up the curves C_+ and C_- of curled tori and leaves in their place a much more complicated arrangement of critical and regular fibres that occupy a 'strip' S of small width. This situation has been studied by Nekhoroshev who introduced the notion of *fuzzy fractional monodromy* and showed that removing the degeneracy does not affect fractional monodromy [27,28]. In this paper we show that the precise structure of the critical sets is irrelevant to the existence of fractional monodromy.

More precisely, we prove fractional monodromy for a very general class $\mathcal{F}_{n_1:n_2}$ of n_1:$(-n_2)$ resonant systems, see Definition 1 in §3.2. The class $\mathcal{L}$ of paths Γ around the origin is made precise in Definition 2. In particular, we use the idea of branched coverings to prove the following result extending previous work on fractional monodromy.

Theorem 1 (Fractional monodromy). *Consider a n_1:$(-n_2)$ resonant system $F = (J, H)$ where $H \in \mathcal{F}_{n_1:n_2}$ and a closed path $\Gamma \in \mathcal{L}$ with $\Gamma(0) = (j, h)$. Then there is an index-$n_1 n_2$ subgroup $\mathsf{H}(j, h)$ of $H_1(F^{-1}(j, h))$ such that*

(i) only homology cycles in $\mathsf{H}(j, h)$ can be parallel transported along Γ,
(ii) the parallel transport of homology cycles in $\mathsf{H}(j, h)$ along Γ is unique and it induces an automorphism μ on $\mathsf{H}(j, h)$,
(iii) if $(a(j, h), b(j, h))$ is an ordered basis of $H_1(F^{-1}(j, h))$, where $b(j, h)$ is the homology cycle represented by an integral curve of X_J on $F^{-1}(j, h)$ and $a(j, h)$ is otherwise arbitrary, then $(n_1 n_2 a(j, h), b(j, h))$ is an ordered basis for $\mathsf{H}(j, h)$ and in this basis the automorphsim μ is written as

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

or formally in the basis $(a(j, h), b(j, h))$ as

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{n_1 n_2} & 1 \end{pmatrix}.$$

1.3. Covering maps. The idea of simplifying the study of a dynamical system by passing to an appropriate covering space has been widely used and has proved to be very fruitful. For example, covering maps have been used in the study of Riemann surfaces, in algebraic topology, and also in dynamical systems. We mention here the study of subharmonic bifurcations of periodic and quasiperiodic tori [4,6–9]. In the context of integrable Hamiltonian fibrations it has been already noted in [10,37] that when the $\mathbf{S}^1$ action has $\mathbf{Z}_2$ isotropy at a hyperbolic critical point p then the $\mathbf{S}^1$ action becomes free by passing to a suitable double covering. Note that the situation of an $\mathbf{S}^1$ action with $\mathbf{Z}_2$ isotropy at a hyperbolic critical point p here corresponds to a 1:2 curled torus.

The 'twist' of the curled tori that exist in the n_1:$(-n_2)$ resonant systems considered in this paper complicates the geometry, not only of the critical fibres themselves, but also of nearby regular fibres. Thus it is natural to pass to an appropriate covering space in order to locally 'untwist' the geometry of the fibration and simplify its study. Furthermore, computations in the covering space related to the parallel transport of homology cycles are considerably simpler than similar computations in the original space.

We discuss now briefly how our covering map ρ, to be introduced in §4.1, simplifies the study of fractional monodromy. The map $\rho : \mathbf{C}^2 \to \mathbf{C}^2$ is a $(n_1 n_2)$-fold branched covering; here we identify $\mathbf{R}^4 \simeq \mathbf{C}^2$. The n_1:$(-n_2)$ resonant $\mathbf{S}^1$ action becomes 1:(-1) resonant in the covering space and is thus free in $\mathbf{C}^2 \backslash \{0\}$. Using ρ we pull back the integrable Hamiltonian fibration of our n_1:$(-n_2)$ resonant systems to the covering space. Each fibre $F^{-1}(j, h)$ lifts to a fibre $\tilde{F}^{-1}(j, h)$ in the covering space. Provided that the 1:(-1) resonant $\mathbf{S}^1$ action on $\tilde{F}^{-1}(j, h)$ is free, that is $\tilde{F}^{-1}(j, h)$ does not contain the origin $0 \in \mathbf{C}^2$, we have the following description of $\tilde{F}^{-1}(j, h)$. The intersection of $\tilde{F}^{-1}(j, h)$ with a suitably chosen Poincaré surface of section is a one-dimensional set $\tilde{\lambda}(j, h)$ which typically is a disjoint union of smooth circles. Furthermore, $\tilde{\lambda}(j, h)$ can be seen as a global section for the restriction of the free 1:(-1) resonant $\mathbf{S}^1$ action on $\tilde{F}^{-1}(j, h)$. Thus $\tilde{F}^{-1}(j, h)$ is diffeomorphic to $\tilde{\lambda}(j, h) \times \mathbf{S}^1$.

If, in particular, $\tilde{F}^{-1}(j, h) \simeq \mathbf{T}^2$ which implies that $\tilde{\lambda}(j, h) \simeq \mathbf{S}^1$ (or if we restrict our attention to a $\mathbf{T}^2$ component of $\tilde{F}^{-1}(j, h)$) then a basis of the homology group $H_1(\tilde{F}^{-1}(j, h)) \simeq \mathbf{Z}^2$ is given by two cycles: the first cycle is represented by $\tilde{\lambda}(j, h)$ while the second cycle is represented by an orbit of the $\mathbf{S}^1$ action. This very simple description of a basis of $H_1(\tilde{F}^{-1}(j, h))$ permits one to easily compute the parallel transport of homology cycles in the covering space.

The deck group $D \simeq \mathbf{Z}_{n_1 n_2}$ of ρ here plays an important role. All objects that are lifted through ρ to the covering space are either invariant or equivariant with respect to the action of D. This in particular allows the use of D in order to simplify the study of the fibres and their homology in the covering space, but also in order to transfer the results from the covering back to the original space.

Note finally that the covering map does not trivialize the *whole* fibration. Indeed the parallel transport of homology cycles in the covering space is not trivial and is reminiscent of standard monodromy, see Proposition 2.

1.4. Overview of the paper. In this paper we prove fractional monodromy for a large class of Hamiltonian functions H that commute with the n_1:$(-n_2)$ resonant oscillator J in Eq. (2). The class of Hamiltonian functions that we consider is defined in §3.2 and it includes as special cases all systems considered in [12,19,22,27–30,32].

We give a short overview. In §2 we give specific examples of resonant systems with standard or fractional monodromy emphasizing their common properties. In §3 we describe in detail the setup for Theorem 1. In particular, in §3.1 we present some common properties of $n_1{:}(-n_2)$ resonant systems focusing on the choice of Poincaré surfaces of section and the isotropy of the induced $\mathbf{S}^1$ action on the phase space and on these surfaces; in §3.2 we precisely define the class of Hamiltonians for which we prove the existence of standard or fractional monodromy; and in §3.3 we define the class of closed paths along which we consider monodromy. In the rest of the paper we set up the methods for proving Theorem 1. In §4 we define the branched covering used for the proof and study the properties of the lift of the fibration to the covering space, including the parallel transport of homology cycles in the covering space. In §5 we show how the covering map acts on the homology of individual fibres. In §6 we define a generalized parallel transport of homology cycles that can be applied to our systems. We compute the parallel transport of homology cycles in the covering space and then push the results back to the original space to obtain the parallel transport there. In §7 we combine the previously obtained results to give the proof of Theorem 1. Finally, in §8 we further discuss our approach and results.

2. Examples

In this section we give a few examples of $n_1{:}(-n_2)$ resonant Hamiltonian systems with standard or fractional monodromy. Most of these examples have already appeared in the literature and we use them here in order to motivate and explain our choices in §3.2.

2.1. Hamiltonian systems. We consider $n_1{:}(-n_2)$ resonant systems described by a Hamiltonian of the form

$$H = \delta R + \operatorname{Im}(z_1^{n_2} z_2^{n_1}) + \varepsilon R^s, \tag{3}$$

where $R = \frac{n_1}{2}|z_1|^2 + \frac{n_2}{2}|z_2|^2$, and $2s > n_1 + n_2$ in order to ensure compactness of the fibres. Here $z_k = q_k + \mathrm{i} p_k$, for $k = 1, 2$. We consider in particular the following cases:

(i) $1{:}(-1)$ resonant system, $n_1 = 1, n_2 = 1, s = 2, \delta = 0$.
(ii) $1{:}(-2)$ resonant system, $n_1 = 1, n_2 = 2, s = 2, \delta = 0$.
(iii) $2{:}(-5)$ resonant system, $n_1 = 2, n_2 = 5, s = 4, \delta = 0$.
(iv) $3{:}(-5)$ resonant system, $n_1 = 3, n_2 = 5, s = 5, \delta = 0$.
(v) Detuned $3{:}(-5)$ resonant system, $n_1 = 3, n_2 = 5, s = 5, \delta = 2/100$.

The parameter ε is a scaling parameter and does not qualitatively affect the geometry of the system.

Note that for the first four systems with $\delta = 0$ the geometry near the origin is determined by the term of order $n_1 + n_2$. In the last case the geometry near the origin is described by the term δR but in a slightly larger neighborhood of the origin the terms of order $n_1 + n_2$ become again important and as we show later the two $3{:}(-5)$ resonant systems have the same type of fractional monodromy although the two fibrations are not equivalent. In the following we refer to the systems (i) and (ii), where $n_1 = 1$, as *lower order resonances* and to the rest of the systems where $n_1 > 1$ as *higher order resonances*.

2.2. Restriction to a Poincaré section. Consider in phase space the Poincaré surface of section

$$\Sigma_0^+ = \{(q, p) \in J^{-1}(0) \,:\, p_1 = 0,\ q_1 \geq 0\},$$

and define on Σ_0^+ coordinates $(x, y) = (q_2, p_2)$. Recall that J, given in (2), describes the $n_1{:}(-n_2)$ resonant oscillator. Denote by H_0^+ the restriction of H on Σ_0^+ and by $\lambda^+(0, h)$

the h-level set of H_0^+. We describe here the common features of the level sets $\lambda^+(0, h)$ for the examples given in §2.1. These features will turn out to play a crucial role in the subsequent discussion of fractional monodromy in the more general setting.

In all cases we find that there is a range of values h for which the level set $\lambda^+(0, h)$ is a smooth circle with winding number[4] equal to 1 and $\mathbf{Z}_{n_1}$ symmetry and another range of values h for which $\lambda^+(0, h)$ consists of n_1 smooth circles, each with winding number 0.

Recall that on Σ_0^+ we have $\mathrm{Arg}(z_1) = 0$, thus $z_1 = |z_1|$, and we use coordinates $z = z_2 \in \mathbf{C}$; in the following we write $z = x + \mathrm{i}y$. Thus from (3) we obtain that the restriction of H to Σ_0^+ is

$$H_0^+ = \delta R_0 + \left(\frac{n_2}{n_1}\right)^{\frac{n_2}{2}} (x^2 + y^2)^{\frac{n_2}{2}} \,\mathrm{Im}[(x + \mathrm{i}y)^{n_1}] + \varepsilon R_0^s,$$

where $R_0 = n_2(x^2 + y^2)$.

In Fig. 5 we show the level sets $\lambda^+(0, h)$ for different values of h for the resonant systems described by the Hamiltonian functions (3). Starting with the 1:(−1) and 1:(−2) resonant systems (Fig. 5(a) and Fig. 5(b) respectively), both having $n_1 = 1$, we observe that the level set $\lambda^+(0, h)$ is always a smooth circle. For $h > 0$ the circle has winding number 1 with respect to the origin, while for $h < 0$ it has winding number 0.

For the rest of the systems where $n_1 > 1$ the situation is different. The level set $\lambda^+(0, h)$ for 2:(−5) and 3:(−5) resonant systems is for $h > 0$ a circle with winding number 1. At $h = 0$ the level set goes through the origin and it is no longer a manifold since the origin has non-trivial isotropy $\mathbf{Z}_{n_1}$. For $h < 0$ the level set breaks into n_1 pieces (2 and 3 respectively for the 2:(−5) and 3:(−5) systems). These pieces are mapped to each other by the $\mathbf{Z}_{n_1}$ action on Σ_0^+. Recall that the level set $\lambda^+(0, h)$ is the intersection with Σ_0^+ of the fibre $F^{-1}(0, h)$. In all cases shown until now, when $h \neq 0$ the fibre $F^{-1}(0, h)$ is a $\mathbf{T}^2$. This means that the n_1 pieces of $\lambda^+(0, h)$ for $h < 0$ are intersections of the same $\mathbf{T}^2$ with Σ_0^+.

Finally, for the detuned 3:(−5) system shown in Fig. 5(e) we observe that for $h > h_c$, where $h_c \simeq 0.015304$ the level set $\lambda^+(0, h)$ is a circle with winding number 1 and $\mathbf{Z}_3$ symmetry, while for $h < 0$ it consists of 3 circles with winding number 0 that are mapped to each other by the $\mathbf{Z}_3$ action. For $0 < h < h_c$ the fibre $F^{-1}(0, h)$ consists of two connected components which are both $\mathbf{T}^2$. The corresponding level set $\lambda^+(0, h)$ consists of a circle with winding number 1 that comes from one $\mathbf{T}^2$ and 3 circles with winding number 0 that come from the second $\mathbf{T}^2$.

2.3. Bifurcation diagrams. In Fig. 6 we show the bifurcation diagrams of the examples of resonant systems given in §2.1. Note that in all cases the origin c_* is a critical value of the integral map $F = (J, H)$ and the corresponding fibre $F^{-1}(c_*)$ is singular. Other critical values are represented by solid curves and regular values by different shades of gray. The white region represents values outside the image of F.

The sets C_+ and C_- appearing in these bifurcation diagrams are defined by Eq. (8) in §3.1. They can consist either of critical or regular values of F depending on the values of n_1 and n_2. In particular, they always correspond to critical values for higher order resonances. When C_+ or C_- consist of regular values they are depicted by a dashed curve in Fig. 6. Note that in all bifurcation diagrams the set $\mathsf{C} = \mathsf{C}_- \cup \{c_*\} \cup \mathsf{C}_+$ separates

[4] By the winding number of a circle we mean the winding number of a closed oriented path that starts at a point on the circle and proceeds counterclockwise until it reaches the initial point. Furthermore, in what follows we consider only the winding number with respect to the origin and we will refer to that as simply the winding number of the curve.

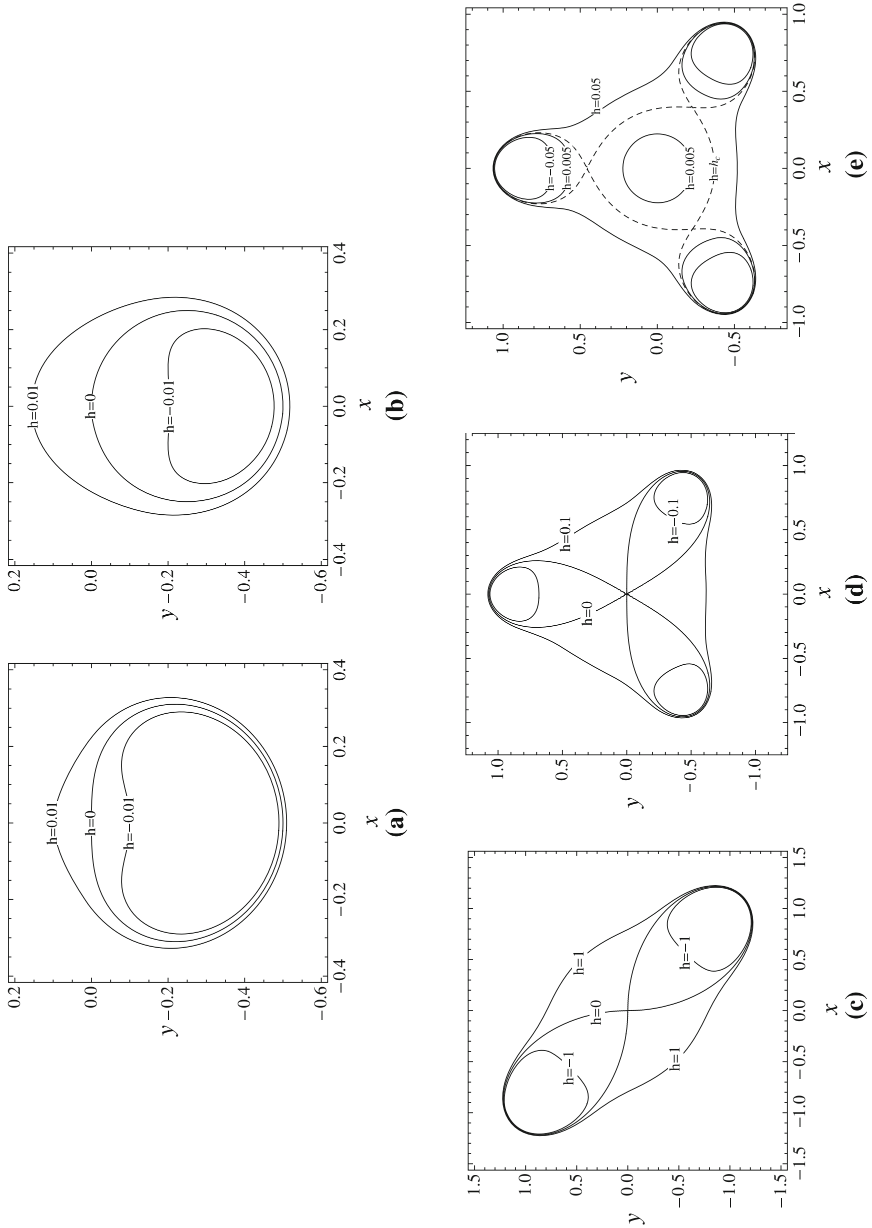

Fig. 5. Level sets $\lambda^+(0, h)$. **(a)** 1:(−1) resonant system ($s = 2$, $\varepsilon = 1$). **(b)** 1:(−2) resonant system ($s = 2$, $\varepsilon = 1$). **(c)** 2:(−5) resonant system ($s = 4$, $\varepsilon = 10^{-2}$). **(d)** 3:(−5) resonant system ($s = 5$, $\varepsilon = 10^{-3}$). **(e)** Detuned 3:(−5) resonant system ($s = 5$, $\varepsilon = 10^{-3}$, $\delta = 0.02$)

