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Monodromy of Hamiltonian systems with complexity 1 torus actions

K. Efstathiou, N. Martynchuk *

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK, Groningen, The Netherlands

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ABSTRACT

We consider the monodromy of n-torus bundles in n degree of freedom integrable Hamiltonian systems with a complexity 1 torus action, that is, a Hamiltonian $\mathbb{T}^{n-1}$ action. We show that orbits with $\mathbb{T}^1$ isotropy are associated to non-trivial monodromy and we give a simple formula for computing the monodromy matrix in this case. In the case of 2 degree of freedom systems such orbits correspond to fixed points of the $\mathbb{T}^1$ action. Thus we demonstrate that, given a $\mathbb{T}^{n-1}$ invariant Hamiltonian H, it is the $\mathbb{T}^{n-1}$ action, rather than H, that determines monodromy.

1. Introduction

The, now classical, work by Duistermaat on obstructions to global action–angle coordinates in integrable Hamiltonian systems [1] highlighted the importance of the non-triviality of torus bundles over circles for such systems. Since then non-trivial monodromy has been demonstrated in several integrable Hamiltonian systems. We indicatively mention the spherical pendulum [1,2], the Lagrange top [3], the Hamiltonian Hopf bifurcation [4], the champagne bottle [5], the coupled angular momenta [6], the two-centers problem [7], and the quadratic spherical pendulum [8,9]. A common aspect of these systems is the presence of a symmetry given by a Hamiltonian $\mathbb{T}^{n-k}$ action, where n is the number of degrees of freedom (for the two-centers problem $k = 2$ and for the other systems $k = 1$).

Remark 1.1. Hamiltonian $\mathbb{T}^{n-k}$ actions on symplectic $2n$ manifolds are called *complexity k torus actions*. Classification of symplectic manifolds with such actions has been studied by Delzant in [10] ($k = 0$), and Karshon and Tolman in [11] ($k = 1$). We note that for integrable systems with a complexity 0 torus action monodromy is always trivial.

In the present paper we consider integrable n degree of freedom systems with a complexity 1 torus action, that is, a Hamiltonian $\mathbb{T}^{n-1}$ action. Monodromy in such systems (along a given curve) is determined by $n-1$ free integer parameters. We will show that these parameters are related to singular orbits of the $\mathbb{T}^{n-1}$ action via the *curvature form* of an appropriate principal $\mathbb{T}^{n-1}$ bundle; see Theorems 3.2 and 3.4. Surprisingly, this relation has not been observed before. The usually adopted approaches to monodromy (see [12–17,2]) do not take into account the differential geometric invariants of the Hamiltonian $\mathbb{T}^{n-1}$ symmetry, such as the curvature form and the *Chern numbers*. Moreover, these approaches are rather concentrated on the study of the whole *integral map*, that is, the Hamiltonian and the momenta that generate the $\mathbb{T}^{n-1}$ action. Our results in this paper show that the Hamiltonian plays a secondary role to the momenta for determining monodromy.

* Corresponding author.
E-mail addresses: K.Efstathiou@rug.nl (K. Efstathiou), N.Martynchuk@rug.nl (N. Martynchuk).

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Specifically, for 2 degrees of freedom systems monodromy is determined in terms of the fixed points of the Hamiltonian $\mathbb{T}^1$ action; see Theorem 3.10. For n degree of freedom systems monodromy is determined in terms of how the $\mathbb{S}^1$ isotropy group is expressed in a given basis of the $\mathbb{T}^{n-1}$ action; see Theorem 3.15.

The paper is organized as follows. In Section 2 we specify our setting and recall necessary definitions from the theory of principal bundles. In Section 3 we formulate our main results that relate monodromy with singularities of the $\mathbb{T}^{n-1}$ action; see Theorems 3.2, 3.4, 3.10 and 3.15. The proof of Theorem 3.2, which is more technical, is postponed to Section 5. In Section 4 we apply our techniques to various integrable systems. The paper is concluded in Section 6 with a discussion.

2. Preliminaries

Let M be a connected $2n$-dimensional manifold with a symplectic form Ω. Since Ω is a non-degenerate 2-form, to every smooth function $F_1 : M \to \mathbb{R}$ one can associate the so-called *Hamiltonian vector field* $X_{F_1} = \Omega^{-1}(dF_1)$. Suppose that we have n almost everywhere independent functions $F_1, \ldots, F_n$ on M such that all *Poisson brackets* vanish:

$$\{F_i, F_j\} = \Omega(X_{F_i}, X_{F_j}) = 0.$$

Then we say that we have an integrable Hamiltonian system on M. The map

$$(F_1, \ldots, F_n) : M \to \mathbb{R}^n$$

is called the *integral map* of the system. Everywhere in the paper we assume that the Assumption 2.1 hold (except for Section 5 where we work in a more general setting of a Hamiltonian $\mathbb{T}^k$ action, $1 \le k \le n-1$).

Assumptions 2.1. The integral map F is assumed to have the following properties.

(1) F is *proper*, that is, for every compact set $K \subset \mathbb{R}^n$ the preimage $F^{-1}(K)$ is a compact subset of M.
(2) The integral map F is invariant under a Hamiltonian $\mathbb{T}^{n-1}$ action.
(3) The $\mathbb{T}^{n-1}$ action is free on $F^{-1}(R)$, where $R \subset \text{image}(F)$ the set of regular values of F.

Consider a regular simple closed curve $\gamma \subset R$ and assume that the fibers $F^{-1}(\xi)$, $\xi \in \gamma$, are connected. By the Arnol'd–Liouville theorem we have a n-torus bundle

$$(E_\gamma = F^{-1}(\gamma), \gamma, F) \tag{1}$$

with respect to F. Take a fiber $F^{-1}(\xi_0)$, $\xi_0 \in \gamma$, and let T^{n-1} be any orbit of the Hamiltonian $\mathbb{T}^{n-1}$ action on $F^{-1}(\xi_0)$. We choose a basis $(e_1, \ldots, e_n)$ of the integer homology group $H_1(F^{-1}(\xi_0))$ so that $(e_1, \ldots, e_{n-1})$ is a basis of $H_1(T^{n-1})$. Since the Hamiltonian $\mathbb{T}^{n-1}$ action is globally defined on E_γ, the generators e_j, $j = 1, \ldots, n-1$, are also 'globally defined', that is they are preserved under the parallel transport along γ. It follows that the monodromy matrix of the bundle (E_γ, γ, F) with respect to the basis $(e_1, \ldots, e_n)$ has the form

$$\begin{pmatrix} 1 & \cdots & 0 & m_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & m_{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We call $\vec{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$ the *monodromy vector*. In Section 3 we relate $\vec{m}$ to the curvature form of an appropriate principal $\mathbb{T}^{n-1}$ bundle and then give a formula that allows us to compute $\vec{m}$ in specific integrable Hamiltonian systems.

The assumption of the existence of a $\mathbb{T}^{n-1}$ action made throughout this paper brings us in the context of principal torus bundles and their Chern numbers. We recall here some relevant definitions. For a detailed exposition of the theory we refer to Postnikov [18].

Consider a principal $\mathbb{T}^{n-1}$ bundle (E, B, ρ). The structure group $\mathbb{T}^{n-1}$ is isomorphic to the direct product of $n-1$ circles:

$$\mathbb{T}^{n-1} = \{(e^{i\varphi_1}, \ldots, e^{i\varphi_{n-1}}) \mid \varphi_j \in \mathbb{R}\} \subset \mathbb{C}^{n-1}.$$

The Lie algebra $T_e\mathbb{T}^{n-1}$ can be thus identified with $i\mathbb{R}^{n-1}$. The Lie bracket is identically zero since $\mathbb{T}^{n-1}$ is a commutative group.

