



Special issue in memory of Hans Duistermaat

Recent advances in the monodromy theory of integrable Hamiltonian systems

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Abstract

The notion of monodromy was introduced by J.J. Duistermaat as the first obstruction to the existence of global action coordinates in integrable Hamiltonian systems. This invariant was extensively studied since then and was shown to be non-trivial in various concrete examples of finite-dimensional integrable systems. The goal of the present paper is to give a brief overview of monodromy and discuss some of its generalizations. In particular, we will discuss the monodromy around a focus–focus singularity and the notions of quantum, fractional and scattering monodromy. The exposition will be complemented with a number of examples and open problems.

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1. Introduction

In the context of finite-dimensional integrable Hamiltonian systems, the notion of monodromy was introduced by Duistermaat in his seminal paper [34] published in 1980. He defined his notion of monodromy as the (usual) monodromy of a certain covering map that can naturally be defined for a given integrable system. To be more specific, assume that we are given n independent functions in involution (F_1, \dots, F_n) on a symplectic manifold M of real dimension

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$2n$.¹ These functions give rise to the so-called *integral* or *momentum map*

$$F = (F_1, \dots, F_n): M \rightarrow \mathbb{R}^n$$

and the (defined on an open subset $U \subset \mathbb{R}^n \times M$) action

$$G: U \subset \mathbb{R}^n \times M \rightarrow M, \quad G(t_1, \dots, t_n)(x) = g_1^{t_1} \dots g_n^{t_n}(x),$$

where g_i^t is the Hamiltonian flow associated to F_i . Observe that the action G leaves the fibers $F^{-1}(f) \subset M$ of F invariant since the functions F_1, \dots, F_n are in involution.

For simplicity, we shall for the moment consider the case when all of the fibers $F^{-1}(f)$ are compact and connected. Then the action G is a global \mathbb{R}^n action on M . Moreover, for each regular value f in the image of F , the isotropy group G_f is an n -dimensional lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. In particular, regular fibers $F^{-1}(f)$ are n -dimensional tori; see Arnol’d–Liouville theorem [2,3,65] for detail. The collection of the lattices G_f , with f in the set $R \subset \text{image}(F)$ of the regular values of F , is a subset of $\mathbb{R}^n \times R$. The natural projection $\text{Pr}: \mathbb{R}^n \times R \rightarrow R$ gives rise to the covering map

$$\text{Pr}: \bigcup_{f \in R} G_f \rightarrow R. \tag{1}$$

This is the covering that we mentioned above. In the paper [34], the monodromy of the torus fibration $F: F^{-1}(R) \rightarrow R$ was defined as the (usual) monodromy of the covering (1), that is, as a representation of the fundamental group $\pi_1(R, f_0)$ in the group of automorphisms of $G_{f_0} \equiv \mathbb{Z}^n$ (the representation is given by lifting paths from $\pi_1(R, f_0)$ to the total space of the covering (1)).

We note that Duistermaat’s original definition included the case of Lagrangian torus fibrations over an arbitrary manifold (not necessarily an open subset of \mathbb{R}^n). We will not pursue this generality here.

Since Duistermaat’s work [34], non-trivial monodromy was found in various concrete integrable systems of physics and classical mechanics. The first such example is the spherical pendulum, which is an integrable system that describes the motion of a particle on the unit sphere in \mathbb{R}^3 in the linear gravitational potential.² The monodromy of the spherical pendulum was observed to be non-trivial by R. Cushman and computed by J. J. Duistermaat in the same paper [34]. It turned out that $\pi_1(R, f_0)$ is isomorphic to \mathbb{Z} in this case (see Fig. 1) and that the monodromy is given by the matrix

$$M_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Here γ corresponds to the generator of the group $\pi_1(R, f_0) \equiv \mathbb{Z}$. We shall return to this example and to the computation of the monodromy matrix later in this paper.