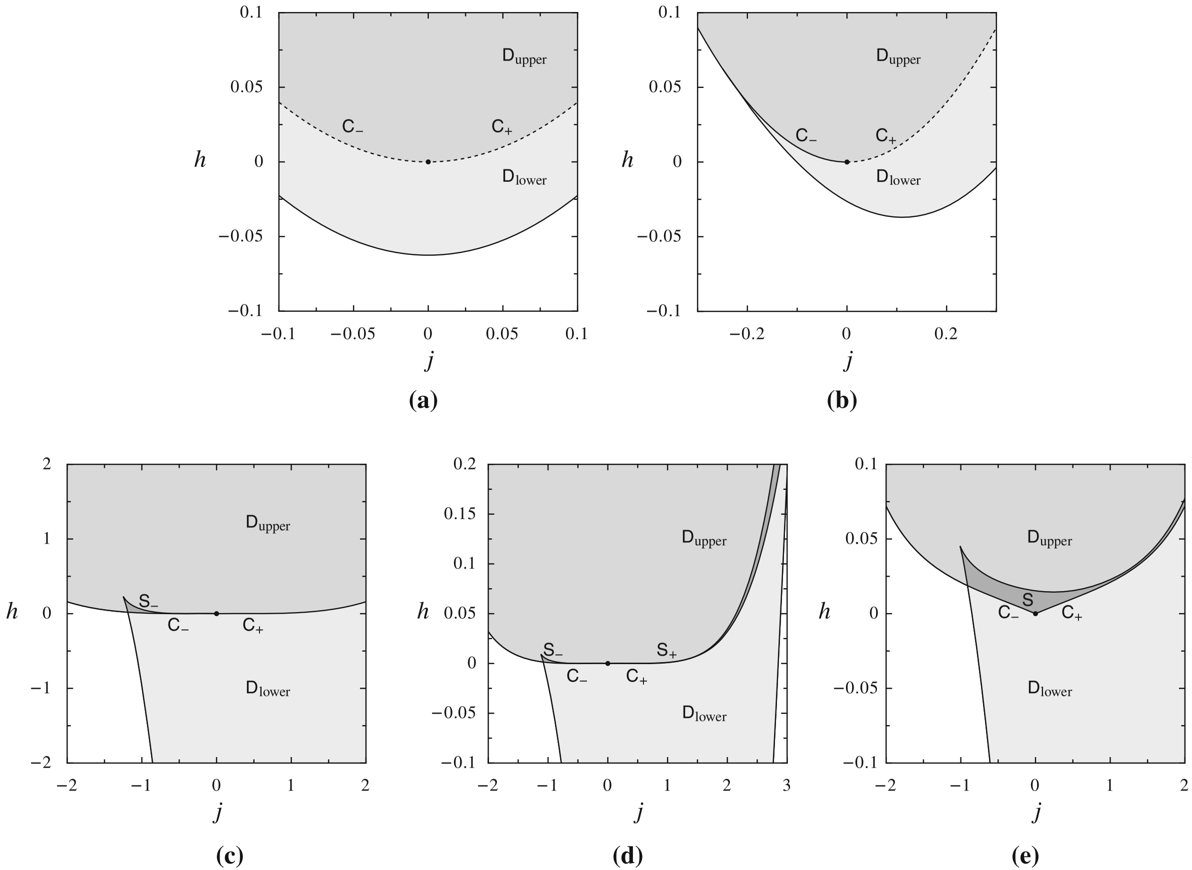

Fig. 6. Bifurcation diagrams. **(a)** 1:(−1) resonant system ($s = 2$, $\varepsilon = 1$). **(b)** 1:(−2) resonant system ($s = 2$, $\varepsilon = 1$). **(c)** 2:(−5) resonant system ($s = 4$, $\varepsilon = 10^{-2}$). **(d)** 3:(−5) resonant system ($s = 5$, $\varepsilon = 10^{-3}$). **(e)** Detuned 3:(−5) resonant system ($s = 5$, $\varepsilon = 10^{-3}$, $\delta = 0.02$)

the image of F into two parts, one above C and one below C. When $n_1 > 1$, which also implies $n_2 > 1$, both C_+ and C_- are sets of critical values of F. Therefore, it is impossible to have a continuous path that goes from a regular value below C to a regular value above C without going through a critical value of F. When $n_1 = 1$ but $n_2 > 1$, then C_- is still a set of critical values, but C_+ is, in our examples, a set of regular values of F and thus the set R of regular values of F is connected. Finally, if $n_1 = n_2 = 1$ then both C_+ and C_- are sets of regular values of F.

In the examples of higher order resonances, 2:(−5), 3:(−5), and detuned 3:(−5), we mark in Fig. 6 regions S_+, S_- and S. In the interior of such regions each regular value corresponds to a fibre that consists of two disjoint $\mathbf{T}^2$, while the boundary of these regions is formed by critical values of F. In particular, C_+ forms the lower part of the boundary of S_+, C_- forms the lower part of the boundary of S_-, and both C_+ and C_-, together with $\{c_*\}$ form the lower part of the boundary of S.

Recall from §2.2 that in all these examples there is a range of values of h, and in particular a value $h_{\text{upper}} > 0$, for which the level set $\lambda^+(0, h_{\text{upper}})$ is a smooth circle with winding number 1 and $\mathbf{Z}_{n_1}$ symmetry and, similarly, a value $h_{\text{lower}} < 0$ for which $\lambda^+(0, h_{\text{lower}})$ consists of n_1 smooth circles, each with winding number 0. In the bifurcation diagrams we mark by $\mathsf{D}_{\text{upper}}$ (respectively $\mathsf{D}_{\text{lower}}$) the maximal connected set of regular values that lies above (respectively below) C and contains h_{upper} (respectively h_{lower}). We also denote $\mathsf{D} = \mathsf{D}_{\text{upper}} \cup \mathsf{D}_{\text{lower}}$. Note that for higher order resonances $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$ are always separated by critical values of F. For the lower order resonances 1:(−1) and 1:(−2) the regions $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$ are separated by the set C which contains also regular values. In this case we distinguish $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$ because the level sets $\lambda^{\pm}(j, h)$ in these regions have different winding numbers. Finally, note that in these cases of lower order resonances we have that $\mathsf{D}' := \mathsf{D} \cup (\partial\mathsf{D} \cap \mathsf{R}) = \mathsf{R}$.

2.4. Monodromy. We now describe the monodromy in our examples, cf. Theorem 1. The 1:(−1) resonant system has only one focus-focus point at the origin and it thus has standard monodromy, see also [18,29,30]. The 1:(−2) resonant system has been studied in [12,19,29,30,32] where it was shown to have fractional monodromy. Resonant 2:(−5) and 3:(−5) systems have been studied in [32] in a degenerate case and in [27] in the non-degenerate case presented here. The case of the detuned 3:(−5) resonant system has not been studied in the literature, although the methods of [27] can be extended also to this case.

In all cases we have the following description of monodromy. Consider any closed path Γ that goes once around the origin and starts and ends at a regular value (j, h) of F with $(j, h) \in \mathsf{D}' := \mathsf{D} \cup (\partial\mathsf{D} \cap \mathsf{R})$. Recall here that in the 1:(−1) and 1:(−2) cases we have $\mathsf{D}' = \mathsf{R}$, while in the higher order resonances an inspection of the bifurcation diagrams shows that $\partial\mathsf{D} \cap \mathsf{R} = \emptyset$ and thus $\mathsf{D}' = \mathsf{D}$. Furthermore, in all cases we assume that Γ crosses the $j = 0$ axis only at points inside D. Then there is a basis $(a(j, h), b(j, h))$ of $H_1(F^{-1}(j, h))$ and a subgroup $\mathsf{H}(j, h)$ spanned over $\mathbf{Z}$ by $(n_1 n_2 a(j, h), b(j, h))$ such that only homology cycles in $\mathsf{H}(j, h)$ can be parallel transported along Γ and such parallel transport induces an automorphism of $\mathsf{H}(j, h)$. The automorphism is written in this basis as

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and in the basis $(a(j,h), b(j,h))$ of $H_1(F^{-1}(j,h))$ as

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{n_1 n_2} & 1 \end{pmatrix}.$$

Note that in the 1:(−1) resonant case $\mathsf{H}(j,h) = H_1(F^{-1}(j,h))$ thus the system has standard monodromy. In all other cases the system has fractional monodromy and its type is determined by $n_1 n_2$.

3. Definitions

We consider an integrable Hamiltonian 2-DOF system on $\mathbf{R}^4$ with canonical coordinates (q_1, q_2, p_1, p_2), given by a smooth Hamiltonian function H which Poisson commutes with the n_1:$(-n_2)$ resonant oscillator J in Eq. (2). We assume that n_1, n_2 are coprime integers with $1 \le n_1 \le n_2$; we allow in particular the case of 1:(−1) resonance. We first give some general facts about Hamiltonians that commute with the n_1:$(-n_2)$ resonant oscillator, define suitable Poincaré surfaces of section for the flow of X_J, and study the corresponding Poincaré maps. Then we specify, in §3.2, the class of Hamiltonian functions H and, in §3.3, the class of closed paths for which we prove Theorem 1 on fractional monodromy.

3.1. Poincaré surfaces of section. We identify $\mathbf{R}^4$ and $\mathbf{C}^2$ by the assignment $z_k = q_k + \mathrm{i} p_k$, $k = 1, 2$. In suitable coordinates (z_1, z_2) the resonant oscillator J is written as

$$J = \frac{n_1}{2}|z_1|^2 - \frac{n_2}{2}|z_2|^2. \tag{4}$$

The flow φ_J^t of the Hamiltonian vector field X_J on $\mathbf{C}^2$ defines the $\mathbf{S}^1$ action

$$t, (z_1, z_2) \mapsto \varphi_J^t(z_1, z_2) = (z_1 \exp(\mathrm{i} n_1 t), z_2 \exp(-\mathrm{i} n_2 t)). \tag{5}$$

The $\mathbf{S}^1$ action (5) has trivial isotropy whenever $z_1 \neq 0$ and $z_2 \neq 0$; it has $\mathbf{Z}_{n_1}$ isotropy when $z_2 = 0$, and $\mathbf{Z}_{n_2}$ isotropy when $z_1 = 0$. The origin $0 \in \mathbf{C}^2$ is a fixed point of the $\mathbf{S}^1$ action.

On the level set $J^{-1}(j)$ consider the two-dimensional Poincaré section given by

$$\Sigma_j^+ = \{(z_1, z_2) \in J^{-1}(j) \,:\, \mathrm{Arg}(z_1) = 0 \text{ or } z_1 = 0\}, \text{ for } j \ge 0,$$

and

$$\Sigma_j^- = \{(z_1, z_2) \in J^{-1}(j) \,:\, \mathrm{Arg}(z_2) = 0 \text{ or } z_2 = 0\}, \text{ for } j \le 0.$$

Note that separately each one of the families Σ_j^+ and Σ_j^- depends continuously on j but the whole family of surfaces of section is discontinuous at $j = 0$ where we have two distinct surfaces Σ_0^+ and Σ_0^-.

Remark 1. The same family $\Sigma_j^\pm$ of Poincaré sections was first used for n_1:$(-n_2)$ resonant systems in [28].

For $j \neq 0$, the surfaces $\Sigma_j^\pm$ are smooth manifolds diffeomorphic to the complex plane $\mathbf{C}$. The surfaces $\Sigma_0^\pm$ are half-cones and are thus only homeomorphic to $\mathbf{C}$. Therefore all surfaces of section can be described using the coordinates $z \in \mathbf{C}$. In particular, for Σ_j^+ we let $z = z_2$, while for Σ_j^- we let $z = z_1$. The corresponding inclusion maps

$$\iota_j^{\pm} : \Sigma_j^{\pm} \to \mathbf{C}^2 : z \mapsto (z_1, z_2) = \iota_j^{\pm}(z)$$

are smooth for $j \neq 0$ and continuous for $j = 0$.

The n_1:$(-n_2)$ resonant $\mathbf{S}^1$ action (5) on $\mathbf{C}^2$ induces the action of the cyclic group $\mathbf{Z}_{n_\pm}$ on $\Sigma_j^{\pm}$. Specifically, the flow φ_J^t of X_J in $J^{-1}(j)$ defines the Poincaré maps $\Pi_j^{\pm} : \Sigma_j^{\pm} \to \Sigma_j^{\pm}$ given by

$$\Pi_j^{\pm}(z) = \varphi_J^{2\pi/n_\pm}(\iota_j^{\pm}(z)),$$

where we let $n_+ = n_1, n_- = n_2$, and here $2\pi/n_\pm$ is the first return time for all points on $\Sigma_j^{\pm}$ (except for $0 \in \Sigma_0^{\pm}$ which is an equilibrium of X_J). Equation (5) then gives that

$$\Pi_j^{\pm}(z) = z \exp(\mp 2\pi \mathrm{i} m_\pm / n_\pm), \tag{6}$$

where on Σ_j^{+} we have $m_+ = n_2, n_+ = n_1$ while on Σ_j^{-} we have $m_- = n_1, n_- = n_2$.

Note that since the Hamiltonian function H on $\mathbf{C}^2$ commutes with J, then $H \circ \varphi_J^t = H$ and thus the restriction $H_j^{\pm}$ of H on $\Sigma_j^{\pm}$ satisfies

$$H_j^{\pm} \circ \Pi_j^{\pm} = H_j^{\pm}. \tag{7}$$

Equation (7), together with the fact that $m_\pm$ and $n_\pm$ are coprime, implies that $H_j^{\pm}$ is a $\mathbf{Z}_{n_\pm}$ invariant function where the action of the cyclic group $\mathbf{Z}_{n_\pm}$ on $\Sigma_j^{\pm}$ is given by

$$\mathbf{Z}_{n_\pm} \times \Sigma_j^{\pm} \to \Sigma_j^{\pm} : (k, z) \mapsto z \exp(2\pi \mathrm{i} k / n_\pm).$$

Note that $0 \in \Sigma_j^{\pm}$ is a fixed point of the $\mathbf{Z}_{n_\pm}$ action and thus has non-trivial isotropy for $n_\pm > 1$. Comparing definitions one sees that the point $0 \in \Sigma_j^{\pm}$ is the intersection with $\Sigma_j^{\pm}$ of an orbit of the $\mathbf{S}^1$ action with the same non-trivial isotropy $\mathbf{Z}_{n_\pm}$.

Denote by $\lambda^{\pm}(j, h)$ the level set $(H_j^{\pm})^{-1}(h)$ on $\Sigma_j^{\pm}$. It follows from the preceding discussion that such level sets are $\mathbf{Z}_{n_\pm}$ invariant. Furthermore, in the interior of the image of F, denoted by image$(F)^\circ$, define the sets

$$\begin{aligned} \mathsf{C}_+ &= \{(j, h) \in \mathrm{image}(F)^\circ \ : \ j > 0, \ h = H_j^{+}(0)\}, \\ \mathsf{C}_- &= \{(j, h) \in \mathrm{image}(F)^\circ \ : \ j < 0, \ h = H_j^{-}(0)\}, \end{aligned} \tag{8}$$

that is, if $(j, h) \in \mathsf{C}_\pm$ then the level set $\lambda^{\pm}(j, h)$ contains the origin on $\Sigma_j^{\pm}$. For $n_- = n_2 > 1$ the $\mathbf{Z}_{n_-}$ action on Σ_j^{-}, $j < 0$, has non-trivial isotropy at the origin. This implies that in this case the level set $\lambda^{-}(j, h)$, and thus also the corresponding fibre $F^{-1}(j, h)$, is not a smooth manifold and furthermore, that C_- is a set of critical values of F. The same is true for C_+ whenever $n_+ = n_1 > 1$ but note that in this case we also have $n_- = n_2 > 1$, so when $n_1 > 1$ both C_+ and C_- are sets of critical values of F.