Let $A^\#$ denote the fundamental vector field generated by $A \in i\mathbb{R}^{n-1}$ and $R_g^\star$ denote the pull-back of the right shift $R_g : E \to E$.

Definition 2.2. A *connection one-form ω on* (E, B, ρ) is a $i\mathbb{R}^{n-1}$-valued one-form on E such that $\omega(A^\#) = A$ and $R_g^\star(\omega) = \omega$.

Remark 2.3. In our setting both E and B are compact manifolds. Hence a connection one-form exists. It separates tangent spaces of E into *vertical* and *horizontal* subspaces.

Let $\{U_\alpha\}_{\alpha \in I}$ be a trivialization cover of B.

Definition 2.4. On each trivialization chart U_α define the curvature form $\mathfrak{F}$ by the following formula:

$$\mathfrak{F}|_{U_\alpha} = ds_\alpha^\star(\omega),$$

where $s_\alpha : U_\alpha \to E$ is a section and $s_\alpha^\star$ denotes the pull-back.

Remark 2.5. Since $\mathbb{T}^{n-1}$ is commutative, the curvature form $\mathfrak{F}$ is well-defined. It is a closed 2-form whose cohomology class $[\mathfrak{F}]$ does not depend on the choices made.

Remark 2.6. If $n = 2$ then $c_1(\mathfrak{F}) = \frac{i}{2\pi}[\mathfrak{F}]$ is the (*first*) *Chern class* of the principal circle bundle (E, B, ρ). But conversely, let $n \geq 2$ be arbitrary. Consider the subgroup $\mathbb{T}^{n-2}_l$ of $\mathbb{T}^{n-1}$ defined by

$$\mathbb{T}^{n-2}_l = \{(e^{i\varphi_1}, \dots, e^{i\varphi_{n-1}}) \mid \varphi_l = 0;\ \varphi_j \in \mathbb{R},\ j \neq l\}.$$

Let $c_1(\mathfrak{F}_l)$ be the first Chern class of the circle bundle $(E/\mathbb{T}^{n-2}_l, B, \rho)$. Then the (cohomology class of a) curvature form $\mathfrak{F}$ of the $\mathbb{T}^{n-1}$ bundle (E, B, ρ) is given by

$$\frac{i}{2\pi}[\mathfrak{F}] = (c_1(\mathfrak{F}_1), \dots, c_1(\mathfrak{F}_{n-1}))\,. \tag{2}$$

We will apply the above theory to the case when B is a 2-dimensional manifold. In this case the following integral is defined:

$$\vec{m} = \frac{i}{2\pi}\int_B \mathfrak{F}.$$

The output $\vec{m}$ is a set of $n - 1$ integers. We call them, in line with Eq. (2), *Chern numbers* of the principal bundle (E, B, ρ).

3. Monodromy and Chern numbers

Recall that we have a n-torus bundle $(E_\gamma = F^{-1}(\gamma), \gamma, F)$. Since the $\mathbb{T}^{n-1}$ action is free on $E_\gamma \subset F^{-1}(R)$, the monodromy vector $\vec{m}$ is defined and we have a *principal* bundle $(E_\gamma, E_\gamma/\mathbb{T}^{n-1}, \rho)$ with respect to the reduction map $\rho : M \to M/\mathbb{T}^{n-1}$. We note that $E_\gamma/\mathbb{T}^{n-1}$ is a 2-torus since it is an orientable circle bundle over the curve γ. Since the base $E_\gamma/\mathbb{T}^{n-1}$ is compact, there exists a curvature form $\mathfrak{F}$ and thus the Chern numbers $\frac{i}{2\pi}\int_{E_\gamma/\mathbb{T}^{n-1}} \mathfrak{F}$ are defined.

Remark 3.1. The (cohomology class) of $\mathfrak{F}$ is given by the Chern classes of the circle bundles $(E_\gamma/\mathbb{T}^{n-2}_l, E_\gamma/\mathbb{T}^{n-1}, \rho)$; see Section 2 and Eq. (2) therein. These Chern classes should not be confused with the Chern class introduced by Duistermaat in [1], which obstructs the existence of a global section of $(F^{-1}(R), R, F)$. It can happen that the Chern class in the sense of Duistermaat is trivial, while the $\mathbb{T}^{n-1}$ action is not.

The entire paper is based on the following result, the proof of which we give in Section 5.

Theorem 3.2. *Let $F : M \to \mathbb{R}^n$ be a proper integral map of an integrable system on M invariant under a Hamiltonian $\mathbb{T}^{n-1}$ action. Consider a regular simple closed curve $\gamma \subset R$ such that the fibers $F^{-1}(\xi)$, $\xi \in \gamma$, are connected and such that the $\mathbb{T}^{n-1}$ action is free on $E_\gamma = F^{-1}(\gamma)$.*

Then the monodromy vector $\vec{m}$ is determined by the Chern numbers of $(E_\gamma, E_\gamma/\mathbb{T}^{n-1}, \rho)$, specifically,

$$\vec{m} = \frac{i}{2\pi}\int_{E_\gamma/\mathbb{T}^{n-1}} \mathfrak{F}.$$

Remark 3.3. Recall that the monodromy vector $\vec{m}$ depends on the choice of the generators $(e_1, \dots, e_{n-1})$. The generators $(e_1, \dots, e_{n-1})$ result in a basis of the Lie-algebra $T_e\mathbb{T}^{n-1} = i\mathbb{R}^{n-1}$. In Theorem 3.2 we implicitly assume that the curvature form $\mathfrak{F}$ is written with respect to this basis.

We will now use Theorem 3.2 in order to show that the monodromy of the bundle (E_γ, γ, F) is related to the orbits of the $\mathbb{T}^{n-1}$ action with $\mathbb{S}^1$ isotropy. In particular, we prove the following result.

Theorem 3.4. *Let F and γ be as in Theorem 3.2. Assume, moreover, that the following conditions hold.*

(1) *There exists a 2-disk U in the image of F with $\partial U = \gamma$.*
(2) *The preimage $F^{-1}(\overline{U})$ is a closed submanifold (with boundary) of M.*
(3) *The $\mathbb{T}^{n-1}$ action is free on $F^{-1}(\overline{U})$ outside ℓ orbits $\mathfrak{p}_1, \dots, \mathfrak{p}_\ell \in F^{-1}(U)$ with $\mathbb{S}^1$ isotropy.*

Then

$$\vec{m} = \frac{i}{2\pi} \sum_{k=1}^{\ell} \int_{S_k^2} \mathfrak{F}, \tag{3}$$

where S_k^2 is an arbitrary small sphere around the point $\rho(\mathfrak{p}_k)$ in the reduced space $F^{-1}(U)/\mathbb{T}^{n-1}$.

Proof. Take sufficiently small open balls $V_k \subset F^{-1}(U)/\mathbb{T}^{n-1}$ around $\rho(\mathfrak{p}_k)$ such that the complement

$$B = \left(F^{-1}(\overline{U})/\mathbb{T}^{n-1}\right) \setminus \bigcup_{k=1}^{\ell} V_k$$

is a compact connected manifold with boundary. Observe that by construction the boundary ∂B is the disjoint union of the spheres $S_k^2 = \partial V_k,\ k = 1, \ldots, \ell$, and the 2-torus $E_\gamma/\mathbb{T}^{n-1}$.