Another example, which is probably the simplest one, is the so-called champagne bottle system (a particle in a Mexican hat potential). For this system, the monodromy was computed by L. Bates in [6]. It turns out that also in this case, the fundamental group $\pi_1(R, f_0)$ is isomorphic to \mathbb{Z} and the corresponding monodromy matrix is given by Eq. (2).

¹ We recall that an integrable Hamiltonian system on a symplectic $2n$ -manifold M is specified by n independent functions in involution F_1, \dots, F_n . Typically, $F_1 = H$ is the Hamiltonian of the system and F_2, \dots, F_n are additional first integrals.

² For this system, the functions $F_1 = H$ and $F_2 = J$ are the restrictions of the functions $H = \frac{1}{2}\|p\|^2 + q_3$ and $J = q_1 p_2 - q_2 p_1$, defined on $T^*\mathbb{R}^3$, to $T^*S^2 \subset T^*\mathbb{R}^3$.

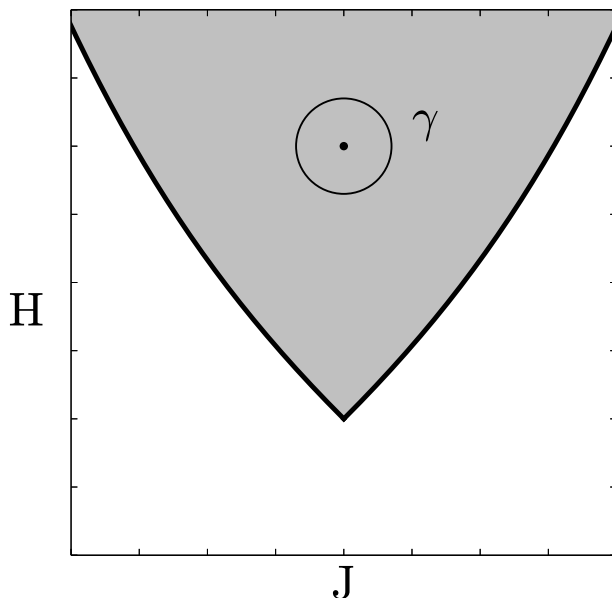


Fig. 1. Bifurcation diagram for the spherical pendulum and the generator γ of $\pi_1(R)$.

Several other examples of integrable Hamiltonian systems with non-trivial monodromy are the quadratic spherical pendulum [8,25,34,38], the coupled angular momenta [83], the Lagrange top [28], the Hamiltonian Hopf bifurcation [35], the Jaynes–Cummings model [36,54,79], the hydrogen atom in crossed fields [29], and the Euler two-center problem [66,97]. We note that monodromy can naturally be generalized to integrable non-Hamiltonian systems [27,104]; see also [14] for a discussion of monodromy in the context of the Hamiltonisation problem. This invariant can also be extended to the setting of nearly-integrable systems [18,20,81], which is relevant for applications since real physical systems are seldom integrable.

It was later understood that most of the known examples of integrable systems with non-trivial monodromy have one common property, namely, the existence of so-called *focus–focus points*. For instance, in the case of the spherical pendulum, this is the unstable equilibrium when the pendulum is at the top of the sphere. In the case of the Mexican hat potential, this is the unstable equilibrium when the particle is on the ‘top of the hat’. The precise result, which is sometimes referred to as the *geometric monodromy theorem*, was obtained first by L. M. Lerman and Ya. L. Umanskiĭ [63] in the case of a single focus–focus point and later by V. S. Matveev [72] and N. T. Zung [103] in the case of arbitrary many focus–focus points on a singular focus–focus fiber. We note that outside the context of integrable Hamiltonian system, this result was already obtained by Y. Matsumoto in [71]. We also note that in the context of complex geometry, the geometric monodromy theorem follows from the Picard–Lefschetz theory; see [5,11,103] for details. We shall come back to case of focus–focus singularities later in this paper, in connection with the *classical* Morse theory and principal circle bundles; this is the content of the recent topological theory of monodromy developed in [66].