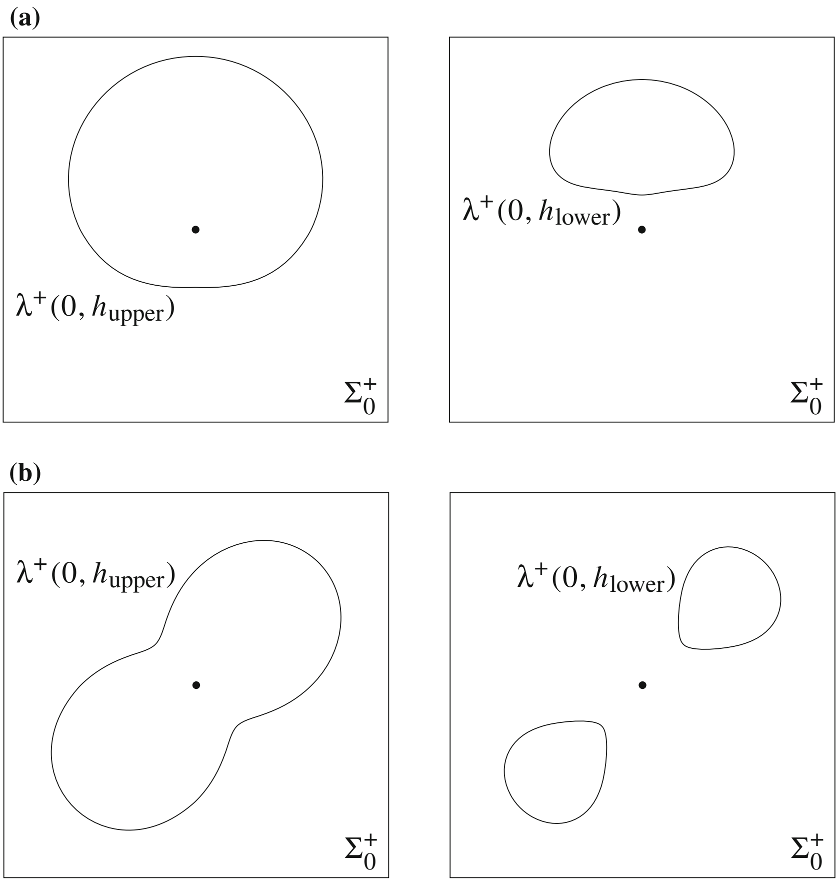

Fig. 7. Schematic representation of level sets $\lambda^+(0, h_{\text{upper}})$ and $\lambda^+(0, h_{\text{lower}})$ for the Hamiltonian system in $\mathcal{F}_{n_1:n_2}$. **(a)** $1{:}(-n_2)$ resonant system. **(b)** $2{:}(-n_2)$ resonant system. Note that in the Poincaré section Σ_0^+ only the value of n_1 is important and that for $n_1 = 1$ both level sets $\lambda^+(0, h_{\text{upper}})$ and $\lambda^+(0, h_{\text{lower}})$ consist of a single connected component but with different winding numbers

3.2. Hamiltonian functions. We now define the class of Hamiltonian systems to be considered in this paper. Recall that all the examples described in §2 share some common characteristics. In particular, in all cases there is a regular value $(0, h_{\text{upper}})$ of F for which the corresponding level set $\lambda^+(0, h_{\text{upper}})$ is a single smooth circle with winding number 1 and a regular value $(0, h_{\text{lower}})$ for which the corresponding level set $\lambda^+(0, h_{\text{lower}})$ consists of n_1 smooth circles with winding number 0. The following definition is motivated by these examples and is further illustrated in Fig. 7 for the cases of $1{:}(-n_2)$ and $2{:}(-n_2)$ systems.

Definition 1 (Hamiltonian functions). *The set $\mathcal{F}_{n_1:n_2}$ consists of those C^∞ Hamiltonian functions $H : \mathbf{C}^2 \to \mathbf{R}$ that satisfy the following:*

(i) H Poisson commutes with the $n_1{:}(-n_2)$ resonance J in Eq. (4).
(ii) For any j the critical points of $H_j^\pm$ on $\Sigma_j^\pm$ are isolated.

(iii) H_0^+ *has regular values* $h_{\text{lower}}, h_{\text{upper}} \in \mathbf{R}$ *with* $h_{\text{lower}} < H(0) < h_{\text{upper}}$ *or* $h_{\text{upper}} < H(0) < h_{\text{lower}}$*, where* $H(0)$ *is the value of* H *at the origin* $0 \in \mathbf{C}^4$ *and is equal to the value of* H_0^+ *at the origin of* Σ_0^+*, such that:*
  (a) the level set $\lambda^+(0, h_{\text{upper}}) := (H_0^+)^{-1}(h_{\text{upper}})$ *is a smooth circle with winding number* 1*, and*
  (b) the level set $\lambda^+(0, h_{\text{lower}}) := (H_0^+)^{-1}(h_{\text{lower}})$ *consists of* n_1 *smooth circles and each one of these circles has winding number* 0.

The first thing to note about Definition 1 is that since H Poisson commutes with J there is an induced $\mathbf{Z}_{n_1}$ action on Σ_0^+ that leaves H_0^+ invariant, see §3.1. This implies that $\lambda^+(0, h_{\text{upper}})$ is $\mathbf{Z}_{n_1}$ invariant while the n_1 circles that constitute $\lambda^+(0, h_{\text{lower}})$ are mapped to each other by this $\mathbf{Z}_{n_1}$ action. Furthermore note that by subtracting $H(0)$ from H and by possibly changing the sign of H we can always arrange so that $h_{\text{lower}} < H(0) = 0 < h_{\text{upper}}$. Thus from now on we assume without any loss of generality that our H has this property.

Remark 2. Although the level set $\lambda^+(0, h_{\text{lower}})$ consists of more than one component when $n_1 > 1$, the corresponding fibre $F^{-1}(0, h_{\text{lower}})$ consists of one single component, i.e., it is a smooth $\mathbf{T}^2$. This can be seen in the examples of §2.2 and it will be proved in §5.1. Furthermore, $F^{-1}(0, h_{\text{upper}})$ is also a $\mathbf{T}^2$.

Let $\mathsf{D}_{\text{lower}}$ be the largest open connected subset of the set R of regular values of F such that $(0, h_{\text{lower}}) \in \mathsf{D}_{\text{lower}}$ and all points $(j, h) \in \mathsf{D}_{\text{lower}}$ satisfy $h < H_j^{\pm}(0)$. Similarly, let $\mathsf{D}_{\text{upper}}$ be the largest open connected subset of R such that $(0, h_{\text{upper}}) \in \mathsf{D}_{\text{upper}}$ and all points $(j, h) \in \mathsf{D}_{\text{upper}}$ satisfy $h > H_j^{\pm}(0)$. Note that $\mathsf{D}_{\text{lower}}$ and $\mathsf{D}_{\text{upper}}$ are not empty. Finally, let $\mathsf{D} = \mathsf{D}_{\text{lower}} \cup \mathsf{D}_{\text{upper}}$.

Remark 3. The conditions $h < H_j^{\pm}(0)$ for $\mathsf{D}_{\text{lower}}$ and $h > H_j^{\pm}(0)$ for $\mathsf{D}_{\text{upper}}$ ensures that the level sets $\lambda^{\pm}(j, h)$ do not contain $0 \in \Sigma_j^{\pm}$ for any $(j, h) \in \mathsf{D}_{\text{lower}}$. This in particular means that the winding number of the level sets $\lambda^{\pm}(j, h)$ with respect to the origin is well defined and constant in each open set $\mathsf{D}_{\text{lower}}$ and $\mathsf{D}_{\text{upper}}$. Furthermore, these conditions will ensure that the lift of any regular fibre $F^{-1}(j, h)$ for $(j, h) \in \mathsf{D}$ to the covering space is also a regular fibre.

3.3. Closed paths. We now define the class of closed paths to be considered for the parallel transport of homology cycles, see also Fig. 8.

Definition 2 (Closed Paths). *Given* $H \in \mathcal{F}_{n_1:n_2}$ *we define the set* $\mathcal{L}$ *of closed smooth paths* $\Gamma : [0, 1] \to \mathit{image}(F)^\circ \backslash \{c_*\} : s \mapsto \Gamma(s) = (\Gamma_j(s), \Gamma_h(s))$ *which satisfy the following:*

(i) $\Gamma(0) = \Gamma(1) = (j, h) \in \mathsf{D}' := \mathsf{D} \cup (\partial\mathsf{D} \cap \mathsf{R})$.
(ii) Γ *has winding number* 1 *with respect to the origin* $c_* = (0, 0)$.
(iii) Γ *crosses the axis* $j = 0$ *only at points that belong in* D.
(iv) For all $s \in (0, 1)$ *and all points* $p \in F^{-1}(\Gamma(s))$ *the* **transversality condition** $\Gamma_h'(s)\, dJ(p) - \Gamma_j'(s)\, dH(p) \neq 0$ *is satisfied.*
(v) There is a smooth path $\Gamma_F : [0, 1] \to \mathbf{R}^4$ *in phase space such that* Γ_F *covers* Γ*, that is,* $F(\Gamma_F(s)) = \Gamma(s)$ *for all* $s \in [0, 1]$.

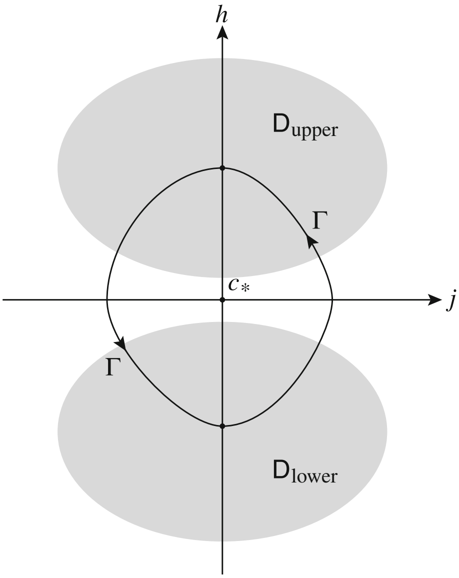

Fig. 8. The path Γ has winding number 1 with respect to the origin and crosses the axis $j = 0$ only at points that belong in D, cf. Fig. 13

Note that the transversality condition (iv) is automatically satisfied whenever p is a regular point of F. Since Γ does not go through the point c_* the case $dJ(p) = dH(p) = 0$, possible only at $p = 0$, is avoided. Finally, in the case that p is a rank-1 critical point of F, condition (iv) implies that whenever Γ crosses a one-parameter family of critical values of F, it does so transversally.

Remark 4. The path Γ is chosen so that its starting and ending point (j, h) is in the set $\mathsf{D}' := \mathsf{D} \cup (\partial\mathsf{D} \cap \mathsf{R})$. Note that each fibre $F^{-1}(j, h)$ for $(j, h) \in \mathsf{D} = \mathsf{D}_{\text{upper}} \cup \mathsf{D}_{\text{lower}}$ is diffeomorphic to a single $\mathbf{T}^2$. This follows from the definition of $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$ and from the fact that each of $F^{-1}(0, h_{\text{upper}})$ and $F^{-1}(0, h_{\text{lower}})$ is a $\mathbf{T}^2$, see Remark 2. This means that if we consider a path Γ with 'initial' point $(j, h) \in \mathsf{D}$ then the 'initial' fibre $F^{-1}(j, h)$ is a $\mathbf{T}^2$. For higher order resonances with $1 < n_1 < n_2$ we have that $\mathsf{D}' = \mathsf{D}$. This follows from the fact that points in $\partial\mathsf{D}$ are, by definition, either critical values of F or points in $\mathsf{C} = \mathsf{C}_- \cup \{c_*\} \cup \mathsf{C}_+$. Since for $1 < n_1 < n_2$ all points in C are critical values of F, all points in $\partial\mathsf{D}$ are also critical values of F, implying that $\partial\mathsf{D} \cap \mathsf{R} = \emptyset$ and thus $\mathsf{D}' = \mathsf{D}$. Thus considering D' instead of D does not add anything in this case. Nevertheless, in the case of lower order resonances there may be regular values in $\partial\mathsf{D}$ and, as the examples in §2 show, it is possible that these regular values lie in the common boundary of $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$. Therefore, in this case the set $\partial\mathsf{D} \cap \mathsf{R}$ 'bridges the gap' between $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$ and D' is the maximally connected set of regular values that contains both $(0, h_{\text{upper}})$ and $(0, h_{\text{lower}})$. Thus for all resonances we have that each fibre $F^{-1}(j, h)$ for $(j, h) \in \mathsf{D}'$ is diffeomorphic to a single $\mathbf{T}^2$ and furthermore there are paths in R that connect (j, h) to one or both of $(0, h_{\text{upper}})$ and $(0, h_{\text{lower}})$.

Remark 5. Condition (v) in Definition 2 ensures that by moving along Γ in the image of F it is possible to find a corresponding path in phase space that connects the fibres of F along Γ. This may not be always the case. Consider for example the bifurcation diagram related to a Hamiltonian swallowtail [20,21] and a path Γ that enters and then exits the two-component region by crossing different curves of elliptic critical values. In this case there is no continuous path in phase space that covers Γ.

Remark 6. Conditions (iv) and (v) in Definition 2 taken together imply that $\Gamma_F(s)$ is not a critical point of F for any $s \in (0, 1)$ or, alternatively, that if $\Gamma_F(s)$ is a critical point of F then Γ cannot be transversal to the corresponding curve of critical values. In order to see this we compute from $F(\Gamma_F(s)) = \Gamma(s)$ that

$$\Gamma_j'(s) = dJ(\Gamma_F(s))\Gamma_F'(s), \quad \Gamma_h'(s) = dH(\Gamma_F(s))\Gamma_F'(s).$$

If $\Gamma_F(s)$ were a critical point of F, with $dH(\Gamma_F(s)) = \kappa dJ(\Gamma_F(s))$, then we would have $\Gamma_h'(s) = \kappa \Gamma_j'(s)$ and

$$\begin{aligned}\Gamma_h'(s)\, dJ(\Gamma_F(s)) - \Gamma_j'(s)\, dH(\Gamma_F(s)) &= (\kappa \Gamma_j'(s))\, dJ(\Gamma_F(s)) - \Gamma_j'(s)\, (\kappa\, dJ(\Gamma_F(s)))\\ &= 0,\end{aligned}$$

therefore condition (iv) in Definition 2 would be violated.

Having given the precise definitions of the class of $n_1{:}(-n_2)$ resonant systems $\mathcal{F}_{n_1:n_2}$ and closed paths $\mathcal{L}$ we proceed in the following sections with the proof of Theorem 1.

4. The Branched Covering Map and Geometry in the Covering Space

In this section we introduce a branched covering map that locally trivializes the geometry of the fibres of the system and we study its properties in detail.

4.1. Definition and basic properties of the branched covering map. We consider the branched covering map

$$\rho : \mathbf{C}^2 \to \mathbf{C}^2 : (w_1, w_2) \mapsto (z_1, z_2) = (w_1^{n_1}, w_2^{n_2}). \tag{9}$$

The map ρ has degree

$$\deg \rho = \begin{cases} n_1 n_2, & \text{for } w_1 w_2 \neq 0,\\ n_1, & \text{for } w_2 = 0,\, w_1 \neq 0,\\ n_2, & \text{for } w_1 = 0,\, w_2 \neq 0,\\ 1, & \text{for } w_1 = w_2 = 0.\end{cases}$$

The $n_1{:}(-n_2)$ resonant $\mathbf{S}^1$ action (5) defined by the flow of X_J lifts to the covering space to the 1:(−1) resonant $\mathbf{S}^1$ action

$$t,\ (w_1, w_2) \mapsto \tilde{\varphi}_J^t(w_1, w_2) = (w_1 \exp(\mathrm{i}t), w_2 \exp(-\mathrm{i}t)), \tag{10}$$

which makes the diagram

$$\begin{array}{ccc} \mathbf{C}^2 & \xrightarrow{\tilde{\varphi}_J^t} & \mathbf{C}^2 \\ \rho\downarrow & & \downarrow\rho \\ \mathbf{C}^2 & \xrightarrow[\varphi_J^t]{} & \mathbf{C}^2 \end{array} \tag{11}$$

commute.

Proceeding just like in §3.1 we define a Poincaré section in $\tilde{J}^{-1}(j)$, where $\tilde{J}$ is the lift of J to the covering space, i.e.,

$$\tilde{J}(w_1, w_2) = J(w_1^{n_1}, w_2^{n_2}) = \frac{n_1}{2}|w_1|^{2n_1} - \frac{n_2}{2}|w_2|^{2n_2}.$$

Specifically, in the covering space we define the families of Poincaré sections

$$\tilde{\Sigma}_j^+ = \{(w_1, w_2) \in \tilde{J}^{-1}(j) \,:\, \mathrm{Arg}(w_1) = 0 \text{ or } w_1 = 0\}, \quad \text{for } j \geq 0,$$

and

$$\tilde{\Sigma}_j^- = \{(w_1, w_2) \in \tilde{J}^{-1}(j) \,:\, \mathrm{Arg}(w_2) = 0 \text{ or } w_2 = 0\}, \quad \text{for } j \leq 0.$$

We describe all surfaces of section with coordinates $w \in \mathbf{C}$ where $w = w_2$ for $\tilde{\Sigma}_j^+$ and $w = w_1$ for $\tilde{\Sigma}_j^-$. The corresponding inclusions are denoted by $\tilde{\iota}_j^\pm : \tilde{\Sigma}_j^\pm \to \mathbf{C}^2$.

The restriction $\rho_j^\pm$ of the map ρ in (9) to $\tilde{\Sigma}_j^\pm$ is a degree $m_\pm$ covering map, where $m_+ = n_2$ and $m_- = n_1$, and is given by

$$\rho_j^\pm \,:\, \tilde{\Sigma}_j^\pm \to \Sigma_j^\pm \,:\, w \mapsto z = w^{m_\pm}. \tag{12}$$

Denote by $\tilde{H} = \rho^* H$ the lift through ρ of the Hamiltonian function H, i.e.,

$$\tilde{H}(w_1, w_2) = H(w_1^{n_1}, w_2^{n_2}).$$

Then the restriction $\tilde{H}_j^\pm = \tilde{H}|_{\tilde{\Sigma}_j^\pm}$ satisfies

$$\tilde{H}_j^\pm(w) = H_j^\pm(w^{m_\pm}) = H_j^\pm(\rho_j^\pm(w)). \tag{13}$$

Consider the action on $\tilde{\Sigma}_j^\pm$ of the cyclic group $\mathbf{Z}_N$, $N = n_1 n_2 = m_\pm n_\pm$, which is given by

$$\mathbf{Z}_N \times \tilde{\Sigma}_j^\pm \to \tilde{\Sigma}_j^\pm : (k, w) \mapsto w \exp(2\pi \mathrm{i} k/N).$$

The function $\tilde{H}_j^\pm$ is invariant under this $\mathbf{Z}_N$ action. Indeed, using that $H_j^\pm$ is $\mathbf{Z}_{n_\pm}$ invariant, see §3.1, we compute that

$$\tilde{H}_j^\pm(w \exp(2\pi \mathrm{i}/N)) = H_j^\pm(w^{m_\pm} \exp(2\pi \mathrm{i}/n_\pm)) = H_j^\pm(w^{m_\pm}) = \tilde{H}_j^\pm(w).$$

Thus we find that the level sets $\tilde{\lambda}^\pm(j, h) := (\tilde{H}_j^\pm)^{-1}(h)$ of $\tilde{H}_j^\pm$ on $\tilde{\Sigma}_j^\pm$ are $\mathbf{Z}_N$ invariant. Furthermore, Eq. (13) implies that

$$\tilde{\lambda}^\pm(j, h) = (\rho_j^\pm)^{-1} \lambda^\pm(j, h). \tag{14}$$

The deck group of the covering map ρ is

$$D = \{(w_1, w_2) \mapsto A_{k_1,k_2}(w_1, w_2) : k_1 = 0, \ldots, n_1 - 1,\ k_2 = 0, \ldots, n_2 - 1\},$$

where

$$A_{k_1,k_2}(w_1, w_2) = (w_1 \exp(2\pi \mathrm{i} k_1/n_1), w_2 \exp(2\pi \mathrm{i} k_2/n_2)).$$

Thus

$$D = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \simeq \mathbf{Z}_N,$$

where $N = n_1 n_2$. A function on the covering space is D invariant if and only if it is the pull-back by ρ of a function in the original space. This implies in particular that the fibres $\tilde{F}^{-1}(j, h)$ are D invariant.