Let $E = \rho^{-1}(B)$. Then the bundle (E, B, ρ) is a principal $\mathbb{T}^{n-1}$ bundle. Let $\mathfrak{F}$ denote its curvature 2-form. Theorem 3.2 implies that

$$\vec{m} = \frac{i}{2\pi} \int_{E_\gamma/\mathbb{T}^{n-1}} \mathfrak{F},$$

and a direct application of Stokes' theorem gives

$$\vec{m} = \frac{i}{2\pi} \sum_{k=1}^{\ell} \int_{S_k^2} \mathfrak{F}. \quad \square$$

Remark 3.5. Since the spheres S_k^2 can be chosen to be arbitrary small, the monodromy vector $\vec{m}$ is determined by the behavior of the $\mathbb{T}^{n-1}$ action near the singular orbits $\mathfrak{p}_k$.

Remark 3.6. In the case $n = 2$ the assumption (2) from Theorem 3.4 can be omitted as it always holds. Moreover, in this case the singular orbits $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ with $\mathbb{S}^1$ isotropy are simply the fixed points of the $\mathbb{S}^1 = \mathbb{T}^1$ action.

Up to the end of this section we assume that F and γ satisfy the conditions of Theorem 3.4. Our goal is to obtain expressions for $\vec{m}$ that can be more easily used in applications. First consider the simplest case $n = 2$.

3.1. The case of 2 degrees of freedom

From the slice theorem [19, Theorem I.2.1] (see also [20]) it follows that in a small equivariant neighborhood of a fixed point the $\mathbb{S}^1 = \mathbb{T}^1$ action can be linearized. Thus in appropriate complex coordinates $(z, w) \in \mathbb{C}^2$ it can be written as

$$(z, w) \mapsto (e^{imt}z, e^{int}w), \quad t \in \mathbb{S}^1,$$

for some integers m, n. By our assumption, the Hamiltonian $\mathbb{S}^1$ action is free everywhere in $F^{-1}(\overline{U})$ except for ℓ singular points $P_1, \ldots, P_\ell \in F^{-1}(U)$. Hence near each such singular point it can be written as

$$(z, w) \mapsto (e^{\pm it}z, e^{it}w), \quad t \in \mathbb{S}^1,$$

in appropriate complex coordinates $(z, w) \in \mathbb{C}^2$. The two cases can be mapped to each other through an orientation-reversing coordinate change.

Definition 3.7. We call a singular point P *positive* if the local $\mathbb{S}^1$ action is given by $(z, w) \mapsto (e^{-it}z, e^{it}w)$ and *negative* if the action is given by $(z, w) \mapsto (e^{it}z, e^{it}w)$ in a coordinate chart having the orientation induced by the symplectic form Ω.

Remark 3.8. The $\mathbb{S}^1$ action $(z, w) \mapsto (e^{it}z, e^{it}w)$ defines the Hopf fibration on the sphere $S^3 = \{(z, w) \in \mathbb{C}^2 \mid 1 = |z|^2 + |w|^2\}$. The $\mathbb{S}^1$ action $(z, w) \mapsto (e^{-it}z, e^{it}w)$ defines a fibration which can be transformed to the Hopf fibration through an orientation-reversing coordinate change. If an orientation is fixed and the $\mathbb{S}^1$ action is given by $(z, w) \mapsto (e^{-it}z, e^{it}w)$ then we talk about an *anti-Hopf fibration* [21].

Lemma 3.9. *The Chern number of the Hopf fibration is equal to* -1, *while for the anti-Hopf fibration it is equal to* 1.

Proof. Consider the case of the Hopf fibration (the anti-Hopf case is analogous). Its projection map $h\colon S^3 \to S^2$ is defined by the formula $h(z, w) = (z : w) \in \mathbb{CP}^1 = S^2$. Put

$$U_1 = \{(u : 1) \mid u \in \mathbb{C},\ |u| < 1\} \quad \text{and} \quad U_2 = \{(1 : v) \mid v \in \mathbb{C},\ |v| < 1\}.$$

Define sections $s_j\colon U_j \to S^3$ by the formulas

$$s_1((u:1)) = \left(\frac{u}{\sqrt{|u|^2+1}}, \frac{1}{\sqrt{|u|^2+1}}\right)$$

and

$$s_2((1:v)) = \left(\frac{1}{\sqrt{|v|^2+1}}, \frac{v}{\sqrt{|v|^2+1}}\right).$$

Now, the cocycle $t_{12}\colon S^1 = \overline{U}_1 \cap \overline{U}_2 \to \mathbb{S}^1$ corresponding to the sections s_1 and s_2 is given by

$$t_{12}((u:1)) = \exp(-i\operatorname{Arg} u).$$

If ω is a connection one-form then

$$s_2^\star(\omega) = s_1^\star(\omega) + t_{12}^{-1} dt_{12} = s_1^\star(\omega) - i \cdot d(\operatorname{Arg} u).$$

It follows that

$$-1 = \frac{i}{2\pi}\int_{S^1} i \cdot d(\operatorname{Arg} u) = \frac{i}{2\pi}\int_{\partial U_1} s_1^\star(\omega) + \frac{i}{2\pi}\int_{\partial U_2} s_2^\star(\omega)$$

and hence

$$-1 = \frac{i}{2\pi}\int_{U_1} ds_1^\star(\omega) + \frac{i}{2\pi}\int_{U_2} ds_2^\star(\omega) = \frac{i}{2\pi}\int_{S^2} \mathfrak{F}. \quad \square$$

Observe that the right hand side of Eq. (3) is the sum of the Chern numbers of Hopf or anti-Hopf fibrations. Thus, in view of Lemma 3.9, the following result holds.

Theorem 3.10. *Let F and γ be as in Theorem 3.4 with $n = 2$. Then the monodromy of the 2-torus bundle (E_γ, γ, F) is given by the number of positive singular points minus the number of negative singular points in $F^{-1}(U)$.*

Remark 3.11. Note that Theorem 3.10 does not require that the singular points are focus–focus singularities of F.

3.2. *The case of $n \geq 2$ degrees of freedom*

In this section we provide two approaches for computing the monodromy vector $\vec{m}$ in the case $n \geq 2$. The first approach is to reduce the number of degrees of freedom and apply techniques from Section 3.1. First, let us reformulate Theorem 3.2 as follows. Consider the subgroup $\mathbb{T}_l^{n-2}$ of $\mathbb{T}^{n-1}$ defined by

$$\mathbb{T}_l^{n-2} = \{(e^{i\varphi_1}, \ldots, e^{i\varphi_{n-1}}) \mid \varphi_l = 0;\ \varphi_j \in \mathbb{R},\ j \neq l\}. \tag{4}$$

Let $c_1(\mathfrak{F}_l)$ be the first Chern class of the circle bundle $(E_\gamma/\mathbb{T}_l^{n-2}, E_\gamma/\mathbb{T}^{n-1}, \rho)$. From Eq. (2) we get

$$m_l = \int_{E_\gamma/\mathbb{T}^{n-1}} c_1(\mathfrak{F}_l). \tag{5}$$

Let J_k^l be smooth functions on M such that their Hamiltonian vector fields generate the $\mathbb{T}_l^{n-2}$ action. Denote by J_c^l the common level set of J_k^l:

$$J_c^l = \{J_1^l = c_1, \ldots, J_{n-2}^l = c_{n-2}\}.$$

Suppose that there exists a regular J_c^l such that $F^{-1}(\overline{U}) \subset J_c^l$. Symplectic reduction with respect to the $\mathbb{T}_l^{n-2}$ action yields a 2 degree of freedom Hamiltonian system on $J_c^l/\mathbb{T}_l^{n-2}$. From Eq. (5) it follows that m_l gives the monodromy along γ in the reduced system, and one can apply the results from Section 3.1 to determine m_l.