Another breakthrough in the monodromy theory was the quantum formulation of this invariant; first, for the quantum spherical pendulum [26,50] and later, in more generality, by S. Vũ Ngọc [92]. The main idea is that in a quantum integrable system, the joint spectrum

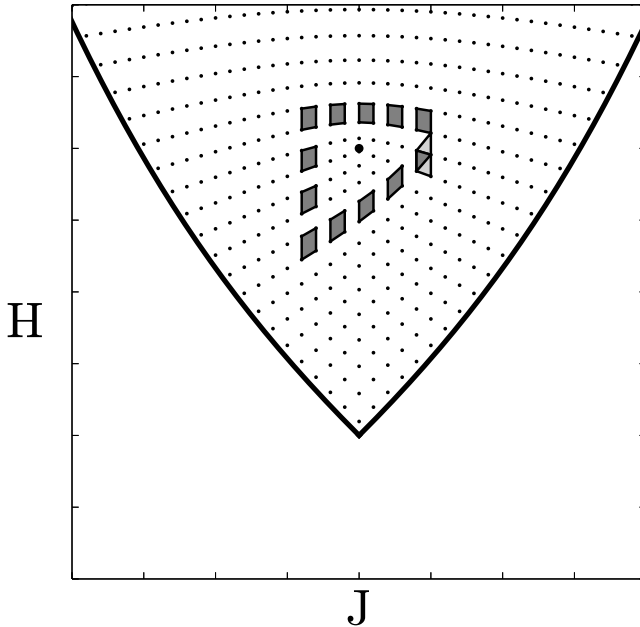


Fig. 2. The joint spectrum of the quantum spherical pendulum ($\hbar = 0.1$), and the transport of an elementary cell around the focus–focus point.

of the commuting operators locally has the form of a lattice. Globally, this does not have to be the case, and one can observe a lattice defect in the joint spectrum when transporting an elementary cell around a singularity; see Fig. 2. This lattice defect is usually interpreted as the non-existence of smooth global quantum number assignment for a given quantum integrable system. We note that this is very similar to what happens classically when one looks at the action coordinates and the so-called integer affine structure [103]. We also note that quantum monodromy is determined by the classical monodromy of the underlying classical integrable Hamiltonian system [92].

This is, in short, what is classically known about monodromy. More recently, several generalized versions of monodromy have been defined. The most important and general of these are the so-called fractional and scattering monodromies as well as their quantum analogs. The notion of fractional monodromy was introduced in the paper [77] as a generalization of the usual Duistermaat’s monodromy (sometimes referred to as Hamiltonian monodromy) to the case of singular fibrations; it naturally appears in integrable systems with hyperbolic singularities. Scattering monodromy appears in completely integrable systems with non-compact invariant manifolds; it was originally defined by L. Bates and R. Cushman in [7] for a two degree of freedom hyperbolic oscillator and later generalized in the works [37,41] and [68].

The main goal of the present paper is to give a concise and systematic overview of the monodromy theory and of some of the recent developments in this field. Our main focus will be on the classical notion of monodromy and some of the generalized versions of this invariant. We will complement our exposition with various concrete examples and formulate a few open problems. For a more thorough exposition of the state of the art of the monodromy theory

and integrable systems, we refer the reader to [11,15,25,66,85,100]. Several parts of this work appeared in a more extended form in [66].

2. Preliminaries on Hamiltonian monodromy

The notion of Hamiltonian monodromy³ was originally introduced as the first obstruction to the existence of global action–angle coordinates in integrable systems [34]. We briefly review a construction of these coordinates here and explain the relation to the definition of Hamiltonian monodromy given in the Introduction. Then we discuss a connection of Hamiltonian monodromy to Picard–Lefschetz theory, the latter being a very classical situation in which monodromy of non-singular hypersurfaces appears. The discussion continues in the next section, where we review the classical theorem which describes the monodromy around a focus–focus singularity and discuss several more recent results.