The relation of the $\mathbf{Z}_N$ action on $\tilde{\Sigma}_j^{\pm}$ to the action of the deck group D on $\mathbf{C}^2$ is the following. Suppose that $(w_1, w_2) \in \tilde{J}^{-1}(j) \subset \mathbf{C}^2$. Then there is a map $\tilde{p}_j^{\pm} : \tilde{J}^{-1}(j) \to \tilde{\Sigma}_j^{\pm}$ that sends (w_1, w_2) to the point $w \in \tilde{\Sigma}_j^{\pm}$ where the $\mathbf{S}^1$ orbit of $\tilde{\varphi}_J^t$ passing through (w_1, w_2) intersects $\tilde{\Sigma}_j^{\pm}$. One can see $\tilde{\Sigma}_j^{\pm}$ as the orbit space of the $\mathbf{S}^1$ action in $\tilde{J}^{-1}(j)$ and $\tilde{p}_j^{\pm}$ as the corresponding reduction map. Let

$$B_k = [w \mapsto w \exp(2\pi \mathrm{i} k/N)] \in \mathbf{Z}_N.$$

Then for A_{k_1,k_2} in the deck group D we have that

$$\tilde{p}_j^{\pm} \circ A_{k_1,k_2} \circ \tilde{\iota}_j^{\pm} = B_{k_1 n_2 + k_2 n_1}. \tag{15}$$

4.2. The pull back of the integrable Hamiltonian fibration to the covering space. In this section we consider the lifted fibres $\tilde{F}^{-1}(j, h)$, where $\tilde{F} := \rho^* F = (\tilde{J}, \tilde{H})$ and their relationship to the level sets $\tilde{\lambda}^{\pm}(j, h)$ of $\tilde{H}_j^{\pm}$.

Note that the $\mathbf{S}^1$ action $\tilde{\varphi}_J^t$ preserves the fibres of $\tilde{F}$. Indeed, if $L : \mathbf{C}^2 \to \mathbf{R}$ is any φ_J^t invariant function in the original space then $\tilde{L} = \rho^* L$ is $\tilde{\varphi}_J^t$ invariant, since $\tilde{L} \circ \tilde{\varphi}_J^t = L \circ \rho \circ \tilde{\varphi}_J^t = L \circ \varphi_J^t \circ \rho = L \circ \rho = \tilde{L}$. We now have the following fundamental result.

Lemma 1 (Product structure of lifted fibres). *If a fibre $\tilde{F}^{-1}(j, h)$ does not contain the origin in $\mathbf{C}^2$ then it is diffeomorphic to $\tilde{\lambda}^{\pm}(j, h) \times \mathbf{S}^1$, where $\tilde{\lambda}^{\pm}(j, h) = (\tilde{H}_j^{\pm})^{-1}(h) = \tilde{F}^{-1}(j, h) \cap \tilde{\Sigma}_j^{\pm}$.*

Proof. We show that the map

$$\Phi : \tilde{\lambda}^{\pm}(j, h) \times \mathbf{S}^1 \to \tilde{F}^{-1}(j, h) : (w, t) \mapsto \tilde{\varphi}_J^t(\tilde{\iota}_j^{\pm}(w)) = (w_1 \exp(\mathrm{i}t), w_2 \exp(-\mathrm{i}t)), \tag{16}$$

where $(w_1, w_2) = \tilde{\iota}_j^{\pm}(w)$, is a diffeomorphism. Recall that $\tilde{\iota}_j^{\pm} : \tilde{\Sigma}_j^{\pm} \to \mathbf{C}^2$ is the inclusion map. Since the $\mathbf{S}^1$ action $\tilde{\varphi}_J^t$ preserves the fibres $\tilde{F}^{-1}(j, h)$ the map Φ is well defined, i.e., $\Phi(w, t) \in \tilde{F}^{-1}(j, h)$.

Φ is smooth, injective, and onto $\tilde{F}^{-1}(j, h)$. We show this for $j \geq 0$, i.e., for level sets $\tilde{\lambda}^{+}(j, h)$ on Σ_j^{+}, the case $j \leq 0$ being similar. If $(w_1', w_2') \in \tilde{F}^{-1}(j, h)$ then there are unique $t = \mathrm{Arg}(w_1') \in [0, 2\pi)$ and $w = w_2' w_1' / |w_1'| \in \tilde{\lambda}^{+}(j, h)$ for which $(w_1', w_2') = \tilde{\varphi}_J^t(\tilde{\iota}_j^{+}(w)) \in \tilde{\Sigma}_j^{+}$. The expressions for t and w are well defined since for $j > 0$ and for $j = 0$, but away from the origin in $\mathbf{C}^2$ we have $|w_1'| > 0$. These expressions show that Φ^{-1} is smooth. $\square$

Remark 7. Lemma 1 implies that the fibre $\tilde{F}^{-1}(j,h)$ has as many connected components as the connected components of $\tilde{\lambda}^{\pm}(j,h)$. Compare this with the situation in the original space where $\lambda^{\pm}(j,h)$ for $(j,h) \in \mathsf{D}_{\mathrm{lower}}$ has $n_{\pm}$ connected components (with $n_{\pm}$ possibly larger than 1) but these are the intersections with $\Sigma_j^{\pm}$ of a connected fibre $F^{-1}(j,h) \simeq \mathbf{T}^2$.

We now relate the geometry of the level sets $\lambda^{\pm}(j,h)$ to the geometry of the corresponding level sets $\tilde{\lambda}^{\pm}(j,h)$. Our aim for the moment is to determine the properties of the level sets $\tilde{\lambda}^{\pm}(j,h)$ for $(j,h) \in \mathsf{D} = \mathsf{D}_{\mathrm{lower}} \cup \mathsf{D}_{\mathrm{upper}}$ from the properties of $\lambda^{+}(0,h_{\mathrm{lower}})$ and $\lambda^{+}(0,h_{\mathrm{upper}})$ given in Definition 1. In Fig. 9 we illustrate our results for the specific cases 1:(−2) and 2:(−5).

We will need the following result that relates the regularity of values of F to the regularity of values of $\tilde{F}$. Recall that we denote by R the set of regular values of F and the sets $\mathsf{C}_{\pm}$ are defined in (8).

Lemma 2 (Regularity of values of $\tilde{F}$). *Points $(j,h) \in \mathsf{R}\backslash(\mathsf{C}_{-} \cup \mathsf{C}_{+})$ are regular values of $\tilde{F}$. If $N = n_1 n_2 \neq 1$ then points $(j,h) \in \mathsf{C}_{-} \cup \mathsf{C}_{+}$ are critical values of $\tilde{F}$ even if they are regular values of F. Critical values of F are also critical values of $\tilde{F}$.*

Proof. Since $\tilde{F} = F \circ \rho$, if (j,h) is a critical value of F it must also be a critical value of $\tilde{F}$.

Assume now that (j,h) is a regular value of F, that is, rank $DF(p') = 2$ for all $p' \in F^{-1}(j,h)$. If $p \in \tilde{F}^{-1}(j,h)$ then $\rho(p) \in F^{-1}(j,h)$ and

$$D\tilde{F}(p) = DF(\rho(p))\, D\rho(p),$$

where $D\rho(p)$ can be computed using Eq. (9). Note that if $D\rho(p)$ has full rank then rank $D\tilde{F}(p) = \operatorname{rank} DF(\rho(p)) = 2$. We find that $D\rho(p)$ with $p = (w_1, w_2) \in \mathbf{C}^2$ has full rank if $w_1 w_2 \neq 0$. The latter is true for all points $p \in \tilde{F}^{-1}(j,h)$ with $(j,h) \in \mathsf{R}\backslash(\mathsf{C}_{+} \cup \mathsf{C}_{-})$. Therefore such (j,h) are regular values of $\tilde{F}$.

Finally, consider the case where $n_1 n_2 \neq 1$. In this case $(j,h) \in \mathsf{C}_{-}$ are critical values of F and thus also critical values of $\tilde{F}$. If $n_1 > 1$ the same is true for $(j,h) \in \mathsf{C}_{+}$. Therefore we are only left to consider values $(j,h) \in \mathsf{C}_{+}$ in the case $n_1 = 1$. Note that if $(j,h) \in \mathsf{C}_{+}$ then there is a point $p = (w_1, 0) \in \mathbf{C}^2$ such that $\tilde{F}(p) = (j,h)$. Then a direct computation shows that

$$D\tilde{F}(p) = \begin{pmatrix} \partial J/\partial q_1 & \partial J/\partial p_1 & 0 & 0 \\ \partial H/\partial q_1 & \partial H/\partial p_1 & 0 & 0 \end{pmatrix},$$

where the partial derivatives are evaluated at $\rho(p) = (w_1, 0)$. Evaluating the Poisson bracket $\{J, H\}(\rho(p)) = 0$ gives

$$\frac{\partial J}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial q_1}\frac{\partial J}{\partial p_1} = 0.$$

Therefore rank $D\tilde{F}(p) \leq 1$ and p is a critical point of $\tilde{F}$. □

Remark 8. Lemma 2 shows that for $N \neq 1$ the set of regular values of $\tilde{F}$ is always separated by the set $\mathsf{C} = \mathsf{C}_{-} \cup \{c_*\} \cup \mathsf{C}_{+}$ into at least two disjoint components. This is the reason we have defined $\mathsf{D}_{\mathrm{upper}}$ and $\mathsf{D}_{\mathrm{lower}}$ as connected components of regular values

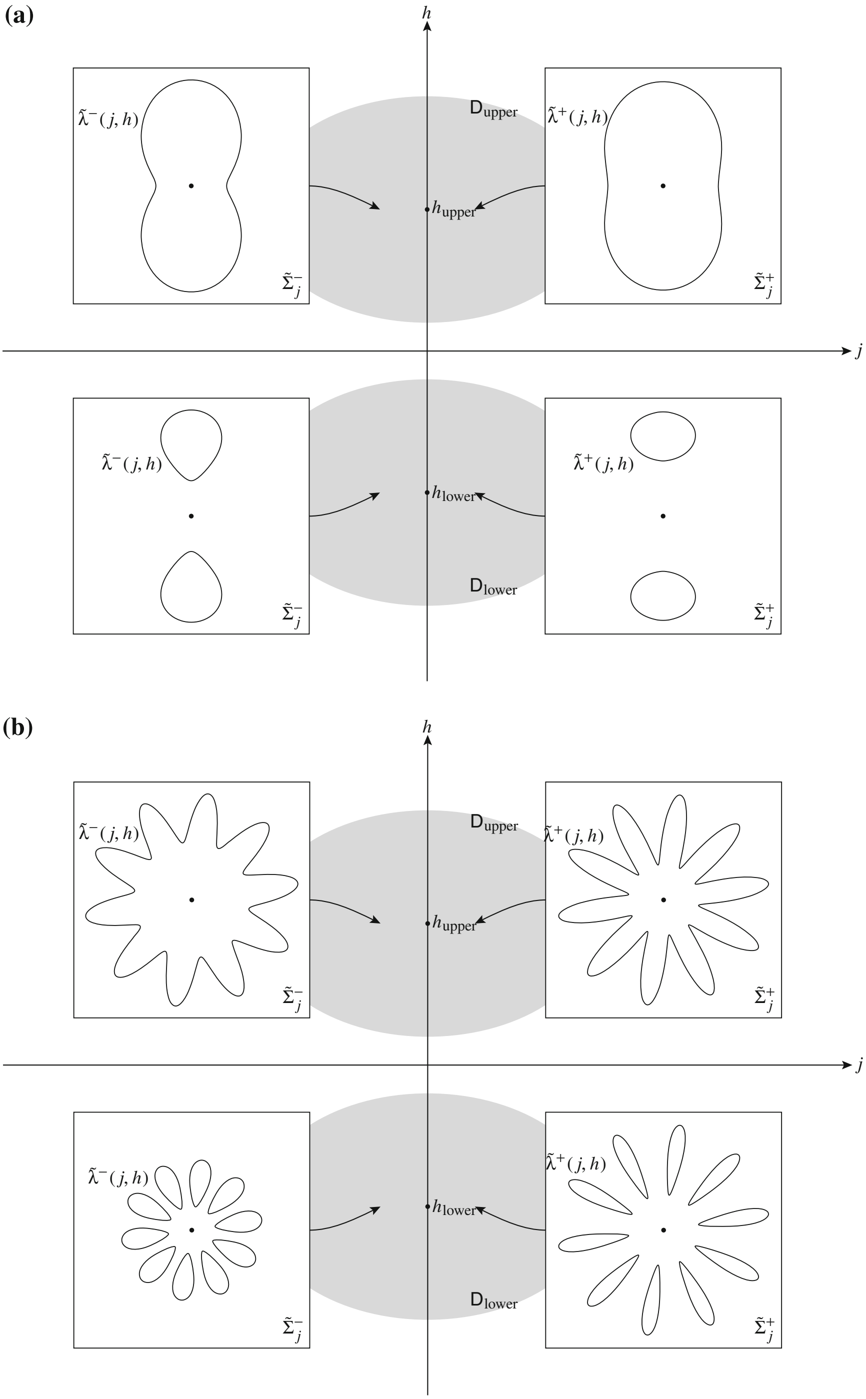

Fig. 9. Schematic representation of level sets $\tilde{\lambda}^{\pm}(j,h)$ for $(j,h) \in \mathsf{D}$. **(a)** 1:(−2) resonant system. **(b)** 2:(−5) resonant system

of F above and below the set C: even though the topology of $F^{-1}(j,h)$ may not change when crossing C this is not in general true for the topology of $\tilde{F}^{-1}(j,h)$. For $N = 1$ (which implies $n_1 = n_2 = 1$) it is not necessary to define the sets $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$ but we can treat the whole set of regular values together. Nevertheless we want to treat all resonances in a uniform way and for this reason we work also in the case $N = 1$ with the sets $\mathsf{D}_{\text{upper}}$ and $\mathsf{D}_{\text{lower}}$.

Lemma 3 (Topology of $\tilde{\lambda}^{\pm}(j,h)$). *For $(j,h) \in \mathsf{D}_{\text{upper}}$ the level sets $\tilde{\lambda}^{\pm}(j,h)$ are smooth $\mathbf{Z}_N$ invariant circles with winding number* 1. *For $(j,h) \in \mathsf{D}_{\text{lower}}$ the level sets $\tilde{\lambda}^{\pm}(j,h)$ are the disjoint union of N smooth circles with winding number* 0 *that are mapped to each other by the $\mathbf{Z}_N$ action.*

Proof. Starting with $\mathsf{D}_{\text{upper}}$ we observe that since $\lambda^{+}(0,h_{\text{upper}})$ on Σ_0^{+} is a circle with winding number 1 it follows from Eqs. (14) and (12) that $\tilde{\lambda}^{+}(0,h_{\text{upper}})$ on $\tilde{\Sigma}_0^{+}$ is also a circle with winding number 1. From Lemma 1 and the fact that $\tilde{\lambda}^{+}(0,h_{\text{upper}}) \simeq \mathbf{S}^1$ we infer that $\tilde{F}^{-1}(0,h_{\text{upper}}) \simeq \mathbf{T}^2$. Since $\mathsf{D}_{\text{upper}}$ is a connected set of regular values of F, Lemma 2 shows that it is also a connected set of regular values of $\tilde{F}$. Therefore all fibres $\tilde{F}^{-1}(j,h)$ for $(j,h) \in \mathsf{D}_{\text{upper}}$ are diffeomorphic to $\tilde{F}^{-1}(0,h_{\text{upper}})$ and thus they are $\mathbf{T}^2$. Using again Lemma 1 we deduce that $\tilde{\lambda}^{\pm}(j,h) \simeq \mathbf{S}^1$. By the definition of $\mathsf{D}_{\text{upper}}$, see §3.2, none of the level sets $\tilde{\lambda}^{\pm}(j,h)$ for $(j,h) \in \mathsf{D}_{\text{upper}}$ goes through $0 \in \tilde{\Sigma}_j^{\pm}$ and thus they all have the same winding number as $\tilde{\lambda}^{+}(0,h_{\text{upper}})$ which is 1.