Remark 3.12. The reduction method just described can be applied only when the functions J_k^l are constant on $F^{-1}(\gamma)$. In practical situations one can use the fact that the monodromy along γ depends only on its homotopy type $[\gamma]$ in $F^{-1}(R)$ to find an appropriate γ and generators J_k^l so that this condition holds.

The second approach starts from Eq. (3). We want to compute integrals $\frac{i}{2\pi}\int_{S_k^2}\mathfrak{F}$, where $1 \leq k \leq \ell$. Represent the acting torus $\mathbb{T}^{n-1}$ as a direct product $\mathbb{T}^{n-1} = \mathbb{S}_1^1 \times \cdots \times \mathbb{S}_{n-1}^1$ in such a way that the isotropy group of $\mathfrak{p}_k$ is $\mathbb{S}_1^1$. This representation leads to a new basis $(e_1', \ldots, e_{n-1}', e_n)$ of $H_1(F^{-1}(\xi_0))$ (cf. the beginning of Section 3).

Consider the subgroup $\mathbb{T}^{n-2}_l$ of $\mathbb{T}^{n-1}$ defined as in (4). Suppose $l > 1$. Then the triple $(\rho^{-1}(V_k)/\mathbb{T}^{n-2}_l, V_k, \rho)$ is a trivial circle bundle since V_k is contractible. Now suppose $l = 1$. It follows from the slice theorem that $\mathbb{S}^1_1$ acts on the quotient $\rho^{-1}(V_k)/\mathbb{T}^{n-2}_1$ linearly as

$$(z, w) \mapsto (e^{\pm it}z, e^{it}w), \quad t \in \mathbb{S}^1_1,$$

in appropriate coordinates (z, w). In accordance to Definition 3.7 we propose the following definition.

Definition 3.13. We call the orbit $\mathfrak{p}_k$ *positive with respect to the* $\mathbb{T}^{n-1}$ *action* if the $\mathbb{S}^1_1$ action on the quotient $\rho^{-1}(V_k)/\mathbb{T}^{n-2}_1$ is given by $(z, w) \mapsto (e^{-it}z, e^{it}w)$ and *negative* otherwise.

Remark 3.14. There is a canonical orientation on the quotient $\rho^{-1}(V_k)/\mathbb{T}^{n-2}_1$ induced by the symplectic form Ω.

With the above conventions we have

$$\frac{i}{2\pi}\int_{S^2_k} \mathfrak{F} = (\pm 1, 0, \ldots, 0)^t \tag{6}$$

depending on whether the singular orbit $\mathfrak{p}_k$ is positive or negative for the $\mathbb{T}^{n-1}$ action.

Note that we made a specific choice of the basis of the Lie algebra of the acting torus $\mathbb{T}^{n-1}$ when we represented it as a direct product $\mathbb{T}^{n-1} = \mathbb{S}^1_1 \times \cdots \times \mathbb{S}^1_{n-1}$. The basis $(e_1, \ldots, e_{n-1})$ associated to the monodromy vector $\vec{m}$ corresponds, in general, to a different basis of the Lie algebra. It can be checked that in the latter basis Eq. (6) becomes

$$\frac{i}{2\pi}\int_{S^2_k} \mathfrak{F} = \pm \vec{u}_k,$$

where $\vec{u}_k = (u^1_k, \ldots, u^{n-1}_k) \in \mathbb{Z}^{n-1}$ is such that $e'_1 = \sum_{j=1}^{n-1} u^j_k e_j$. In other words, the coefficients $(u^1_k, \ldots, u^{n-1}_k)$ are the expansion coefficients of the isotropy group $\mathfrak{p}_k$ with respect to the generators $(e_1, \ldots, e_{n-1})$.

Finally we get the following theorem.

Theorem 3.15. *Let F and γ be as in Theorem 3.4 with $n \geq 2$ arbitrary. Then the monodromy of the n-torus bundle (E_γ, γ, F) is given by*

$$\vec{m} = \sum_{k=1}^{\ell} \pm \vec{u}_k,$$

where the sign for the kth term depends on whether the orbit $\mathfrak{p}_k$ is positive or negative with respect to the $\mathbb{T}^{n-1}$ action.

4. Examples

In this section we use Theorems 3.10 and 3.15 to determine the monodromy of torus bundles in specific cases.

4.1. Monodromy around a focus–focus singularity

Suppose that $\xi_0 = (0, 0)$ is a focus–focus critical value of the 2 degree of freedom Hamiltonian system defined by a proper integral map F. Then $F^{-1}(\xi_0)$ is a singular fiber of complexity $\ell \in \mathbb{N}$, i.e., it contains ℓ singular focus–focus points. Let γ be a circle around ξ_0 of sufficiently small radius $r > 0$. Then there is the following result.

Theorem 4.1 ([13,14]). *The monodromy of the bundle (E_γ, γ, F) is given by the matrix* $\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$.

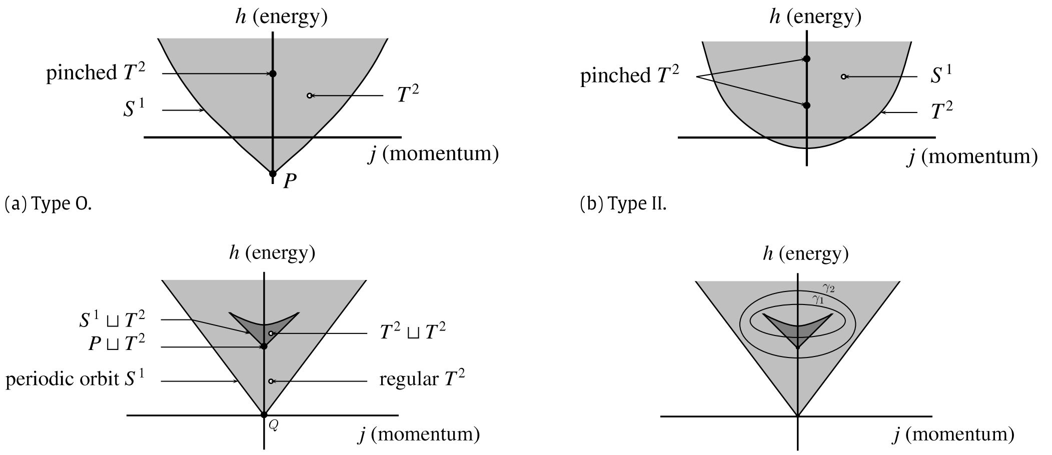
The above monodromy theorem was proved for $\ell = 1$ by Lerman and Umanskiĭ in [12] and then for arbitrary ℓ by Matveev in [13]. In [14] Zung gave a proof of the monodromy theorem based on the fact that all focus–focus singularities of the same complexity are semi-locally C^0-equivalent and on the existence of a unique Hamiltonian $\mathbb{S}^1$ action as described in the following result.

Theorem 4.2 ([14]). *In a neighborhood of the singular fiber $F^{-1}(\xi_0)$ there is a unique (up to orientation reversing) Hamiltonian $\mathbb{S}^1$ action which is free everywhere except for the singular points of F. Near each singular point of F in $F^{-1}(\xi_0)$ there is a local symplectic coordinate system (q_1, p_1, q_2, p_2) in which the momentum of the $\mathbb{S}^1$ action is*

$$J = \frac{1}{2}(q_1^2 + p_1^2) - \frac{1}{2}(q_2^2 + p_2^2).$$

Corollary 4.3. *Given the existence of a circle action for granted, the monodromy theorem follows directly from our results. Namely,*

Theorems 3.10 and 4.2 $\Rightarrow$ Theorem 4.1.