2.1. Liouville integrability, action–angle coordinates and monodromy

We recall that a Hamiltonian system

$$\dot{x} = X_H, \quad \omega(X_H, \cdot) = -dH,$$

on a $2n$ -dimensional symplectic manifold (M, ω) is called Liouville integrable if there exist almost everywhere independent functions $F_1 = H, \dots, F_n$ that are in *involution* with respect to the symplectic form ω :

$$\{F_i, F_j\} = \omega(X_{F_i}, X_{F_j}) = 0.$$

We note that by definition, for each i and j , the function F_i is invariant with respect to the Hamiltonian flow of F_j ; in particular, the functions F_i are first integrals of the flow of X_H . Various Hamiltonian systems, such as the Kepler problem, the spherical pendulum, the geodesic flow on an ellipsoid, Euler, Lagrange and Kovalevskaya tops, the Calogero–Moser systems, are integrable in this sense.

The map $F = (F_1, \dots, F_n)$ consisting of the integrals F_i is called the *integral map* (or the *energy–momentum map*) of the integrable system. It encodes both the dynamics ($F_1 = H$) and the symmetry associated to the system. A central problem in the theory of integrable systems is to understand the geometry of such integral maps; in other words, to classify them up to a topological, smooth or symplectic equivalence.

It is well-known that, in the case when the map F is proper, any regular fiber $F^{-1}(\xi_0)$ is an n -dimensional torus (or a union of several n -tori). Moreover, a small tubular neighborhood of any such torus is a trivial torus bundle $D^n \times T^n$ admitting *action–angle coordinates*

$$I \in D^n \text{ and } \varphi \text{ mod } 2\pi \in T^n, \quad \omega = dI \wedge d\varphi.$$

This is the content of the Arnol’d–Liouville theorem [2,3,65]. It follows from the existence of action–angle coordinates that the motion (that is, the flow of X_H) is quasi-periodic on each torus $\{\xi\} \times T^n$.

The above coordinates are sometimes referred to as semi-local since they exist in a neighborhood of a given invariant torus. The global situation (of when do such coordinates

³ Duistermaat’s notion of monodromy is usually referred to as Hamiltonian monodromy to distinguish it from other types of monodromy, such as fractional monodromy or monodromy of a covering map.

exist globally) was clarified by Nekhoroshev [76] and Duistermaat [34]. We briefly review a few main results of these works below.

Let $R \subset \text{image}(F)$ denote the set of the regular values of F that are in the image of F . Assume for the moment that all of the fibers $F^{-1}(f)$ are compact and connected. Then global action–angle coordinates exist if the following two conditions are satisfied (see [76]):

$$\pi_1(R, f_0) = 0 \quad \text{and} \quad H^2(R, \mathbb{R}) = 0.$$

Otherwise, the torus bundle $F: F^{-1}(R) \rightarrow R$ is not necessarily globally trivial, and certain obstructions to the triviality of this bundle appear; see [34]. One of such obstructions is monodromy, which we have briefly discussed in the introduction. It is an obstruction in the sense that its non-triviality entails the non-existence of global action coordinates. To see this, let us assume for simplicity that the symplectic form ω is exact: $\omega = d\eta$. Then the action coordinates $I = (I_1, \dots, I_n)$ can be defined by the formula

$$I_i = \frac{1}{2\pi} \int I_i d\varphi_i = \frac{1}{2\pi} \int_{\alpha_i} I d\varphi = \frac{1}{2\pi} \int_{\alpha_i} \eta + c_i,$$

where α_i is the φ_i -cycle on the corresponding Liouville torus $F^{-1}(f)$ and c_i does not depend on f . The cycles $\alpha_1, \dots, \alpha_n$ form a basis of the first integer homology group of $F^{-1}(f)$. But this homology group can be identified with the isotropy group G_f of the global \mathbb{R}^n action on $F^{-1}(f)$; see Introduction (Section 1). Thus, the non-triviality of monodromy of the covering, Eq. (1), formed by the lattices G_f implies that it is not possible to choose the cycles $\alpha_1, \dots, \alpha_n$ in a continuous way over R : transports of these homology cycles along different paths do not give the same result. In particular, it is not possible to choose the action coordinates in a globally smooth way: transports along different paths result in different sets of action coordinates I and I' related by a transformation $I = MI'$, where $M \in SL(n, \mathbb{Z})$. After excursions along elements of $\pi_1(R, f_0)$, we get the monodromy automorphisms, described in the Introduction.