For $\mathsf{D}_{\text{lower}}$ we observe that since $\lambda^{+}(0,h_{\text{lower}})$ on Σ_0^{+} consists of $n_+ = n_1$ circles with winding number 0 it follows that $\tilde{\lambda}^{+}(0,h_{\text{lower}})$ on $\tilde{\Sigma}_0^{+}$ consists of $N = m_+ n_+ = n_1 n_2$ circles with winding number 0. For $(j,h) \in \mathsf{D}_{\text{lower}}$, using a similar argument as for $\mathsf{D}_{\text{upper}}$, we find that the fibre $\tilde{F}^{-1}(j,h)$ is diffeomorphic to $\tilde{F}^{-1}(0,h_{\text{lower}})$ which is the union of N disjoint $\mathbf{T}^2$. Thus $\tilde{\lambda}^{\pm}(j,h)$ consists of N disjoint circles and again an argument similar to the case $(j,h) \in \mathsf{D}_{\text{upper}}$ shows that their winding number is 0. □

We denote by $\tilde{\lambda}^{\pm}(j,h)^{(k)}$ with $k = 1,\ldots,K$ the connected components of $\tilde{\lambda}^{\pm}(j,h)$. Note that for $(j,h) \in \mathsf{D}_{\text{upper}}$ we have $K = 1$, while for $(j,h) \in \mathsf{D}_{\text{lower}}$ we have $K = N$. Similarly, we denote by $\tilde{F}^{-1}(j,h)^{(k)} \simeq \tilde{\lambda}^{\pm}(j,h)^{(k)} \times \mathbf{S}^1$ with $k = 1,\ldots,K$ the corresponding connected components of $\tilde{F}^{-1}(j,h)$. Lemma 1 implies that for each connected component $\tilde{F}^{-1}(j,h)^{(k)}$ there is a diffeomorphism

$$\Phi^{\pm}_{j,h;k} : \mathbf{T}^2 \to \tilde{F}^{-1}(j,h)^{(k)} : (s,t) \mapsto \Phi^{\pm}_{j,h;k}(s,t) = \tilde{\varphi}^t_J(\tilde{\iota}^{\pm}_j(\zeta(s))), \tag{17}$$

where $\zeta : \mathbf{S}^1 \to \tilde{\lambda}^{\pm}(j,h)^{(k)}$ is a parameterization of the level set $\tilde{\lambda}^{\pm}(j,h)^{(k)}$. In particular for $(j,h) \in \mathsf{D}_{\text{upper}}$ the parameterization ζ is chosen so that $\zeta(s + 2\pi/N) = \zeta(s)\exp(2\pi i/N)$. When $K = 1$ we often drop the index k from our notation.

Remark 9. Using Eq. (14) we can also determine the properties of the level sets $\lambda^{\pm}(j,h)$ in the original space. We will not need these properties further and for this reason we only describe them briefly here. For $(j,h) \in \mathsf{D}_{\text{upper}}$ each level set $\lambda^{\pm}(j,h)$ is a circle with winding number 1 and $\mathbf{Z}_{n_\pm}$ symmetry. For $(j,h) \in \mathsf{D}_{\text{lower}}$ each level set $\lambda^{\pm}(j,h)$ consists of $n_\pm$ circles with winding number 0 and the whole level set is $\mathbf{Z}_{n_\pm}$ invariant. In Fig. 10 we illustrate these properties for the specific cases 1:(−2) and 2:(−5).

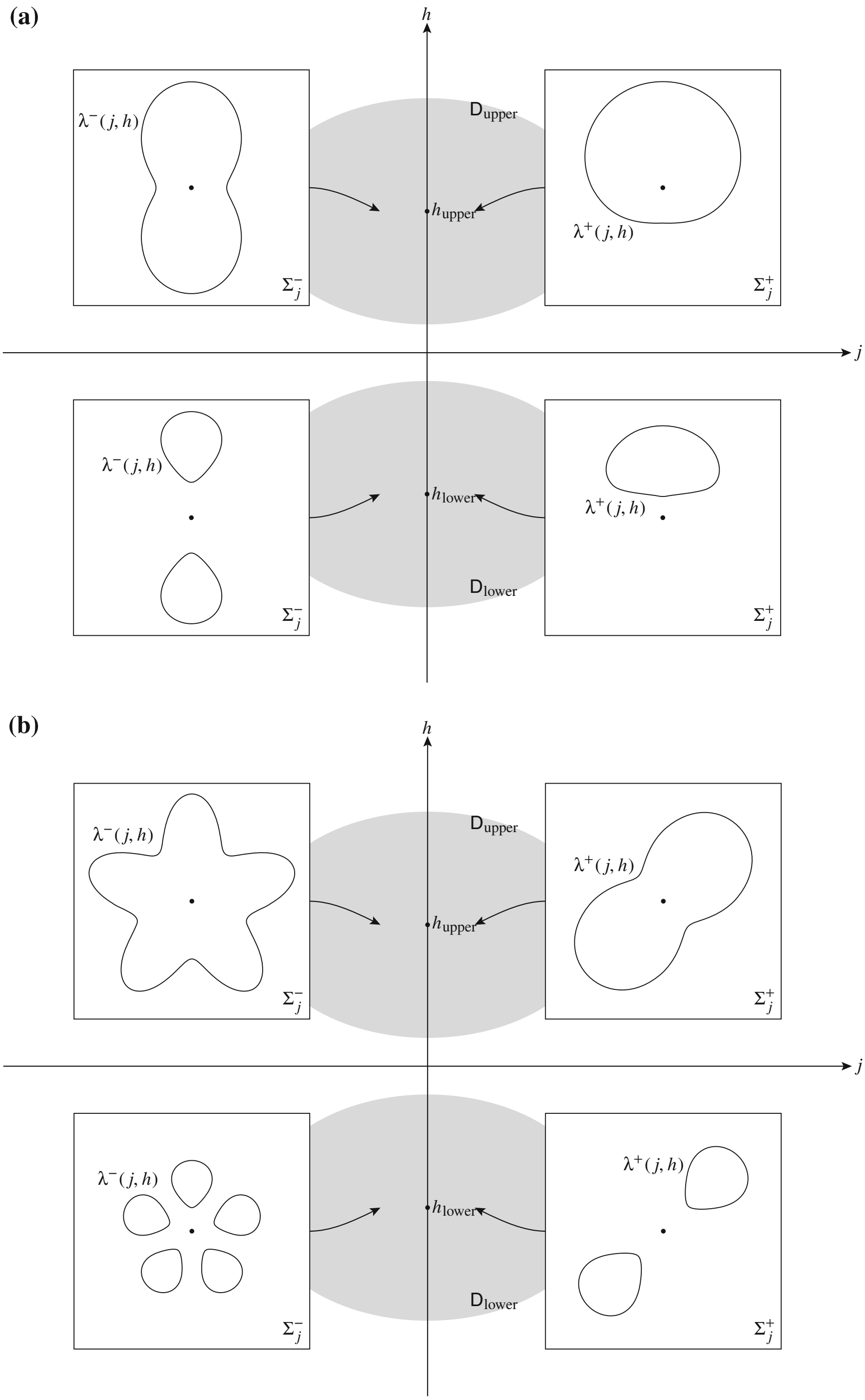

Fig. 10. Schematic representation of level sets $\lambda^{\pm}(j,h)$ for $(j,h) \in \mathsf{D} = \mathsf{D}_{\text{lower}} \cup \mathsf{D}_{\text{upper}}$. **(a)** 1:(−2) resonant systems. **(b)** 2:(−5) resonant systems

4.3. Homology of fibres in the covering space. We now turn our attention to the first homology group $H_1(\tilde{F}^{-1}(j,h))$. For $(j,h) \in \mathsf{D}$ the fibre $\tilde{F}^{-1}(j,h)$ consists of K connected components $\tilde{F}^{-1}(j,h)^{(k)}$ with $k = 1, \dots, K$. Recall that $K = 1$ for $(j,h) \in \mathsf{D}_{\text{upper}}$ and $K = N$ for $(j,h) \in \mathsf{D}_{\text{lower}}$. Thus

$$H_1(\tilde{F}^{-1}(j,h)) \simeq \bigoplus_{k=1}^{K} H_1(\tilde{F}^{-1}(j,h)^{(k)}) \simeq \bigoplus_{k=1}^{K} \mathbf{Z}^2. \tag{18}$$

We now want to fix a basis for each $H_1(\tilde{F}^{-1}(j,h)^{(k)})$. For each $k = 1, \dots, K$ we define the ordered basis $(\tilde{g}^{\pm}(j,h)^{(k)}, \tilde{b}^{\pm}(j,h)^{(k)})$ by

$$\tilde{g}^{\pm}(j,h)^{(k)} = [s \mapsto \Phi^{\pm}_{j,h;k}(s,0)], \quad \tilde{b}^{\pm}(j,h)^{(k)} = [t \mapsto \Phi^{\pm}_{j,h;k}(0,t)], \tag{19}$$

where $[s \mapsto \gamma(s)]$ denotes the homology cycle represented by the closed path γ. By construction, each cycle $\tilde{g}^{\pm}(j,h)^{(k)}$ corresponds to the level set $\tilde{\lambda}^{\pm}(j,h)^{(k)}$ on the surface of section $\tilde{\Sigma}^{\pm}_{j}$ traversed in a counterclockwise direction. Furthermore, each cycle $\tilde{b}^{\pm}(j,h)^{(k)}$ is generated by the $\mathbf{S}^1$ action (10) on the component $\tilde{F}^{-1}(j,h)^{(k)}$. Finally, we define

$$\tilde{g}^{\pm}(j,h) = \sum_{k=1}^{K} \tilde{g}^{\pm}(j,h)^{(k)}. \tag{20}$$

For $(j,h) \in \mathsf{D}_{\text{upper}}$, $\tilde{g}^{\pm}(j,h) = \tilde{g}^{\pm}(j,h)^{(1)}$.

Note that for $j = 0$ there are two cycles $\tilde{g}^{+}(0,h)^{(k)}$ and $\tilde{g}^{-}(0,h)^{(k)}$ and two cycles $\tilde{b}^{+}(0,h)^{(k)}$ and $\tilde{b}^{-}(0,h)^{(k)}$ on each component $\tilde{F}^{-1}(0,h)^{(k)}$. We aim to find the relation between these cycles.

Lemma 4. *On $\tilde{F}^{-1}(0,h)^{(k)}$ we have*

$$\tilde{g}^{-}(0,h)^{(k)} = \begin{cases} \tilde{g}^{+}(0,h)^{(k)} + \tilde{b}^{+}(0,h)^{(k)}, & \textit{for } (0,h) \in \mathsf{D}_{\text{upper}}, \\ \tilde{g}^{+}(0,h)^{(k)}, & \textit{for } (0,h) \in \mathsf{D}_{\text{lower}}, \end{cases}$$

and

$$\tilde{b}^{-}(0,h)^{(k)} = \tilde{b}^{+}(0,h)^{(k)}.$$

Because of the last lemma from now on we write $\tilde{b}(j,h)^{(k)}$ instead of $\tilde{b}^{\pm}(j,h)^{(k)}$.

Proof. Fixing the component $\tilde{F}^{-1}(0,h)^{(k)}$ we temporarily simplify notation to $\tilde{g}^{\pm} = \tilde{g}^{\pm}(0,h)^{(k)}$ and $\tilde{b}^{\pm} = \tilde{b}^{\pm}(0,h)^{(k)}$. The cycles $\tilde{b}^{+}$ and $\tilde{b}^{-}$ are generated by the $\mathbf{S}^1$ action and lie on the same connected component. Therefore these cycles are homologous. We give now a proof of the relation between the cycles $\tilde{g}^{+}$ and $\tilde{g}^{-}$. From the same proof we recover the fact about $\tilde{b}^{+}$ and $\tilde{b}^{-}$ being homologous.

Consider a parameterization ζ^{+} of $\tilde{\lambda}^{+}(0,h)^{(k)}$ given by

$$\zeta^{+}(s) = R_2(s)\exp(\mathrm{i}\theta(s)),$$

where $\theta : \mathbf{S}^1 \to \mathbf{S}^1$ is a map whose degree d is the winding number of $\tilde{\lambda}^+(0,h)^{(k)}$. This means that $d = 1$ for $(0,h) \in \mathsf{D}_{\text{upper}}$ and $d = 0$ for $(0,h) \in \mathsf{D}_{\text{lower}}$. Then, applying Eq. (17), we obtain the parameterization

$$\Phi^+ : \mathbf{T}^2 \to \tilde{F}^{-1}(0,h)^{(k)} : (s,t) \mapsto (R_1(s)\exp(\mathrm{i}t),\ R_2(s)\exp(\mathrm{i}(\theta(s)-t))),$$

where we used the fact that $\tilde{\iota}_0^+(\zeta^+(s))$ has the form $(R_1(s), R_2(s)\exp(\mathrm{i}\theta(s)))$ with $R_1(s) > 0$. It follows from the expression for Φ^+ that the set $\tilde{\lambda}^-(0,h)^{(k)} = \tilde{F}^{-1}(0,h)^{(k)} \cap \tilde{\Sigma}_0^-$, where $\operatorname{Arg}(w_2) = 0$, is given by $t = \theta(s)$ and can therefore be parameterized as

$$\tilde{\iota}_0^-(\zeta^-(s)) = (R_1(s)\exp(\mathrm{i}\theta(s)),\ R_2(s)).$$

Then, applying again Eq. (17), we obtain an alternative parameterization for $\tilde{F}^{-1}(0,h)^{(k)}$ given by

$$\Phi^- : \mathbf{T}^2 \to \tilde{F}^{-1}(0,h)^{(k)} : (s,t) \mapsto (R_1(s)\exp(\mathrm{i}(\theta(s)+t)),\ R_2(s)\exp(-\mathrm{i}t)).$$

The diffeomorphisms Φ^+ and Φ^- are related by

$$\Phi^- = \Phi^+ \circ \psi,$$

where

$$\psi : \mathbf{T}^2 \to \mathbf{T}^2 : (s,t) \mapsto \psi(s,t) = (s, t+\theta(s))$$

is a diffeomorphism on $\mathbf{T}^2$.

Let (c_1, c_2) be the standard ordered basis of $H_1(\mathbf{T}^2) \simeq \mathbf{Z}^2$. Then, by definition, we have that $\tilde{g}^\pm = \Phi_*^\pm c_1$ and $\tilde{b}^\pm = \Phi_*^\pm c_2$. Furthermore, $\psi_* c_1 = c_1 + \deg(\theta)\, c_2$, where $\deg(\theta)$ is the degree of the map θ, and $\psi_* c_2 = c_2$. Thus we obtain

$$\tilde{g}^- = \Phi_*^- c_1 = \Phi_*^+ \psi_* c_1 = \Phi_*^+(c_1 + \deg(\theta)\, c_2) = \tilde{g}^+ + \deg(\theta)\, \tilde{b}^+,$$

and

$$\tilde{b}^- = \Phi_*^- c_2 = \Phi_*^+ \psi_* c_2 = \Phi_*^+ c_2 = \tilde{b}^+.$$

□

5. Fibres and Homology in the Original Space

In this section we study the action of the covering map on the homology groups of individual fibres and determine bases for the homology groups of fibres in the original space. Consider the restriction of the covering map ρ to the fibre $\tilde{F}^{-1}(j,h)$, that is,

$$\rho_{j,h} : \tilde{F}^{-1}(j,h) \to F^{-1}(j,h).$$

Then in this section we prove the following result concerning the basis of $H_1(F^{-1}(j,h))$ for $(j,h) \in \mathsf{D}$. Recall that we denote by $b(j,h)$ the homology cycle on $F^{-1}(j,h) \simeq \mathbf{T}^2$ which is represented by any closed orbit of the $\mathbf{S}^1$ action.

Proposition 1. *The homology group $H_1(F^{-1}(j,h))$ for $(j,h) \in \mathsf{D}$ is spanned over $\mathbf{Z}$ by an ordered basis $(a^\pm(j,h), b(j,h))$ for which we have that*

$$(\rho_{j,h})_* \tilde{g}^\pm(j,h) = N a^\pm(j,h) + M b(j,h), \quad \textit{and} \quad (\rho_{j,h})_* \tilde{b}^{(k)}(j,h) = b(j,h)$$
$$\textit{for } k = 1, \ldots, K.$$

Here, $N = n_1 n_2$ and $M \in \mathbf{Z}$ is determined in the course of the proof.

In the rest of this section we give the proof of Proposition 1.

5.1. The action of the covering map on individual fibres. Consider a fibre $\tilde{F}^{-1}(j,h)$ in the covering space, where $(j,h) \in \mathsf{D}$. Recall that if $(j,h) \in \mathsf{D}_{\text{upper}}$ then $\tilde{F}^{-1}(j,h) \simeq \mathbf{T}^2$ and that if $(j,h) \in \mathsf{D}_{\text{lower}}$ then $\tilde{F}^{-1}(j,h)$ is the union of N disjoint $\mathbf{T}^2$ and the action of the deck group D maps each connected component to another one. The last fact follows from Eq. (15) that intertwines the action of D with the $\mathbf{Z}_N$ action on the surface of section $\tilde{\Sigma}_j^{\pm}$ and the fact that the latter action maps each connected component of $\tilde{\lambda}^{\pm}(j,h)$ to another one.

The fibre $F^{-1}(j,h)$ in the original space is the image under the covering map ρ of the fibre $\tilde{F}^{-1}(j,h)$. Since D is a subset of the set R of regular values of F we conclude using the Arnol'd-Liouville theorem that $F^{-1}(j,h)$ is a $\mathbf{T}^2$ or a disjoint union of $\mathbf{T}^2$. For $(j,h) \in \mathsf{D}_{\text{upper}}$ we immediately find that $F^{-1}(j,h) \simeq \mathbf{T}^2$, since $\tilde{F}^{-1}(j,h) \simeq \mathbf{T}^2$ and ρ is continuous. For $(j,h) \in \mathsf{D}_{\text{lower}}$ each $\mathbf{T}^2$ component $\tilde{F}^{-1}(j,h)^{(k)}$ of $\tilde{F}^{-1}(j,h)$ is mapped under ρ to the same set. Therefore for all $k = 1, \ldots, N$ we have that $\rho(\tilde{F}^{-1}(j,h)^{(k)}) = F^{-1}(j,h)$ and the latter is again a $\mathbf{T}^2$.