(a) Type O. (b) Type II. (c) Type I. (d) Paths in a type I system.

Fig. 1. (a–c) Bifurcation diagrams for different regimes of the quadratic spherical pendulum: (a) type O; (b) type II; (c) type I. (d) Paths in a type I system.

Proof. Let $z = q_1 + ip_1$ and $w = q_2 + ip_2$, where (q_1, p_1, q_2, p_2) are as in Theorem 4.2. Then the local chart (z, w) is positively oriented with respect to the orientation induced by $\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. It can be checked that the $\mathbb{S}^1$ action near each focus–focus point has the form $(z, w) \mapsto (e^{-it}z, e^{it}w)$. It follows from Lemma 3.9 that each focus–focus point is positive. $\square$

Remark 4.4. In the work [22] of Zung (cf. Cushman and Duistermaat [23]) monodromy was generalized to the case of an integrable non-Hamiltonian system of *bi-index* (2, 2), that is a 4-dimensional symplectic manifold M together with 2 vector fields X_j and 2 functions J_j such that $[X_j, X_l] = 0$ and $X_j(J_l) = 0$.

Just as in the Hamiltonian case, one can define monodromy for the map $F = (J_1, J_2)$, define the notion of a (possibly degenerate) focus–focus critical value and prove the existence, in a tubular neighborhood of the singular fiber, of a system-preserving $\mathbb{S}^1$ action; see [22]. The $\mathbb{S}^1$ action turns out to be free outside fixed points.

It is not difficult to see (details will be published in a forthcoming paper) that Theorem 3.10 is applicable in this setting and thus monodromy around a possibly degenerate focus–focus critical value is given by the number of positive singular points minus the number of negative singular points, cf. [23] and [22]. Note that negative singular points might appear which means that some of the Chern numbers in Eq. (3) will be equal to -1.

4.2. Quadratic spherical pendula

Consider a particle moving on the unit sphere

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

in a quadratic potential $V(x_3) = bx_3^2 + cx_3$. The corresponding Hamiltonian system $(TS^2, \Omega|_{TS^2}, H)$, where $H(x, v) = \frac{1}{2}\langle v, v\rangle + V(x)$ is the total energy, is called *quadratic spherical pendulum* [9]. This system is integrable since the x_3 component J of the angular momentum is conserved. Moreover, J generates a global Hamiltonian $\mathbb{S}^1$ action on TS^2. As we change the parameters b and c of the potential the system goes through different regimes characterized by qualitatively different bifurcation diagrams of the integral map $F = (H, J)$. In [9] these regimes were classified as follows:

Type O The image of F has one isolated critical value that lifts to a pinched torus containing one focus–focus point (Fig. 1(a)). The spherical pendulum $V(x_3) = x_3$ belongs to this category.

Type II The integral map F has two focus–focus critical values, isolated in the set of critical values (Fig. 1(b)). Each such critical value lifts to a singly pinched torus.

Type I The set of regular values F consists of two disjoint regions (Fig. 1(c)). Fibers of F over points in the outer region are T^2 while fibers over points in the inner region are disjoint unions of two T^2. We call the inner region "island". The common boundary of the two regions consists of critical values of F. Fibers of F over the *hyperbolic* critical values at the top of the island are the topological product of a 2-bouquet with $\mathbb{S}^1$. At the two top ends of the island these fibers degenerate to cuspidal tori, see [24]. The fiber over the lowest point of the island is the disjoint union of a T^2 and a single point P. The latter is an elliptic–elliptic singularity for the integrable system. The remaining *elliptic* critical fibers at the boundary of the island are the disjoint union of a T^2 and an S^1.

The monodromy for systems of type O or II is standard. As discussed in Section 4.1, the monodromy matrix for any path in the image of the integral map that once encircles exactly one focus–focus critical value is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The monodromy of type I systems is determined in [9] using the fact that such systems can be obtained through a subcritical Hamiltonian Hopf bifurcation of type O systems; the (local) bifurcation does not affect the (global) geometry of the T^2 bundle. Here, applying Theorem 3.10, we can compute the monodromy of such systems in a direct way even though they do not have focus–focus points and their monodromy cannot be determined by the monodromy theorem (Theorem 4.1). Indeed, consider the path γ_2 shown in Fig. 1(d) which encircles the inner region of regular values of F. Then

Proposition 4.5 ([9]). *The monodromy along γ_2 is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

Proof. Let D_2 be the 2-disk with $\partial D_2 = \gamma_2$. Then in the preimage $F^{-1}(D_2)$ there is only one fixed point of the global $\mathbb{S}^1$ action induced by J, namely, the elliptic–elliptic point P. Since the function J does not depend on the parameters b and c, and P is positive when it is a focus–focus point (in type O systems) we deduce that P is also positive with respect to the $\mathbb{S}^1$ action in type I systems. It is left to apply Theorem 3.10. $\square$

Remark 4.6. Since P is an elliptic–elliptic point of the integrable system (H, J) there exists a $\mathbb{T}^2$ action in its neighborhood. Thus one can consider two $\mathbb{S}^1$ actions in a neighborhood of P such that P is positive with respect to one and negative with respect to the other. Nevertheless, to apply Theorem 3.10 we must consider an $\mathbb{S}^1$ action defined on $F^{-1}(D_2)$ and the only such action is the global $\mathbb{S}^1$ action generated by J. Therefore, checking whether P is positive or negative must be done with respect to the global $\mathbb{S}^1$ action.

The path γ_2 we considered does not cross any critical values of F. However, we can also consider closed paths, such as γ_1 in Fig. 1(d), that encircle the curve of hyperbolic critical values and cross the curves of elliptic critical values. Then the preimage $F^{-1}(\gamma_1)$ consists of two connected components [9,24]. One of these components is diffeomorphic to S^3 while the other is the total space of a T^2 bundle over γ_1.

Proposition 4.7 ([9]). *The monodromy of the T^2 bundle over γ_1 is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

Proof. Consider the preimage of the interior of γ_1 and of the island. This preimage is a smooth manifold containing a fixed point P of the $\mathbb{S}^1$ action. Cut out a small 4-ball around P to get a manifold E, invariant under the $\mathbb{S}^1$ action. The quotient $E/\mathbb{S}^1$ is a 3-manifold with boundary $T^2 \sqcup S^2$. Applying Eq. (5) we find that the monodromy is

$$\int_{T^2} c_1(\mathfrak{F}) = \int_{S^2} c_1(\mathfrak{F}) = 1. \quad \square$$

Remark 4.8. Alternatively, to compute monodromy along γ_1 one can switch from working with values of F in $\mathbb{R}^2$ to the *unfolded momentum domain* [24]. Each value of F in the unfolded momentum domain corresponds to exactly one connected component of the fibration defined by F. Theorem 3.10 holds also in this setting.