2.2. Picard–Lefschetz theory

In the context of fibrations by complex tori, the notion of Hamiltonian monodromy is essentially the classical monodromy that appears in Picard–Lefschetz theory.

Let \mathbb{C}^2 be the complex two-plane with complex coordinates (z, w) . Following [11], consider the symplectic transformation

$$A(z, w) \rightarrow (w^{-1}, zw^2) \tag{3}$$

(defined for $w \neq 0$). Let the compact manifold M be defined by gluing the boundary solid tori of

$$U_1 = \{(z, w) \in \mathbb{C}^2 \mid |zw| \leq \varepsilon, |z| \leq 1, |w| \leq 1\} \tag{4}$$

using this transformation. (The boundary solid tori of U_1 are given by the sets $\{(z, w) \in U_1 \mid |z| = 1\}$ and $\{(z, w) \in U_1 \mid |w| = 1\}$.) Observe that the function $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ defined by

$$f(z, w) = zw$$

descends to a smooth function on this manifold. It has one critical fiber: the preimage of the origin in \mathbb{C} . All of the other fibers are regular two-tori. Let γ be a small circle in \mathbb{C} around the

origin. According to the Picard–Lefschetz formula [4], the monodromy of f along γ is given by the matrix

$$M_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now observe that the holomorphic function f can be viewed as an energy–momentum map of a real integrable Hamiltonian system on M : the functions in involution are given by the real and imaginary part of the function f ; see [46]. By a topological definition of Hamiltonian monodromy in terms of homology cycles, this matrix is the monodromy matrix along γ associated to this integrable system.

For the above argument, it is important that the phase space is a complex manifold and that f is a holomorphic (meromorphic) function on this manifold. We note that in a general situation, an integrable Hamiltonian system is only defined on a real symplectic manifold and, even if the manifold can be endowed with a complex structure, the integrals of motion are not always meromorphic functions. Therefore, the Picard–Lefschetz formula is not always applicable; at least, not directly. Nonetheless, in various examples of integrable systems the integrals of motions are polynomials and it is possible to complexify them. Then one can use the Picard–Lefschetz theory in the complexified domain and deduce information about monodromy in the original system. We refer to [5,9,88] for more information.

3. Hamiltonian monodromy

In this section, we continue our discussion of Hamiltonian monodromy. We review the *geometric monodromy theorem*, which describes the monodromy around a focus–focus singularity. This central result in monodromy theory allows one to compute monodromy in various concrete integrable systems by computing the complexity of the focus–focus fibers of such systems. We then explain a dynamical manifestation of non-trivial Hamiltonian monodromy. Afterwards, we come back to the spherical pendulum and discuss the monodromy from a different point of view based on Morse theory and Chern numbers (a general situation is treated in the work [67]). We conclude this section with an extension of Hamiltonian monodromy to nearly integrable systems.

3.1. Monodromy around a focus–focus singularity

Hamiltonian monodromy was first observed to be non-trivial in concrete integrable systems of classical mechanics and molecular physics. It was later observed that in the typical case of $n = 2$ degrees of freedom, non-trivial monodromy is manifested by the presence of the so-called *focus–focus* points of the integral fibration F ; see [63,72,103]. (The singular point $z = w = 0$ of the function $f = zw$ from Section 2.2 is an example of a focus–focus point.) Such a result is often referred to as the *geometric monodromy theorem*. Below we discuss a few different approaches to this theorem.

First, let us recall the notion of the focus–focus singularity.