We now study in more detail the action of the covering map on individual fibres. Specifically, we want to find a diffeomorphism $\Psi^{\pm}_{j,h;k} : \mathbf{T}^2 \to F^{-1}(j,h)$ and a map $r^{\pm}_{j,h;k} : \mathbf{T}^2 \to \mathbf{T}^2$ such that the diagram

$$\begin{array}{ccc} \mathbf{T}^2 & \xrightarrow{\Phi^{\pm}_{j,h;k}} & \tilde{F}^{-1}(j,h)^{(k)} \\ {\scriptstyle r^{\pm}_{j,h;k}}\downarrow & & \downarrow{\scriptstyle \rho_{j,h;k}} \\ \mathbf{T}^2 & \xrightarrow[\Psi^{\pm}_{j,h;k}]{} & F^{-1}(j,h) \end{array} \tag{21}$$

commutes. Here $\Phi^{\pm}_{j,h;k} : \mathbf{T}^2 \to \tilde{F}^{-1}(j,h)^{(k)}$ is the diffeomorphism given in Eq. (17) and $\rho_{j,h;k} : \tilde{F}^{-1}(j,h)^{(k)} \to F^{-1}(j,h)$ is the restriction of ρ to $\tilde{F}^{-1}(j,h)^{(k)}$. Our main interest here is determining $r^{\pm}_{j,h;k}$, since then it becomes trivial to study the relation between homology cycles in the covering space and the original space. The two cases $(j,h) \in \mathsf{D}_{\text{lower}}$ and $(j,h) \in \mathsf{D}_{\text{upper}}$ are different and we treat them separately.

5.1.1. The case $(j,h) \in \mathsf{D}_{\text{lower}}$. Recall that in this case the fibre $\tilde{F}^{-1}(j,h)$ consists of N connected components $\tilde{F}^{-1}(j,h)^{(k)}$ with $k = 1, \ldots, N$ and each such component is diffeomorphic to $\mathbf{T}^2$. We have that $\rho_{j,h;k}(\tilde{F}^{-1}(j,h)^{(k)}) = F^{-1}(j,h)$ and the map $\rho_{j,h;k}$ is injective since the N points in the covering space that are mapped under ρ to the same point belong to the N different connected components of $\tilde{F}^{-1}(j,h)$. Finally, $\rho_{j,h;k}$ is smooth and its inverse $\rho^{-1}_{j,h;k} : F^{-1}(j,h) \to \tilde{F}^{-1}(j,h)^{(k)}$ is also smooth. Therefore $\rho_{j,h;k}$ is a diffeomorphism. Then the diagram (21) can be made commutative by choosing $r^{\pm}_{j,h;k}$ to be the identity map on $\mathbf{T}^2$ and $\Psi^{\pm}_{j,h;k} = \rho_{j,h;k} \circ \Phi^{\pm}_{j,h;k}$.

5.1.2. The case $(j,h) \in \mathsf{D}_{\text{upper}}$. Consider the action of the deck group D on $\tilde{F}^{-1}(j,h)$. Recall that for $A_{k_1,k_2} \in D$ we have that

$$A_{k_1,k_2}(w_1, w_2) = (w_1 \exp(2\pi \mathrm{i} k_1/n_1), w_2 \exp(2\pi \mathrm{i} k_2/n_2)).$$

The action of D on $\tilde{F}^{-1}(j,h)$ induces through $\Phi^{\pm}_{j,h}$ an action of D on $\mathbf{T}^2$ given by

$$A'_{k_1,k_2} = (\Phi^{\pm}_{j,h})^{-1} \circ A_{k_1,k_2} \circ \Phi^{\pm}_{j,h},$$

and which we compute to be

$$A'_{k_1,k_2}(s,t) = \left(s + 2\pi\left(\frac{k_1}{n_1} + \frac{k_2}{n_2}\right), t \pm 2\pi\frac{k_\pm}{n_\pm}\right),$$

where, in analogy with $n_\pm$, we define $k_+ = k_1$ and $k_- = k_2$. Then $\mathbf{T}^2/D \simeq \mathbf{T}^2$ and the required map $r^\pm_{j,h}$ is the reduction map of the discrete action of D on $\mathbf{T}^2$. In particular, giving coordinates (u, v) on $\mathbf{T}^2/D \simeq \mathbf{T}^2$ the map $r^\pm_{j,h}$ is given by

$$(u, v) = r^\pm_{j,h}(s,t) = (m_\pm(s \mp t), n_\pm t). \tag{22}$$

Note that $r^\pm_{j,h}$ is an N-fold covering map. The map $\Psi^\pm_{j,h} : \mathbf{T}^2/D \to \tilde{F}^{-1}(j,h)/D$ induced by $\Phi^\pm_{j,h}$ is then a diffeomorphism and since $\tilde{F}^{-1}(j,h)/D \simeq F^{-1}(j,h)$ the diffeomorphism $\Psi^\pm_{j,h}$ is the required diffeomorphism. The corresponding commutative diagram is

$$\begin{array}{ccc} \mathbf{T}^2 & \xrightarrow{\Phi^\pm_{j,h}} & \tilde{F}^{-1}(j,h) \\ {\scriptstyle r^\pm_{j,h}}\downarrow & & \downarrow{\scriptstyle \rho_{j,h}} \\ \mathbf{T}^2/D \simeq \mathbf{T}^2 & \xrightarrow[\Psi^\pm_{j,h}]{} & F^{-1}(j,h) \simeq \tilde{F}^{-1}(j,h)/D \end{array} \tag{23}$$

5.2. Bases of the homology groups and the action of the covering map on homology. Recall that for each $\tilde{F}^{-1}(j,h)^{(k)} \simeq \mathbf{T}^2$ in the covering space we have defined ordered bases $(\tilde{g}^\pm(j,h)^{(k)}, \tilde{b}(j,h)^{(k)})$ of $H_1(\tilde{F}^{-1}(j,h)^{(k)}) \simeq \mathbf{Z}^2$ given by

$$\tilde{g}^\pm(j,h)^{(k)} = [s \mapsto \Phi^\pm_{j,h;k}(s,0)], \quad \tilde{b}(j,h)^{(k)} = [t \mapsto \Phi^\pm_{j,h;k}(0,t)],$$

where $\Phi^\pm_{j,h;k}$ is a parameterization of $\tilde{F}^{-1}(j,h)^{(k)}$. Denote by $c_1 = [s \mapsto (s,0)]$ and $c_2 = [t \mapsto (0,t)]$ the two standard basis cycles of $H_1(\mathbf{T}^2)$. Then

$$\tilde{g}^\pm(j,h)^{(k)} = (\Phi^\pm_{j,h;k})_* c_1, \quad \tilde{b}(j,h)^{(k)} = (\Phi^\pm_{j,h;k})_* c_2.$$

It follows from the commutative diagram (11) that the cycle $\tilde{b}(j,h)^{(k)}$ on $\tilde{F}^{-1}(j,h)^{(k)}$ is mapped under $(\rho_{j,h;k})_*$ to the cycle $b(j,h)$ generated by the X_J flow on $F^{-1}(j,h)$. We want to construct an ordered basis of $H_1(F^{-1}(j,h))$ where the second element in the basis is the cycle $b(j,h)$. We discuss again separately the cases $(j,h) \in \mathsf{D}_{\text{lower}}$ and $(j,h) \in \mathsf{D}_{\text{upper}}$.

5.2.1. The case $(j,h) \in \mathsf{D}_{\text{lower}}$. Here $r^\pm_{j,h;k}$ is the identity and $\Psi^\pm_{j,h;k} = \rho_{j,h;k} \circ \Phi^\pm_{j,h;k}$. Since $\Psi^\pm_{j,h;k}$ is a diffeomorphism, the cycles $(\Psi^\pm_{j,h;k})_* c_i$, $i = 1, 2$, form a basis of $H_1(F^{-1}(j,h))$. Furthermore, we have

$$(\Psi^\pm_{j,h;k})_* c_1 = (\rho_{j,h;k})_* \tilde{g}^\pm(j,h)^{(k)}, \quad (\Psi^\pm_{j,h;k})_* c_2 = (\rho_{j,h;k})_* \tilde{b}(j,h)^{(k)} = b(j,h).$$

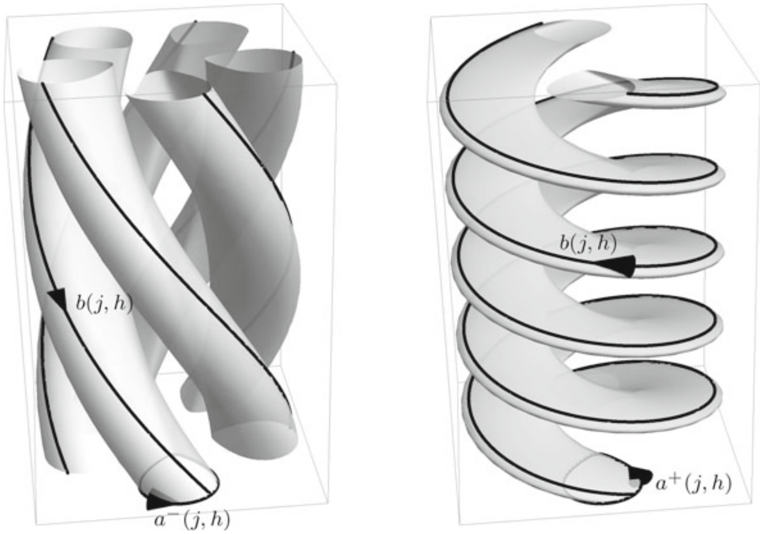

Fig. 11. Schematic representation of fibres $F^{-1}(j,h)$ for $f = (j,h) \in \mathsf{D}_{\text{lower}}$ and $n_1 = 2, n_2 = 5$. The upper and lower boundaries of the cylinders should be identified. At the left side $j < 0$. A basis of the first homology group is provided by the cycles $a^-(j,h)$ and $b(j,h)$ shown here. The cycle $b(j,h)$ is represented by the black curves that start at the upper side of the cylinder and end at the lower side. We have to take all 5 such curves in order to obtain a closed curve. At the right side $j > 0$. Here the cycle $b(j,h)$ is represented by the two black curves that start at the lower side of the cylinder and end at the upper side. Note that in both cases the fibre is a single $\mathbf{T}^2$

Thus we have for $(j,h) \in \mathsf{D}_{\text{lower}}$ the ordered basis $(a^{\pm}(j,h), b(j,h))$ of $H_1(F^{-1}(j,h))$, where

$$a^{\pm}(j,h) = (\rho_{j,h;k})_* \tilde{g}^{\pm}(j,h)^{(k)},$$

see also Fig. 11. Note that

$$(\rho_{j,h})_* \tilde{g}^{\pm}(j,h) = \sum_{k=1}^{N} (\rho_{j,h;k})_* \tilde{g}^{\pm}(j,h)^{(k)} = N a^{\pm}(j,h).$$

5.2.2. The case $(j,h) \in \mathsf{D}_{\text{upper}}$. In this case, using Eq. (22), we find that

$$\begin{aligned} b(j,h) &= (\rho_{j,h})_* \tilde{b}(j,h) = (\rho_{j,h})_* (\Phi^{\pm}_{j,h})_* c_2 = (\Psi^{\pm}_{j,h})_* (r^{\pm}_{j,h})_* c_2 \\ &= (\Psi^{\pm}_{j,h})_* (\mp m_{\pm} c_1 + n_{\pm} c_2) = \mp m_{\pm} e_1^{\pm}(j,h) + n_{\pm} e_2^{\pm}(j,h), \end{aligned}$$

where $e_i^{\pm}(j,h) := (\Psi^{\pm}_{j,h})_* c_i$ for $i = 1, 2$, and the two cycles $e_1^{\pm}(j,h)$ and $e_2^{\pm}(j,h)$ give a basis of $H_1(F^{-1}(j,h))$, see Fig. 12. An ordered basis $(a^{\pm}(j,h), b(j,h))$ of $H_1(F^{-1}(j,h))$ can then be obtained from $(e_1^{\pm}(j,h), e_2^{\pm}(j,h))$ if there is a matrix $A \in \mathrm{SL}(2,\mathbf{Z})$ such that

$$\begin{pmatrix} a^{\pm}(j,h) \\ b(j,h) \end{pmatrix} = A \begin{pmatrix} e_1^{\pm}(j,h) \\ e_2^{\pm}(j,h) \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_2 \\ \mp m_{\pm} & n_{\pm} \end{pmatrix} \begin{pmatrix} e_1^{\pm}(j,h) \\ e_2^{\pm}(j,h) \end{pmatrix}.$$

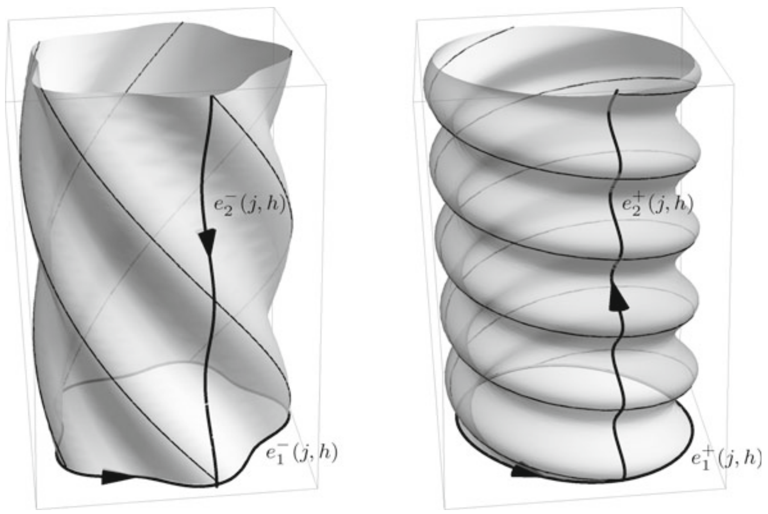

Fig. 12. Schematic representation of a torus $F^{-1}(j,h)$ for $f \in \mathsf{D}_{\text{upper}}$ for the specific choice $n_1 = 2, n_2 = 5$. The upper and lower boundaries of the cylinders should be identified. At the left side $j < 0$. A basis of the first homology group is provided by the cycles $e_1^-(j,h)$ and $e_2^-(j,h)$ shown here. The cycle $b(j,h)$ is represented by the thin black curves that start at the upper side of the cylinder and end at the lower side while winding by 2/5 of a full circle around the cylinder. We have to take all 5 such curves in order to obtain a closed curve so we obtain that in this case $b(j,h) = 2e_1^-(j,h) + 5e_2^-(j,h)$. At the right side $j > 0$. Here the cycle $b(j,h)$ is represented by the two thin black curves that start at the lower side of the cylinder and end at the upper side after winding by two and a half circles around the cylinder. Here $b(j,h) = -5e_1^+(j,h) + 2e_2^+(j,h)$. In both cases $b(j,h) = \mp m_\pm e_1^\pm(j,h) + n_\pm e_2^\pm(j,h)$

Since $m_\pm, n_\pm$ are coprime, it is always possible to find $\ell_1, \ell_2 \in \mathbf{Z}$ such that $\det A = n_\pm \ell_1 \pm m_\pm \ell_2 = 1$, so $A \in \mathrm{SL}(2,\mathbf{Z})$. Finally note that

$$(\rho_{j,h})_* \tilde{g}^\pm(j,h) = (\rho_{j,h})_*(\Phi^\pm_{j,h})_* c_1 = (\Psi^\pm_{j,h})_*(r^\pm_{j,h})_* c_1 = (\Psi^\pm_{j,h})_* m_\pm c_1 = m_\pm e_1^\pm(j,h).$$

Thus we find

$$(\rho_{j,h})_* \tilde{g}^\pm(j,h) = m_\pm(n_\pm a^\pm(j,h) - \ell_2 b(j,h)) = N a^\pm(j,h) + M b(j,h), \qquad (24)$$

where $N = m_\pm n_\pm = n_1 n_2$ and $M = -\ell_2 m_\pm$.

6. Parallel Transport of Homology Cycles

The standard notion of parallel transport of homology cycles in [11,16] applies to the situation where the path Γ is in the set R of regular values of F. In this section we give a definition of parallel transport of homology cycles which generalizes the standard notion of parallel transport to cases where Γ goes through critical values of F. Furthermore, we compute the parallel transport of homology cycles in the covering space and we push these results down to the original space.

6.1. Definition of parallel transport. We first recall and then reformulate the standard notion of parallel transport. Assume that $(j_0,h_0) \in \mathsf{R}$ and that $F^{-1}(j_0,h_0) \simeq \mathbf{T}^2$. This implies that if Γ is a path in R that starts at (j_0,h_0) and ends at a regular value (j_1,h_1)

then for all points (j, h) along Γ the fibre $F^{-1}(j, h)$ is also diffeomorphic to $\mathbf{T}^2$. Focusing on the homology groups $H_1(F^{-1}(j, h)) \simeq \mathbf{Z}^2$ we find over the image of Γ a smooth bundle with discrete fibre $\mathbf{Z}^2$. Given a cycle $\alpha_0 \in H_1(F^{-1}(j_0, h_0))$ the path Γ lifts to a unique path with initial point α_0 in the $\mathbf{Z}^2$ bundle. Then the parallel transport of α_0 along Γ is the endpoint $\alpha_1 \in H_1(F^{-1}(j_1, h_1))$ of the lifted path. For further details see [16,26].

For any path $\Gamma : [0, 1] \to \text{image}(F)$ we define the set

$$M_\Gamma = \{(p, s) \in \mathbf{R}^4 \times [0, 1] \; : \; F(p) = \Gamma(s)\}. \tag{25}$$

Let $M_0 = \{(p, 0) \in \mathbf{R}^4 \times [0, 1] \; : \; F(p) = \Gamma(0)\} \simeq F^{-1}(\Gamma(0))$ and similarly $M_1 = \{(p, 1) \in \mathbf{R}^4 \times [0, 1] \; : \; F(p) = \Gamma(1)\} \simeq F^{-1}(\Gamma(1))$. Finally, consider the corresponding inclusions $j : M_0 \to M_\Gamma$ and $i : M_1 \to M_\Gamma$.