4.3. Monodromy in the 1:1:(−2) resonance

Consider the Hamiltonian system $(\mathbb{R}^6, dq \wedge dp, H)$ defined by the Hamiltonian

$$H = \mathrm{Re}(z_1 z_2 z_3) + |z_1|^2 |z_2|^2.$$

Here we introduced complex coordinates $z_k = q_k + ip_k,\ k = 1, 2, 3$. The system is called 1:1:(−2) *resonance* due to the fact that H Poisson commutes with the *resonant* 1:1:(−2) *oscillator*; see [25]. Here by the resonant 1:1:(−2) oscillator we mean the following function

$$N = \frac{1}{2}(|z_1|^2 + |z_2|^2) - |z_3|^2.$$

Let

$$J = \frac{1}{2}(|z_1|^2 - |z_3|^2), \qquad L = \frac{1}{2}(|z_1|^2 - |z_2|^2),$$

and note that $N = 2J - L$. It can be checked that L, N and H Poisson commute. Hence the map

$$F\colon \mathbb{R}^6 \to \mathbb{R}^3 : F = (L, N, H)$$

is the integral map of an integrable system on $\mathbb{R}^6$. Furthermore, the Hamiltonian flows of integrals J and L define an *effective* Hamiltonian $\mathbb{T}^2$ action $\Phi : \mathbb{T}^2 \times \mathbb{C}^3 \to \mathbb{C}^3$. This action is given by the formula

$$\Phi(t_1, t_2, z) = (z_1 \exp[i(t_1 + t_2)],\ z_2 \exp[-it_2],\ z_3 \exp[-it_1]).$$

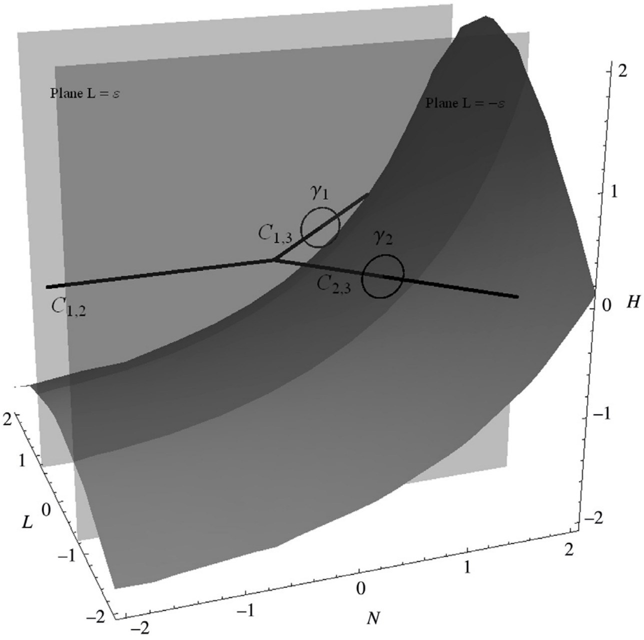


Fig. 2. BD for 1:1:(−2) resonance.

The *bifurcation diagram* (set of critical values of F) has the form shown in Fig. 2. For each pair of values (N, L) there is a minimum permissible value of the energy, giving a surface S of critical values of F. The image of the integral map (N, L, H) consists of values above S. In the interior of the image of F the only critical values are the sets

$$C_1 = \{(s, s, 0) : s > 0\}, \qquad C_2 = \{(s, -s, 0) : s > 0\}, \qquad C_3 = \{(-s, 0, 0) : s > 0\},$$

and the origin.

The fundamental group of the set of regular values is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. Its generators are closed paths γ_1 encircling C_1 and γ_2 encircling C_2. We want to find the monodromy matrices M_{γ_1} and M_{γ_2}.

Recall from Section 3 that in a basis (e_J, e_L, e) of $H_1(T^3)$, with e_J and e_L the generators corresponding to the flow of X_J and X_L respectively, the monodromy matrices have the form

$$M_{\gamma_k} = \begin{pmatrix} 1 & 0 & m_1^{(k)} \\ 0 & 1 & m_2^{(k)} \\ 0 & 0 & 1 \end{pmatrix},$$

where $m_1^{(k)}$ and $m_2^{(k)}$ are integers.

Proposition 4.9. *The integers $m_j^{(k)}$ are as follows:*

$$m_1^{(1)} = 1, \qquad m_2^{(1)} = -1, \qquad m_1^{(2)} = 1, \quad \textit{and} \quad m_2^{(2)} = 0.$$

Proof. Let us compute $m_1^{(1)}$ and $m_2^{(1)}$ first. Since M_{γ_1} depends only on the homotopy type of γ_1, we can assume that γ_1 lies on a constant $L = \varepsilon,\ \varepsilon > 0$, plane. Let U_1 be the interior of γ_1 in the plane $L = \varepsilon$. Since $\varepsilon > 0$ is a regular value of L, the preimage $F^{-1}(\overline{U}_1)$ is a submanifold of $\mathbb{R}^6$. The $\mathbb{T}^2$ action is free everywhere in $F^{-1}(\overline{U}_1)$ except for one singular orbit

$$\mathfrak{p}_1 = \{(z_1, z_2, z_3) \mid z_2 = z_3 = 0 \text{ and } |z_1|^2 = \varepsilon\}.$$

The isotropy group of $\mathfrak{p}_1$ is $\mathbb{S}^1$ and it corresponds to the flow of $J - L$. Therefore the generator of the isotropy is written in the basis (e_J, e_L) as $(1, -1)^t$. From Theorem 3.15 it follows that $m_1^{(1)} = 1$ and $m_2^{(1)} = -1$.

Analogously we can compute $m_1^{(2)}$ and $m_2^{(2)}$. In this case we assume that γ_2 lies on a constant $L = -\varepsilon,\ \varepsilon > 0$, plane. Just as before we let U_2 be the interior of γ_2 in the plane $L = -\varepsilon$. It can be checked that the only singular orbit of the $\mathbb{T}^2$ action in $F^{-1}(\overline{U}_2)$ is

$$\mathfrak{p}_2 = \{(z_1, z_2, z_3) \mid z_1 = z_3 = 0 \text{ and } |z_2|^2 = \varepsilon\}.$$

The isotropy group of $\mathfrak{p}_2$ is also $\mathbb{S}^1$ and it corresponds to the flow of J, thus the generator of the isotropy is written in the basis (e_J, e_L) as $(1, 0)^t$. Direct application of Theorem 3.15 yields $m_1^{(2)} = 1$ and $m_2^{(2)} = 0$. □

Remark 4.10. Resonances n:n:$(-m)$ are studied in [25] in more detail. In particular, the monodromy of the 1:1:(−2) resonance is computed there with the help of symplectic reductions.

5. Proof of Theorem 3.2

In this section we give the proof of Theorem 3.2 working in a more general setting than in Section 3. In particular, we have an integrable Hamiltonian system $F = (F_1, \ldots, F_n)$ defined on a connected $2n$-dimensional symplectic manifold M and we assume that F is proper and invariant under a Hamiltonian $\mathbb{T}^k$ action, where $k \le n-1$. The $\mathbb{T}^k$ action is assumed to be free on $F^{-1}(R)$, where $R \subset \operatorname{image}(F)$ is the set of regular values of F.

Consider a regular simple closed curve $\gamma \subset R$ and assume that the fibers $F^{-1}(\xi)$, $\xi \in \gamma$, are connected. Being compact, connected and invariant under the $\mathbb{R}^n$ action (generated by the flows of X_{F_j}), these fibers are homeomorphic to a n-torus. Specifically,

$$F^{-1}(\xi) = \mathbb{R}^n/\mathbb{Z}^n_\xi,$$

where $\mathbb{Z}^n_\xi$ is the isotropy group of the $\mathbb{R}^n$ action on $F^{-1}(\xi)$. We note that $\mathbb{Z}^n_\xi$ can be identified with $H_1(F^{-1}(\xi))$ for each $\xi \in \gamma$.