Definition 3.1. Consider a two-degree of freedom integrable system $F = (H, J): M \rightarrow \mathbb{R}^2$ on a 4-manifold M . Let x_0 be a rank zero singular point of F , that is, $dF_{x_0} = 0$. The point x_0 is called a focus–focus point of $F = (H, J)$ if the Hessians $d^2_{x_0} H$ and $d^2_{x_0} J$ are independent

and there exist local canonical coordinates near x_0 such that

$$\begin{aligned} d_{x_0}^2 H &= A_1(dp_1dq_1 + dp_2dq_2) + B_1(dp_1dq_2 - dp_2dq_1) \\ d_{x_0}^2 J &= A_2(dp_1dq_1 + dp_2dq_2) + B_2(dp_1dq_2 - dp_2dq_1). \end{aligned}$$

Remark 3.2. The focus–focus singularity is an example of a non-degenerate singularity of an integrable system. Alongside focus–focus points, there are also other types of non-degenerate singular points of integrable two-degrees of freedom systems: elliptic–elliptic, hyperbolic–hyperbolic, elliptic–regular, etc.; see [11].

Remark 3.3. We note that by Eliasson’s theorem [44,45,95], not only the quadratic parts of H and J , but also the map $F = (H, J)$ itself can be put into a normal form near a singular focus–focus point x_0 : there exist local canonical coordinates near x_0 such that

$$\begin{aligned} H &= H(p_1q_1 + p_2q_2, p_1q_2 - p_2q_1) \\ J &= J(p_1q_1 + p_2q_2, p_1q_2 - p_2q_1). \end{aligned}$$

In particular, in a neighborhood of a focus–focus point x_0 , the (singular) fibration induced by F is locally the same (up to a regular change of coordinates on the base of this fibration) as the fibration induced by the function $\tilde{F} = (Re(zw), Im(zw)): \mathbb{C}^2 \rightarrow \mathbb{R}^2$ near the origin.

We note that Eliasson’s theorem describes the local symplectic normal form also of other types of non-degenerate singularities; see [44,45] for details.

Assume that we are given a proper integral map F with an isolated critical value f_0 such that the singular fiber $F^{-1}(f_0)$ contains a (finite) number m of focus–focus points. Note that in this case, the singular fiber $F^{-1}(f_0)$ is homeomorphic to a 2-torus with m “pinches”, that is, a 2-torus, where m parallel homology cycles are shrunk to a single point each; see [11, Lemma 9.7]. (In the case when $m = 1$, this is also a sphere with two points identified). According to the geometric monodromy theorem, the monodromy of F around f_0 is completely determined by the topology of the singular fiber $F^{-1}(f_0)$ and is essentially given by the number of “pinches” (the focus–focus points).

Theorem 3.4 (*Geometric Monodromy Theorem, [63,71,72,92,103]*). *Monodromy around a focus–focus singularity is given by the matrix*

$$M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

where m is the number of the focus–focus points on the singular fiber.

One way to prove this theorem is to prove that the number m of the focus–focus points on a singular focus–focus fiber $F^{-1}(f_0)$ (also called the *complexity* of this fiber) is a complete topological invariant of the Liouville fibration in a tubular neighborhood of this fiber $F^{-1}(f_0)$; see [72,103]. The monodromy is a particular invariant of this fibration, and is thus a function of the number m of the focus–focus points. To prove the geometric monodromy theorem, it is sufficient to prove the statement for a particular example of an integrable system with m focus–focus points. The rest follows from Picard–Lefschetz theory; cf. Section 2.2. We refer to [103] for details.

Remark 3.5. We have noted above that the complexity is a complete semi-local topological invariant of a focus–focus singularity; see [11,103]. This is not the case symplectically: there

exist infinitely many (semi-locally) non-symplectomorphic Lagrangian fibrations even in the case of complexity $m = 1$; see [94]. We note that a similar result does not hold even in the smooth category: there exist smoothly non-equivalent Lagrangian fibrations in the case of $m \geq 2$ focus–focus points on a given focus–focus fiber; see the works [11,12,52] for details.

Remark 3.6. We note that in concrete problems of physics and classical mechanics, the complexity of focus–focus fibers is usually small. This can be proven rigorously in many cases in terms of the topology of the underlying symplectic manifold [86]. For instance, in \mathbb{R}^4 one can only have complexity $m = 1$ focus–focus fibers (\mathbb{R}^4 does not contain Lagrangian spheres [15]). For integrable systems on T^*S^2 , one can have complexity $m = 1$ or $m = 2$, but not 3 or more. We refer to the work [86] for details.