If Γ is a path in R then M_Γ is diffeomorphic to $[0, 1] \times \mathbf{T}^2$ and the corresponding bundle of homology groups is diffeomorphic to $[0, 1] \times \mathbf{Z}^2$. Furthermore $M_0 \simeq \mathbf{T}^2$ and $M_1 \simeq \mathbf{T}^2$. Given an isomorphism $H_1(M_0) \simeq \mathbf{Z}^2$ the parallel transport along Γ fixes the corresponding isomorphism $H_1(M_1) \simeq \mathbf{Z}^2$. Also whenever $\alpha_0 \in H_1(M_0)$ is parallel transported to $\alpha_1 \in H_1(M_1)$ it means that α_0 and α_1 are homologous in M_Γ. Thus in the case where Γ is in R we can define α_1 as the parallel transport of α_0 if these are homologous in M_Γ.

After the description of the parallel transport of homology cycles along paths Γ in R we now generalize the notion of parallel transport of homology cycles to paths Γ that go through critical values of F. Note that in this case M_Γ is no longer diffeomorphic to $[0, 1] \times \mathbf{T}^2$ and the corresponding bundle of homology groups is not diffeomorphic to $[0, 1] \times \mathbf{Z}^2$.

Definition 3 (Parallel transport of homology cycles). *Given a path $\Gamma : [0, 1] \to$ image(F) we say that the homology cycle $\alpha_1 \in H_1(M_1)$ is a* **parallel transport along $\boldsymbol{\Gamma}$** *of the homology cycle $\alpha_0 \in H_1(M_0)$ if α_0 and α_1 are homologous as cycles in M_Γ, that is, if $j_*\alpha_0 = i_*\alpha_1 \in H_1(M_\Gamma)$.*

Note that $\partial M_\Gamma = M_0 \sqcup M_1$ and thus $H_1(\partial M_\Gamma) = H_1(M_0) \oplus H_1(M_1)$. Consider the long exact sequence

$$\cdots \longrightarrow H_2(M_\Gamma, \partial M_\Gamma) \xrightarrow{\partial_*} H_1(\partial M_\Gamma) \longrightarrow H_1(M_\Gamma) \longrightarrow \cdots,$$

where ∂_* is the corresponding connecting homomorphism, see [25]. Since α_0 and α_1 are homologous in M_Γ the element (α_0, α_1) of $H_1(\partial M_\Gamma)$ belongs in the kernel of the inclusion map $H_1(\partial M_\Gamma) \to H_1(M_\Gamma)$ and thus there is a relative cycle $C \in H_2(M_\Gamma, \partial M_\Gamma)$ such that $\partial_* C = (\alpha_0, \alpha_1)$. Therefore each parallel transport from α_0 to α_1 can be associated to the image under ∂_* of a relative cycle in $H_2(M_\Gamma, \partial M_\Gamma)$.

Definition 4 (Parallel transport group). *The* **parallel transport group along the path $\boldsymbol{\Gamma}$ for the fibration $\boldsymbol{F}$** *is the subgroup of $H_1(\partial M_\Gamma) = H_1(M_0) \oplus H_1(M_1)$ given by*

$$\mathrm{PT}(F, \Gamma) = \partial_*(H_2(M_\Gamma, \partial M_\Gamma)).$$

Thus the cycle α_1 is a parallel transport along Γ of the cycle α_0 if and only if $(\alpha_0, \alpha_1) \in \mathrm{PT}(F, \Gamma)$.

Remark 10 (On the uniqueness of parallel transport). Note that when Γ goes through critical values of F then the parallel transport of homology cycles may not be unique. One example where this occurs is given in [21]. In that case, the path Γ goes through a curve of critical values of F that in phase space lift to bitori (the cartesian product of a Fig. 8 and a circle). Nevertheless, for our systems we show that the parallel transport along closed paths $\Gamma \in \mathcal{L}$ is unique, see Theorem 1 and §7.1.

We close this section with two remarks concerning the relation of the parallel transport of homology cycles as given by Definition 3 to other approaches that have appeared in the literature.

Remark 11 (The parallel transport as cobordism). Definition 3 of parallel transport of homology cycles is a refinement of a closely related notion from [22]. In [22] the cycle α_1 is the parallel transport of α_0 along Γ if these cycles can be realized as 1-dimensional manifolds that are cobordant. It is implied in [22] that the cobordism lies, not in M_Γ but, in $F^{-1}(\text{image}(\Gamma))$. The approach of [22] and our definition coincide whenever M_Γ and $F^{-1}(\text{image}(\Gamma))$ are diffeomorphic, and all elements of $H_2(M_\Gamma, \partial M_\Gamma)$ can be realized as embedded 2-dimensional submanifolds of M_Γ. The latter is true whenever M_Γ is a three dimensional topological manifold, see [24, Lemma 3.6].

Remark 12 (Admissible deformations of homotopy classes). Nekhoroshev in [27] uses the notion of admissible deformations of homotopy classes instead of the parallel transport of homology cycles. Such deformations include, for example, the breaking of a single closed curve to several closed curves, or the vanishing of closed curves. Note that in the case of paths in R, where the first homology group and the fundamental group of each fibre are always isomorphic it makes no difference whether we consider homotopy classes or homology cycles. Furthermore, as it has been clarified in [22], fractional monodromy is properly described in the context of homology groups.

6.2. The covering map and parallel transport. The following result shows that it is possible to compute the parallel transport of homology cycles in the covering space and then use the covering map to compute the corresponding parallel transport in the original space.

Lemma 5 (Commutation of the covering map and parallel transport). *Consider a cycle $\alpha_0 \in H_1(M_0)$ and assume that there is a cycle $\tilde{\alpha}_0 \in H_1(\tilde{M}_0)$ such that $(\rho_0)_*\tilde{\alpha}_0 = \alpha_0$. Let $\tilde{\alpha}_1 \in H_1(\tilde{M}_1)$ be the parallel transport in the covering space of $\tilde{\alpha}_0$ along Γ. Then $\alpha_1 := (\rho_1)_*\tilde{\alpha}_1$ is the parallel transport in the original space of α_0 along Γ.*

Proof. Let $\tilde{\imath}_s : \tilde{M}_s \to \tilde{M}_\Gamma$ be the standard inclusion in the covering space and $i_s : M_s \to M_\Gamma$ the corresponding inclusion in the original space. Thus for $p \in \tilde{M}_s$ we have that

$$(\rho \circ \tilde{\imath}_s)(p) = \rho(p) = (i_s \circ \rho_s)(p).$$

Thus we obtain

$$\rho_* \circ (\tilde{\imath}_s)_* = (i_s)_* \circ (\rho_s)_*,$$

and the following diagram commutes:

$$\begin{array}{ccccc} H_1(\tilde{M}_0) & \xrightarrow{(\tilde{\imath}_0)_*} & H_1(\tilde{M}_\Gamma) & \xleftarrow{(\tilde{\imath}_1)_*} & H_1(\tilde{M}_1) \\ {\scriptstyle (\rho_0)_*}\big\downarrow & & {\scriptstyle \rho_*}\big\downarrow & & {\scriptstyle (\rho_1)_*}\big\downarrow \\ H_1(M_0) & \xrightarrow[(i_0)_*]{} & H_1(M_\Gamma) & \xleftarrow[(i_1)_*]{} & H_1(M_1) \end{array}$$

Now take $\alpha_0 \in H_1(M_0)$ and let $\tilde{\alpha}_0$ be a homology cycle in $H_1(\tilde{M}_0)$ such that $(\rho_0)_*\tilde{\alpha}_0 = \alpha_0$. Let $\tilde{\alpha}_1 \in H_1(\tilde{M}_1)$ be the parallel transport of $\tilde{\alpha}_0$ along Γ. This means that $(\tilde{\imath}_1)_*\tilde{\alpha}_1 = (\tilde{\imath}_0)_*\tilde{\alpha}_0$. Let $\alpha_1 = (\rho_1)_*\tilde{\alpha}_1$. Then

$$(i_1)_*\alpha_1 = (i_1)_*(\rho_1)_*\tilde{\alpha}_1 = \rho_*(\tilde{\imath}_1)_*\tilde{\alpha}_1 = \rho_*(\tilde{\imath}_0)_*\tilde{\alpha}_0 = (i_0)_*(\rho_0)_*\tilde{\alpha}_0 = (i_0)_*\alpha_0,$$

and thus α_1 is a parallel transport along Γ of α_0. □

6.3. Computation of the parallel transport of homology cycles in the covering space. We consider the parallel transport along a closed path Γ of homology cycles in the covering space. Our main result is the following.

Proposition 2 (Transport along a closed path in the covering space). *Consider a closed path $\Gamma \in \mathcal{L}$, see Definition 2, with the restriction that $(j,h) := \Gamma(0) = \Gamma(1) \in \mathsf{D}$. Consider also the cycles*

$$\tilde{b}(j,h)^{(k)} \textit{ for } 1 \le k \le K, \textit{ and } \tilde{g}^{\pm}(j,h) = \sum_{k=1}^{K} \tilde{g}^{\pm}(j,h)^{(k)},$$

where K is the number of connected components of $\tilde{F}^{-1}(j,h)$. Then, for any k' and k'' with $1 \le k', k'' \le K$, we have that

$$\left(\tilde{b}(j,h)^{(k)}, \tilde{b}(j,h)^{(k')}\right) \in \mathrm{PT}(\tilde{F},\Gamma), \textit{ and } \left(\tilde{g}^{\pm}(j,h), \tilde{g}^{\pm}(j,h) - \tilde{b}(j,h)^{(k'')}\right) \in \mathrm{PT}(\tilde{F},\Gamma),$$

where $\mathrm{PT}(\tilde{F},\Gamma)$ is the parallel transport group along Γ for $\tilde{F}$.

Remark 13. In the case $(j,h) \in \mathsf{D}_{\text{upper}}$, where $K = 1$, Proposition 2 reads that the cycles $\tilde{b}(j,h)$ and $\tilde{g}^{\pm}(j,h) - \tilde{b}(j,h)$ are parallel transports along Γ of the cycles $\tilde{b}(j,h)$ and $\tilde{g}^{\pm}(j,h)$ respectively. In the case $(j,h) \in \mathsf{D}_{\text{lower}}$, where $K = N$, we have to take into account that two cycles $\tilde{b}(j,h)^{(k)}$ and $\tilde{b}(j,h)^{(k')}$ with $k \ne k'$ are not homologous in the fibre $\tilde{F}^{-1}(j,h)$ but they are homologous in $\tilde{M}_\Gamma$ since, as we show in Lemma 6, all boundary components of $\tilde{M}_\Gamma$ belong to the same connected component of $\tilde{M}_\Gamma$. This further implies that for $(j,h) \in \mathsf{D}_{\text{lower}}$ the parallel transport of homology cycles along Γ in the covering space is not unique, cf. Remark 10. This situation does not arise in the original space where, as we show in §7.1, parallel transport along Γ is unique.

Before giving the proof of Proposition 2 we prove the following result asserting that the boundary components $\tilde{M}_0$ and $\tilde{M}_1$ are connected in $\tilde{M}_\Gamma$ even though they may be disconnected when considered in isolation.

Lemma 6 (On the boundary components of $\tilde{M}_\Gamma$). *The boundary components $\tilde{M}_0$ and $\tilde{M}_1$ of $\tilde{M}_\Gamma$ for the closed path Γ of Proposition 2 belong to the same connected component $\tilde{M}^c_\Gamma$ of $\tilde{M}_\Gamma$.*

Proof. Recall from Definition 2 that for the path Γ there exists a smooth path $\Gamma_F : [0,1] \to \mathbf{C}^2 \simeq \mathbf{R}^4$ such that $F(\Gamma_F(s)) = \Gamma(s)$. Consider now the lift of Γ_F through the branched covering map ρ. Note that if a point $p \in \mathbf{C}^2$ belongs in the branching locus of ρ then, for $n_1 > 1$, it is also a critical point of F. Recall also from Remark 6 that Γ_F cannot contain any critical points of F. Therefore we conclude that Γ_F does not go through the branching locus of ρ for $n_1 > 1$. For $n_1 = 1$ it may occur that branching points of ρ are regular points of F. In this case, Γ_F can still be chosen so that it avoids going through the branching locus of ρ. If, for example, we are given a path Γ_F that goes through a branching point then the path can be smoothly deformed in order to avoid such points, while at the same time continuing to satisfy the relation $F(\Gamma_F(s)) = \Gamma(s)$. The fact that Γ_F can always be chosen so that it does not go through a branching point implies that for each $s \in [0,1]$ the preimage $\rho^{-1}(\Gamma_F(s))$ consists of $N = n_1 n_2$ distinct points and since ρ is a local homeomorphism away from the preimage of the branching locus it follows that there exist N continuous paths $\tilde{\Gamma}_F^{(k)}$, $k = 1, \ldots, N$ in the covering space such that $\rho(\tilde{\Gamma}_F^{(k)}(s)) = \Gamma_F(s)$ for all $k = 1, \ldots, N$. In particular, for $\Gamma(s) \in \mathsf{D}_{\text{lower}}$ the points $\tilde{\Gamma}_F^{(k)}(s)$ belong to the N distinct connected components of $\tilde{F}^{-1}(\Gamma(s))$ while for $\Gamma(s) \in \mathsf{D}_{\text{upper}}$ the points $\tilde{\Gamma}_F^{(k)}(s)$ belong to the only connected component of $\tilde{F}^{-1}(\Gamma(s))$.

Assume now that $\Gamma(0) = \Gamma(1) \in \mathsf{D}_{\text{lower}}$. This means that each of $\tilde{M}_0$ and $\tilde{M}_1$ is the disjoint union of N two-dimensional tori. Denote the connected components of $\tilde{M}_0$ and $\tilde{M}_1$ by $\tilde{M}_0^{(k)}$ and $\tilde{M}_1^{(k)}$ respectively with $k = 1, \ldots, N$. Consider now points $p_0 \in \tilde{M}_0^{(k_0)}$ and $p_1 \in \tilde{M}_1^{(k_1)}$. Then we can construct a path in $\tilde{M}_\Gamma$ connecting p_0 to p_1 in the following way. First connect p_0 to $\tilde{\Gamma}_F^{(k_0)}(0)$ with a path on the two-dimensional torus $\tilde{M}_0^{(k_0)}$. Then follow the path $\tilde{\Gamma}_F^{(k_0)}$ until it reaches the point $\tilde{\Gamma}_F^{(k_0)}(s')$ such that $\Gamma(s') \in \mathsf{D}_{\text{upper}}$. Consequently on $\tilde{M}_{s'}$, which is a single $\mathbf{T}^2$, connect $\tilde{\Gamma}_F^{(k_0)}(s')$ to $\tilde{\Gamma}_F^{(k_1)}(s')$. Then follow the path $\tilde{\Gamma}_F^{(k_1)}$ until the point $\tilde{\Gamma}_F^{(k_1)}(1) \in \tilde{M}_1^{(k_1)}$. Finally, connect $\tilde{\Gamma}_F^{(k_1)}(1)$ to p_1. The crucial part of the argument is that the path Γ goes through $\mathsf{D}_{\text{upper}}$, where $\tilde{M}_s \simeq \tilde{F}^{-1}(\Gamma(s)) \simeq \mathbf{T}^2$ is connected. In a similar way one can show that all components $\tilde{M}_0^{(k)}$, $k = 1, \ldots, N$ belong to the same connected component of $\tilde{M}_\Gamma$ and that the same is true for $\tilde{M}_1^{(k)}$, $k = 1, \ldots, N$. Furthermore, if $\Gamma(0) = \Gamma(1) \in \mathsf{D}_{\text{upper}}$ a similar argument shows that $\tilde{M}_0$ and $\tilde{M}_1$ (each diffeomorphic to $\mathbf{T}^2$) belong to the same connected component of $\tilde{M}_\Gamma$.

We denote by $\tilde{M}_\Gamma^c$ the connected component of $\tilde{M}_\Gamma$ that contains $\tilde{M}_0$ and $\tilde{M}_1$. The sets $\tilde{M}_0$ and $\tilde{M}_1$ form the boundary of $\tilde{M}_\Gamma$ and since they both belong to the connected component $\tilde{M}_\Gamma^c$ they also form the boundary of the latter. □

Proof (Proposition 2). Consider first a path Γ', not necessarily closed, that starts at a point (j_0, h_0) and ends at a point (j_1, h_1). Recall that, according to Definition 3, a cycle $\alpha_1 \in H_1(\tilde{F}^{-1}(j_1, h_1))$ is the parallel transport along Γ' of a cycle $\alpha_0 \in H_1(\tilde{F}^{-1}(j_0, h_0))$ if α_1 and α_0 are homologous in

$$\tilde{M}_{\Gamma'} = \{(p, s) \in \mathbf{C}^2 \times [0,1] \,:\, \tilde{F}(p) = \Gamma'(s)\}.$$

If $\tilde{F}^{-1}(j_0, h_0)^{(k_0)}$ and $\tilde{F}^{-1}(j_1, h_1)^{(k_1)}$ belong to the same connected component of $\tilde{M}_{\Gamma'}$ then there is a continuous path $c : [0,1] \to \mathbf{C}^2$ such that $c(s) \in \tilde{F}^{-1}(\Gamma'(s))$ for

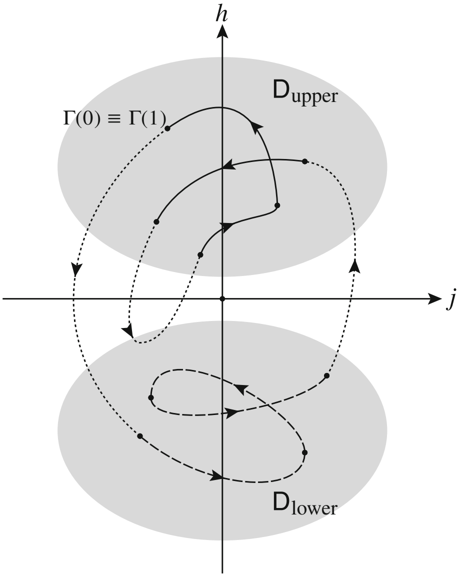

Fig. 13. An example of the decomposition of a path Γ. The components of Γ represented by solid curves correspond to types (iv) and (v) and are thus the only components that give a non-trivial contribution to the homology cycle $\tilde{g}^{\pm}$. The dotted curves represent paths of type (i) and (ii). The dashed curves represent paths of type (iii)

all $s \in [0, 1]$ and $c(0) \in \tilde{F}^{-1}(j_0, h_0)^{(k_0)}$ while $c(1) \in \tilde{F}^{-1}(j_1, h_1)^{(k_1)}$. The set

$$C = \{(\tilde{\varphi}_J^t(c(s)), s) : s \in [0, 1],\ t \in [0, 2\pi]\}$$

is a cylinder in $\tilde{M}_{\Gamma'}$ and represents a 2-chain $[C]$ with boundary $\partial[C] = \tilde{b}(j_1, h_1)^{(k_1)} - \tilde{b}(j_0, h_0)^{(k_0)}$.