Let γ be parametrized by an angle coordinate χ and let $(e_1(0), \ldots, e_n(0))$ be a basis of $\mathbb{Z}^n_{\chi=0}$. Extend this basis into a smooth family

$$(e_1 = e_1(\chi), \ldots, e_n = e_n(\chi)),$$

where $\chi \in (-\varepsilon, \pi + \varepsilon)$ and $(e_1(\chi), \ldots, e_n(\chi))$ is a basis of $\mathbb{Z}^n_\chi$. Analogously define the family

$$(e'_1 = e'_1(\chi), \ldots, e'_n = e'_n(\chi)),$$

where $\chi \in (\pi - \varepsilon, 2\pi + \varepsilon)$. By the construction,

$$e_1 = e'_1, \ldots, e_n = e'_n \quad \text{when } \chi = 0.$$

Because of the $\mathbb{T}^k$ action we can assume that $e_1, \ldots, e_k$ are globally defined. In particular,

$$e_1 = e'_1, \ldots, e_k = e'_k \quad \text{when } \chi = \pi. \tag{7}$$

Generally speaking, $(e_1, \ldots, e_n)$ are not globally defined since monodromy of the bundle $(F^{-1}(\gamma), \gamma, F)$ can be non-trivial. The corresponding monodromy matrix, which is a transformation matrix between the bases $(e_1(\pi), \ldots, e_n(\pi))$ and $(e'_1(\pi), \ldots, e'_n(\pi))$, has the form

$$\begin{pmatrix} E & M \\ O & N \end{pmatrix} \in \mathrm{SL}(n, \mathbb{Z}). \tag{8}$$

Here the $(E, O)^t$ block corresponds to Eq. (7), that is, E is the $k \times k$ identity matrix and O is the $(n-k) \times k$ zero matrix, and the matrices $M = (m_{lj})$, $N = (n_{ji})$ are unknown.

Consider a smooth section of the bundle $(F^{-1}(\gamma), \gamma, F)$. Let $(\varphi_1, \ldots, \varphi_n)$ be angle coordinates on $F^{-1}(-\varepsilon, \pi + \varepsilon)$ corresponding to this section and the family $(e_1, \ldots, e_n)$. Analogously define $(\varphi'_1, \ldots, \varphi'_n)$. Since $e_1 = e'_1, \ldots, e_n = e'_n$ when $\chi = 0$, we also have

$$\varphi_1 = \varphi'_1, \ldots, \varphi_n = \varphi'_n \quad \text{when } \chi = 0.$$

From (8) it follows that

$$\varphi_l = \varphi'_l + \sum_{i=1}^{n-k} m_{li}\varphi'_{i+k} \quad \text{and} \quad \varphi_{k+j} = \sum_{i=1}^{n-k} n_{ji}\varphi'_{k+i}, \tag{9}$$

when $l \le k$, $j \le n-k$ and $\chi = \pi$.

Let $U_1 = F^{-1}(-\varepsilon, \pi + \varepsilon)/\mathbb{T}^k$ and $U_2 = F^{-1}(\pi - \varepsilon, 2\pi + \varepsilon)/\mathbb{T}^k$. Define a section $s_1 \colon U_1 \to F^{-1}(-\varepsilon, \pi + \varepsilon)$ by the following rule:

$$(\chi, \varphi_{k+1}, \ldots, \varphi_n) \mapsto \left(\chi, \sum_{j=1}^{n-k} \varphi_{k+j} e_{k+j}\right).$$

Analogously, define a section $s_2 \colon U_2 \to F^{-1}(\pi - \varepsilon, 2\pi + \varepsilon)$ given by

$$(\chi, \varphi'_{k+1}, \ldots, \varphi'_n) \mapsto \left(\chi, \sum_{j=1}^{n-k} \varphi'_{k+j} e'_{k+j}\right).$$

We will need the following lemma.

Lemma 5.1. *Let ω be a connection one-form on $(E_\gamma, E_\gamma/\mathbb{T}^k, \rho)$. Define a loop $\gamma'_{k+j} \subset F^{-1}(\pi)/\mathbb{T}^k$ by setting $\varphi'_{k+i} = 0$, $i \neq j$. Then the columns $\vec{m}_j$ of the matrix M satisfy*

$$\vec{m}_j = \frac{i}{2\pi} \int_{\gamma'_{k+j}} s_1^\star(\omega) - s_2^\star(\omega).$$

Proof. On one hand, recall that on $F^{-1}(\pi)$ we have

$$e'_{k+j} = \sum_{i=1}^{k} m_{ij} e_i + \sum_{i=1}^{n-k} n_{ij} e_{k+i}.$$

Therefore,

$$s_2|_{\gamma'_{k+j}} = \varphi'_{k+j} e'_{k+j} = \sum_{i=1}^{k} m_{ij} \varphi'_{k+j} e_i + \sum_{i=1}^{n-k} n_{ij} \varphi'_{k+j} e_{k+i}.$$

On the other hand, Eq. (9) implies

$$s_1|_{\gamma'_{k+j}} = \sum_{i=1}^{n-k} \varphi_{k+i} e_{k+i} = \sum_{i=1}^{n-k} n_{ij} \varphi'_{k+j} e_{k+i}.$$

Thus

$$s_2|_{\gamma'_{k+j}} = s_1|_{\gamma'_{k+j}} + \sum_{i=1}^{k} m_{ij} \varphi'_{k+j} e_i.$$

We see that the cocycle $t_{12} \colon U_1 \cap U_2 \to \mathbb{T}^k$ corresponding to the sections s_1 and s_2 is given by the vector

$$\left(\exp(i m_{1j} \varphi'_{k+j}), \ldots, \exp(i m_{kj} \varphi'_{k+j})\right) = \exp(i \vec{m}_j \varphi'_{k+j}).$$

Hence, the compatibility condition

$$s_2^\star(\omega) = s_1^\star(\omega) + t_{12}^{-1} dt_{12} = s_1^\star(\omega) + i \vec{m}_j \, d\varphi'_{k+j}$$

implies

$$\frac{i}{2\pi} \int_{\gamma'_{k+j}} s_1^\star(\omega) - s_2^\star(\omega) = \frac{\vec{m}_j}{2\pi} \int_{\gamma'_{k+j}} d\varphi'_{k+j} = \vec{m}_j. \quad \square$$

Analogous to the loop $\gamma'_{k+j} \subset F^{-1}(\pi)/\mathbb{T}^k$, we have a loop $\gamma_{k+j} \subset F^{-1}(\pi)/\mathbb{T}^k$ defined by $\varphi_{k+i} = 0$, $i \neq j$.

Remark 5.2. Suppose that $\gamma'_{k+j} = \gamma_{k+j}$. Then the generator e_{k+j} is globally defined on the quotient $E_\gamma/\mathbb{T}^k$. It spans a 2-torus $T_j^2 \subset E_\gamma/\mathbb{T}^k$. We can thus form the principal $\mathbb{T}^k$ bundle $(\rho^{-1}(T_j^2), T_j^2, \rho)$ with a curvature form $\mathfrak{F}$.

Theorem 5.3. *Assume that $\gamma_{k+j} = \gamma'_{k+j}$ for some $j \leq n-k$. Let T_j^2 and $\mathfrak{F}$ be as in Remark 5.2. Then the column $\vec{m}_j$ of the matrix M satisfies*

$$\vec{m}_j = \frac{i}{2\pi} \int_{T_j^2} \mathfrak{F}.$$

Proof. Consider the cylinders $C_1 = [0, \pi] \times \gamma_{k+j}$ and $C_2 = [\pi, 2\pi] \times \gamma_{k+j}$. Lemma 5.1 implies

$$\vec{m}_j = \frac{i}{2\pi} \int_{\partial C_1} s_1^\star(\omega) - s_2^\star(\omega) = \frac{i}{2\pi} \int_{\partial C_1} s_1^\star(\omega) + \frac{i}{2\pi} \int_{\partial C_2} s_2^\star(\omega).$$

Therefore

$$\vec{m}_j = \frac{i}{2\pi} \int_{C_1} ds_1^\star(\omega) + \frac{i}{2\pi} \int_{C_2} ds_2^\star(\omega) = \frac{i}{2\pi} \int_{T_j^2} \mathfrak{F}. \quad \square$$

We now observe that Theorem 3.2 follows directly from Theorem 5.3. Indeed, in Theorem 3.2 we consider a Hamiltonian $\mathbb{T}^{n-1}$ action, that is $k = n-1$. The quotient $E_\gamma/\mathbb{T}^{n-1}$ is a 2-torus since it is an orientable circle bundle over the curve γ. It follows that $\gamma_n = \gamma'_n$ and $T_1^2 = E_\gamma/\mathbb{T}^{n-1}$. This completes the proof of Theorem 3.2.

6. Discussion

In this paper we considered n degree of freedom integrable Hamiltonian systems $F : M \to \mathbb{R}^n$ with a $\mathbb{T}^{n-1}$ action. We studied the monodromy of n-torus bundles over closed paths γ in the set of regular values of F. We have shown that if we consider the pre-image $F^{-1}(U)$ of a 2-disk U bounded by γ and the only singular orbits of the $\mathbb{T}^{n-1}$ action are orbits with $\mathbb{S}^1$ isotropy then the latter completely determine the monodromy of the n-torus bundle. Suppose, in particular, that the manifold M and the $\mathbb{T}^{n-1}$ action are fixed and write F in the form $(J_1, \ldots, J_{n-1}, H)$, where $J_1, \ldots, J_{n-1}$ are momenta for the $\mathbb{T}^{n-1}$ action and H is the $\mathbb{T}^{n-1}$ invariant energy function which can be varied. Then our results imply that the only way in which H affects monodromy is by determining whether there exist paths γ encircling critical values of F corresponding to singular $\mathbb{T}^{n-1}$ orbits with $\mathbb{S}^1$ isotropy; if such paths exist then the monodromy for the n-torus bundle over γ depends only on these singular orbits of the $\mathbb{T}^{n-1}$ action.

This point of view is different than the one usually adopted in studies of Hamiltonian monodromy which have until now focused on the behavior near focus–focus singularities or near families of hyperbolic orbits, that is, near singularities of the map F. In such cases one starts with a singularity of F, proves the existence of a $\mathbb{T}^{n-1}$ action in a saturated neighborhood of such singularity, and uses this to prove monodromy for nearby n-torus bundles. In our approach we assume that a $\mathbb{T}^{n-1}$ action exists and that F defines n-torus bundles near singular orbits of the $\mathbb{T}^{n-1}$ action with $\mathbb{T}^1$ isotropy. Then we prove the non-triviality of the monodromy without having to analyze what type of singularity of F we have.

In a forthcoming paper we provide a natural extension of this point of view to systems with fractional monodromy. Such an extension would also be a natural continuation of the results in [26] where fractional monodromy was proved for specific $\mathbb{S}^1$ actions in 2 degree of freedom systems but with weak requirements on the form of the energy function H.

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References

[1] J.J. Duistermaat, On global action-angle coordinates, Comm. Pure Appl. Math. 33 (6) (1980) 687–706.
[2] R.H. Cushman, L.M. Bates, Global Aspects of Classical Integrable Systems, second ed., Birkhäuser, 2015.
[3] R.H. Cushman, H. Knörrer, The energy momentum mapping of the Lagrange top, in: Differential Geometric Methods in Mathematical Physics, in: Lecture Notes in Mathematics, vol. 1139, Springer, 1985, pp. 12–24.
[4] J.J. Duistermaat, The monodromy in the Hamiltonian Hopf bifurcation, Z. Angew. Math. Phys. 49 (1) (1998) 156.
[5] L.M. Bates, Monodromy in the champagne bottle, J. Appl. Math. Phys. 42 (6) (1991) 837–847.
[6] D.A. Sadovskií, B.I. Zhilinskií, Monodromy, diabolic points, and angular momentum coupling, Phys. Lett. A 256 (4) (1999) 235–244.
[7] H. Waalkens, H.R. Dullin, P.H. Richter, The problem of two fixed centers: bifurcations, actions, monodromy, Physica D 196 (3–4) (2004) 265–310.
[8] L.M. Bates, M. Zou, Degeneration of Hamiltonian monodromy cycles, Nonlinearity 6 (2) (1993) 313–335.
[9] K. Efstathiou, Metamorphoses of Hamiltonian Systems with Symmetries, Springer, Berlin Heidelberg New York, 2005.
[10] T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, Bulletin de la S. M. F. 116 (3) (1988) 315–339.
[11] Y. Karshon, S. Tolman, Centered complexity one Hamiltonian torus actions, Trans. Amer. Math. Soc. 353 (2001) 4831–4861.
[12] L.M. Lerman, Ya. L. Umanskiĭ, Classification of four-dimensional integrable Hamiltonian systems and Poisson actions of $\mathbb{R}^2$ in extended neighborhoods of simple singular points. I, Russian Academy of Sciences. Sb. Math. 77 (2) (1994) 511–542.
[13] V.S. Matveev, Integrable Hamiltonian system with two degrees of freedom. The topological structure of saturated neighbourhoods of points of focus-focus and saddle-saddle type, Sb. Math. 187 (4) (1996) 495–524.
[14] N.T. Zung, A note on focus-focus singularities, Differential Geom. Appl. 7 (2) (1997) 123–130.
[15] S. Vũ Ngọc, Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type, Comm. Pure Appl. Math. 53 (2) (2000) 143–217.
[16] M. Audin, Hamiltonian monodromy via Picard-Lefschetz theory, Comm. Math. Phys. 229 (3) (2002) 459–489.
[17] D. Sugny, P. Mardešić, M. Pelletier, A. Jebrane, H.R. Jauslin, Fractional Hamiltonian monodromy from a Gauss–Manin monodromy, J. Math. Phys. 49 (4) (2008) 042701.
[18] M.M. Postnikov, Differential geometry IV, MIR, 1982.
[19] M. Audin, Torus Actions on Symplectic Manifolds, Birkhäuser, 2004.
[20] S. Bochner, Compact groups of differentiable transformations, Ann. of Math. 46 (3) (1945) 372–381.
[21] H.K. Urbantke, The Hopf fibration—seven times in physics, J. Geom. Phys. 46 (2) (2003) 125–150.
[22] N.T. Zung, Another note on focus-focus singularities, Lett. Math. Phys. 60 (1) (2002) 87–99.
[23] R.H. Cushman, J.J. Duistermaat, Non-Hamiltonian monodromy, J. Differential Equations 172 (1) (2001) 42–58.
[24] K. Efstathiou, A. Giacobbe, The topology associated to cusp singular points, Nonlinearity 25 (12) (2012) 3409–3422.
[25] K. Efstathiou, H. Hanßmann, A. Marchesiello, Bifurcations and monodromy of the axially symmetric 1:1: − 2 resonance, in preparation.
[26] K. Efstathiou, H.W. Broer, Uncovering fractional monodromy, Comm. Math. Phys. 324 (2) (2013) 549–588.