Considering now the closed path Γ of Proposition 2 recall from Lemma 6 that all boundary components of $\tilde{M}_\Gamma$ belong in the same connected component $\tilde{M}_\Gamma^c$ of $\tilde{M}_\Gamma$. This implies that for any choice of k and k' in $\{1, \dots, K\}$ there is a 2-chain $[C]$ in $\tilde{M}_\Gamma$ with boundary $\partial[C] = \tilde{b}(j, h)^{(k')} - \tilde{b}(j, h)^{(k)}$, and thus $\tilde{b}(j, h)^{(k')}$ is the parallel transport of $\tilde{b}(j, h)^{(k)}$ along Γ.

We now turn our attention to the parallel transport of the homology cycles $\tilde{g}^{\pm}(j, h) = \sum_{k=1}^{K} \tilde{g}^{\pm}(j, h)^{(k)}$. Note first that we can decompose Γ as the sum of successive paths $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_\ell$, where each Γ_i, $i = 1, \dots, \ell$ is of one of the following types, see also Fig. 13:

(i) along Γ_i we have $j \geq 0$ and both $\Gamma_i(0)$ and $\Gamma_i(1)$ are in D,
(ii) along Γ_i we have $j \leq 0$ and both $\Gamma_i(0)$ and $\Gamma_i(1)$ are in D,
(iii) Γ_i is a path in $\mathsf{D}_{\text{lower}}$ and it crosses the $j = 0$ axis exactly once,
(iv) Γ_i is a path in $\mathsf{D}_{\text{upper}}$ and it crosses the $j = 0$ axis exactly once from $j > 0$ to $j < 0$,
(v) Γ_i is a path in $\mathsf{D}_{\text{upper}}$ and it crosses the $j = 0$ axis exactly once from $j < 0$ to $j > 0$.

Since the winding number of Γ around the origin is 1, if the number of paths of type (iv) is ℓ' then the number of paths of type (v) is $\ell' - 1$.

We study the parallel transport of $\tilde{g}^{\pm}(j,h)$ for each one of these types of paths separately. In particular, we show that for each type of path Γ_i we can make the following statement concerning the parallel transport. In the following list the numbering indicates the type of the path Γ_i:

(i) $(\tilde{g}^{+}(j_0,h_0), \tilde{g}^{+}(j_1,h_1)) \in \mathrm{PT}(\tilde{F},\Gamma_i)$,
(ii) $(\tilde{g}^{-}(j_0,h_0), \tilde{g}^{-}(j_1,h_1)) \in \mathrm{PT}(\tilde{F},\Gamma_i)$,
(iii) $(\tilde{g}^{\mp}(j_0,h_0), \tilde{g}^{\pm}(j_1,h_1)) \in \mathrm{PT}(\tilde{F},\Gamma_i)$,
(iv) $(\tilde{g}^{+}(j_0,h_0), \tilde{g}^{-}(j_1,h_1) - \tilde{b}(j_1,h_1)) \in \mathrm{PT}(\tilde{F},\Gamma_i)$,
(v) $(\tilde{g}^{-}(j_0,h_0), \tilde{g}^{+}(j_1,h_1) + \tilde{b}(j_1,h_1)) \in \mathrm{PT}(\tilde{F},\Gamma_i)$.

Consider first a path Γ' of type (i) from (j_0,h_0) to (j_1,h_1). The two dimensional subset

$$C = \{(w_1,w_2,t) \in \tilde{M}_{\Gamma'} \ : \ \mathrm{Arg}(w_1) = 0\}$$

of $\tilde{M}_{\Gamma'}$ represents a 2-chain $[C]$ with $\partial[C] = \tilde{g}^{+}(j_1,h_1) - \tilde{g}^{+}(j_0,h_0)$. Thus the cycle $\tilde{g}^{+}(j_1,h_1)$ is a parallel transport along Γ' of the cycle $\tilde{g}^{+}(j_0,h_0)$. The same argument shows that the cycle $\tilde{g}^{-}(j_1,h_1)$ is the parallel transport along a path Γ' type (ii) of the cycle $\tilde{g}^{-}(j_0,h_0)$. The results for paths of type (iii), (iv), and (v) follow directly from Lemma 4.

Then the proposition follows from the parallel transports for each Γ_i and the fact that there is one more path of type (iv) than paths of type (v) that make up the complete path Γ. □

6.4. Computation of the parallel transport of homology cycles in the original space. We consider here the parallel transport along a closed path Γ as given by Definition 2 and we show that the cycle $b(j,h)$ is a parallel transport of itself along Γ, while $Na^{\pm}(j,h) - b(j,h)$ is a parallel transport along Γ of $Na^{\pm}(j,h)$.

Proposition 3 (Transport along a closed path in the original space). *Consider a closed path $\Gamma \in \mathcal{L}$, see Definition 2. Consider also the cycles $Na^{\pm}(j,h)$ and $b(j,h)$. Then we have that*

$$(b(j,h), b(j,h)) \in \mathrm{PT}(F,\Gamma), \ \textit{and} \ \left(Na^{\pm}(j,h), Na^{\pm}(j,h) - b(j,h)\right) \in \mathrm{PT}(F,\Gamma),$$

where $\mathrm{PT}(F,\Gamma)$ *is the parallel transport group along Γ for F.*

Proof. Recall that for paths $\Gamma \in \mathcal{L}$ we have that $(j,h) := \Gamma(0) = \Gamma(1) \in \mathsf{D} \cup (\mathsf{R} \cap \partial \mathsf{D})$ but Proposition 2 can be applied only for paths Γ for which $(j,h) \in \mathsf{D}$. For this reason and in order to be able to apply Proposition 2 we consider first the case where $(j,h) \in \mathsf{D}$. In this case the cycle $\tilde{b}(j,h)^{(k_1)}$ is the parallel transport along Γ of $\tilde{b}(j,h)^{(k_0)}$, where $1 \le k_0, k_1 \le K$. Since $(\rho_{j,h})_* \tilde{b}(j,h)^{(k)} = b(j,h)$ for all $k \in \{1,\dots,K\}$ we conclude using Lemma 5 that $b(j,h)$ is a parallel transport of itself along Γ. The cycle $\tilde{g}^{\pm}(j,h) - \tilde{b}(j,h)^{(k)}$ with $1 \le k \le K$ is the parallel transport along Γ of $\tilde{g}^{\pm}(j,h)$. Using again Lemma 5 we obtain that the cycle

$$(\rho_{j,h})_*(\tilde{g}^{\pm}(j,h) - \tilde{b}(j,h)^{(k)}) = (\rho_{j,h})_* \tilde{g}^{\pm}(j,h) - b(j,h),$$

is the parallel transport along Γ of the cycle $(\rho_{j,h})_* \tilde{g}^{\pm}(j,h)$.

Recall from §5.2 that

$$(\rho_{j,h})_* \tilde{g}^{\pm}(j,h) = Na^{\pm}(j,h) + Mb(j,h),$$

where $M = -\ell m_{\pm}$ for $(j,h) \in \mathsf{D}_{\text{upper}}$ and $M = 0$ for $(j,h) \in \mathsf{D}_{\text{lower}}$. Since the cycles $Na^{\pm}(j,h) + (M-1)b(j,h)$ and $b(j,h)$ are the parallel transports along Γ of the cycles $Na^{\pm}(j,h) + Mb(j,h)$ and $b(j,h)$ respectively, we conclude that $Na^{\pm}(j,h) - b(j,h)$ is the parallel transport along Γ of $Na^{\pm}(j,h)$.

Consider now the case where $(j,h) \in \mathsf{R} \cap \partial\mathsf{D}$. This means that (j,h) belongs in at least one of $\partial\mathsf{D}_{\text{upper}}$ or $\partial\mathsf{D}_{\text{lower}}$. Assume here for concreteness that $(j,h) \in \mathsf{R} \cap \partial\mathsf{D}_{\text{upper}}$. Then since $(j,h) \in \mathsf{R}$ there is an open set $U \subset \mathsf{R}$ with $(j,h) \in U$. Given an arbitrary $(j',h') \in \mathsf{D}_{\text{upper}} \cap U$ and the basis $(a^{\pm}(j',h'), b(j',h'))$ of $H_1(F^{-1}(j',h'))$ there is a unique parallel transport of this basis to all points in U. Thus for $(j,h) \in \mathsf{R} \cap \partial\mathsf{D}_{\text{upper}}$ it is possible to define the basis $(a^{\pm}(j,h), b(j,h))$ as the parallel transport of $(a^{\pm}(j',h'), b(j',h'))$ along a path Γ_0 in $(\mathsf{D}_{\text{upper}} \cap U) \cup \{(j,h)\}$ that connects (j',h') to (j,h). Then the path Γ that starts and ends at $(j,h) \in \mathsf{R} \cap \partial\mathsf{D}$ can be decomposed as $\Gamma = -\Gamma_0 + \Gamma' + \Gamma_0$ with the path Γ' joining $(j',h') \in \mathsf{D}_{\text{upper}}$ to itself and $\Gamma' \in \mathcal{L}$. Thus $Na^{\pm}(j',h') - b(j',h')$ is the parallel transport along Γ' of $Na^{\pm}(j',h')$ and since parallel transport along $\pm\Gamma_0$ is trivial we conclude that $Na^{\pm}(j,h) - b(j,h)$ is the parallel transport along Γ of $Na^{\pm}(j,h)$. □

7. Proof of Theorem 1

Proposition 3 gives the parallel transport of homology cycles in the index-N subgroup $\mathsf{H}(j,h)$ of $H_1(F^{-1}(j,h))$ spanned over $\mathbf{Z}$ by the cycles $Na^{\pm}$ and b, that is, the computational part (iii) of Theorem 1. We now show that the parallel transport of cycles in $\mathsf{H}(j,h)$ is unique and that only cycles in $\mathsf{H}(j,h)$ can be parallel transported along Γ. This completes the proof of Theorem 1. Thus the parallel transport along Γ gives a well-defined automorphism of the index-N subgroup $\mathsf{H}(j,h)$ of the full homology group $H_1(F^{-1}(j,h))$. In the basis $(Na^{\pm}(j,h), b(j,h))$ of $\mathsf{H}(j,h)$ this automorphism is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

7.1. Uniqueness of parallel transport along Γ. As we have mentioned in §6.1 the parallel transport of homology cycles along a path Γ may not be unique. We show here that this situation does not occur in our problem.

Recall that if $\alpha_1 \in H_1(F^{-1}(\Gamma(1)))$ is a parallel transport along Γ of $\alpha_0 \in H_1(F^{-1}(\Gamma(0)))$ then there is a relative cycle $C \in H_2(M_\Gamma, \partial M_\Gamma)$ such that

$$\partial_* C = (\alpha_0, \alpha_1),$$

where ∂_* is the connecting homomorphism in the long exact sequence

$$\cdots \longrightarrow H_2(M_\Gamma, \partial M_\Gamma) \xrightarrow{\partial_*} H_1(\partial M_\Gamma) \longrightarrow H_1(M_\Gamma) \longrightarrow \cdots.$$

For our specific Γ we have that $\partial M_\Gamma = \mathbf{T}^2 \cup \mathbf{T}^2$ and thus $H_1(\partial M_\Gamma) = \mathbf{Z}^2 \oplus \mathbf{Z}^2$. Furthermore, because of the transversality condition (iv) in Definition 2 the set M_Γ is here a 3-dimensional compact orientable smooth manifold. Then the rank of the image of $H_2(M_\Gamma, \partial M_\Gamma)$ under ∂_*, i.e., the rank of $\mathrm{PT}(F, \Gamma)$, is half of the rank of $H_1(\partial M_\Gamma)$, see [24, Lemma 3.5]. Thus

$$\operatorname{rank} \mathrm{PT}(F, \Gamma) = 2.$$

Furthermore since $\mathrm{PT}(F, \Gamma)$ is a subgroup of the free abelian group $\mathbf{Z}^2 \oplus \mathbf{Z}^2$, $\mathrm{PT}(F, \Gamma)$ is also free abelian and thus $\mathrm{PT}(F, \Gamma) \simeq \mathbf{Z} \oplus \mathbf{Z}$.

It follows from Proposition 3 that $W_1 = (Na, Na - b)$ and $W_2 = (b, b)$ belong in $\mathrm{PT}(F, \Gamma)$; here we write $a := a^{\pm}(j, h)$ and $b := b(j, h)$. Then W_1 and W_2 are linearly independent over $\mathbf{Z}$ since if $k_1 W_1 + k_2 W_2 = (0, 0)$, then

$$(k_1 Na + k_2 b, k_1 Na + (k_1 + k_2)b) = (0, 0),$$

which implies $k_1 = k_2 = 0$. Thus W_1 and W_2 span a rank-2 subgroup of $\mathrm{PT}(F, \Gamma)$.

In order to show that parallel transport along Γ is unique we have to show that $\mathrm{PT}(F, \Gamma)$ does not contain any elements of the form $(0, c)$ with $c \neq 0$. Assume for a contradiction that $(0, c) \in \mathrm{PT}(F, \Gamma)$. Then, since W_1 and W_2 span a rank-2 subgroup of $\mathrm{PT}(F, \Gamma)$, there are $\ell, k_1, k_2 \in \mathbf{Z}$ with $\ell \neq 0$ such that $\ell(0, c) = k_1 W_1 + k_2 W_2$. This implies $k_1 = k_2 = 0$ and thus also $\ell = 0$.

7.2. Only cycles in $\mathsf{H}(j, h)$ can be parallel transported along Γ. In order to complete the proof of Theorem 1 we need to show that only cycles in $\mathsf{H}(j, h)$ can be parallel transported along Γ. In order to see this, consider a cycle $\gamma \in H_1(F^{-1}(j, h)) \backslash \mathsf{H}(j, h)$ and assume that there is a cycle $\gamma' \in H_1(F^{-1}(j, h))$ such that γ' is the parallel transport of γ along Γ. Then $\gamma = k_1 a^{\pm}(j, h) + k_2 b(j, h)$, where $k_1, k_2 \in \mathbf{Z}$ and k_1 is not an integer multiple of N. The cycle $N\gamma = Nk_1 a^{\pm}(j, h) + Nk_2 b(j, h)$ belongs in $\mathsf{H}(j, h)$ and is thus parallel transported to $Nk_1 a(j, h) - k_1 b(j, h) + Nk_2 b(j, h) = N\gamma - k_1 b(j, h)$. Furthermore, since γ is parallel transported to γ' we deduce that $N\gamma$ is parallel transported to $N\gamma'$. Since parallel transport along Γ is unique we conclude that $N(\gamma - \gamma') = k_1 b(j, h)$. This implies that k_1 is an integer multiple of N which contradicts our initial assumption.

8. Discussion

We proved the existence of fractional monodromy in $n_1{:}(-n_2)$ resonant systems by passing to an appropriate covering space. This allowed us to study the parallel transport of homology cycles in the covering space without the complications presented by the ‘twisting’ of the fibres in the original space.

Our method of proof of standard and fractional monodromy has a particularly geometric character. In this respect it is very close to the approach of [27–30] and uses some of the same techniques, such as the choice of appropriate surfaces of section, that were first used in these earlier works.

One remarkable feature of the present approach is that the conclusions do not depend on the detailed knowledge of the structure of the critical sets of the integral map F. Indeed in this paper we have only studied the regular fibres in the sets $\mathsf{D}_{\mathrm{upper}}$ and $\mathsf{D}_{\mathrm{lower}}$ and ignored the critical fibres existing between these regular fibres. It is the $n_1{:}(-n_2)$ resonant action that ‘forces’ the geometry of the fibration. Note in particular that both in the original and in the covering space it is possible to have several different critical sets but we always obtain the same kind of parallel transport and the same type of standard or fractional monodromy.

This clearly shows that our results are persistent for perturbations that commute with the resonant $n_1{:}(-n_2)$ oscillator. Furthermore, our approach and results firmly establish that standard and fractional monodromy are purely geometric properties of this system that nevertheless can be approached from a dynamical point of view as earlier work [5,12,19,31,32] has shown.

A complete understanding of the absolutely essential properties of a dynamical system that lead to fractional monodromy is still missing. Such understanding will prove to be fundamental for generalizing the notion of fractional monodromy to non-Hamiltonian, e.g. non-holonomic, contexts. The simplification of the study of the geometry by passing to a covering space and the resulting improved understanding of the geometry is one of the first steps toward such goal.

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