Remark 3.7. For a generalization of the geometric monodromy theorem to the case of integrable systems with many degrees of freedom, we refer the reader to [51,104].

A related result in the context of the focus–focus singularities is that they come with a Hamiltonian circle action [103,104].

Theorem 3.8 (Circle Action Near Focus-Focus, [103,104]). *In a neighborhood of a singular focus–focus fiber, there exists a unique (up to orientation) Hamiltonian circle action which is free outside the singular focus–focus points. Near each focus–focus point, the momentum of the circle action can be written as*

$$J = \frac{1}{2}(q_1^2 + p_1^2) - \frac{1}{2}(q_2^2 + p_2^2)$$

for some local canonical coordinates (q_1, p_1, q_2, p_2) . In particular, the circle action defines the anti-Hopf fibration on every sufficiently small 3-sphere $S_\varepsilon^3 = \{q_1^2 + p_1^2 + q_2^2 + p_2^2 = \varepsilon\}$ around each focus–focus point.

One implication of Theorem 3.8 is that it allows one to give a different proof of the geometric monodromy theorem by looking at the circle action. For example, one can apply the Duistermaat–Heckman theorem; see [104]. A related and purely topological proof will be given below on the example of the spherical pendulum, following the point of view of [43,66,67,69]. For other approaches to the geometric monodromy theorem, we refer the reader to [5,25,41,93].

3.2. Dynamical manifestation of monodromy

In this subsection we briefly comment on implications of non-trivial monodromy for dynamics. More specifically, we make a connection to the so-called *rotation number* [25, Section IV.4].

We assume that the energy–momentum map $F = (H, J)$ is such that all of the fibers $F^{-1}(f)$ are compact and connected. Moreover, we assume that F is invariant under the Hamiltonian circle action given by the Hamiltonian flow φ_J^t of J . Let $F^{-1}(f)$ be a regular torus. Consider a point $x \in F^{-1}(f)$ and the orbit of the circle action passing through this point. The trajectory $\varphi_H^t(x)$ leaves the orbit of the circle action at $t = 0$ and then returns back to the same orbit at some time $T > 0$. The time T is called the *the first return time*. The *rotation number* $\Theta = \Theta(f)$ is defined by $\varphi_J^{2\pi\Theta}(x) = \varphi_H^T(x)$. With this notation, there is the following result.

Theorem 3.9 (*Monodromy and Rotation Number, [25]*). *The Hamiltonian monodromy of the torus bundle $F: F^{-1}(\gamma) \rightarrow \gamma$ is given by*

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),$$

where $-m = \int_{\gamma} d\Theta$ is the variation of the rotation number Θ .

We note that this theorem can be used as a powerful analytic tool for the computation of monodromy in specific examples of integrable systems with a circle action. We refer to [25,41,93] for details. For another dynamical manifestation of monodromy, see [32].

3.3. The spherical pendulum

We now come back to the case of the spherical pendulum and prove that the monodromy matrix of this system is given by Eq. (2). We shall mainly focus on a topological idea which goes back to R. Cushman and F. Takens and which has been developed in the work [67], see also [43,69].

We recall that the spherical pendulum is a mechanical Hamiltonian system that describes the motion of a particle moving on the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

in the linear gravitational potential $V(x, y, z) = z$. The phase space is T^*S^2 with the standard symplectic structure. The Hamiltonian is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$$

the total energy of the pendulum. Since (the component of) the angular momentum $J = xp_y - yp_x$ is conserved, the system is Liouville integrable. The bifurcation diagram of the energy–momentum map

$$F = (H, J): T^*S^2 \rightarrow \mathbb{R}^2,$$

that is, the set of the critical values of this map, is shown in Fig. 1.

Consider the closed path γ around the isolated critical value; see Fig. 1. It was shown by Duistermaat in [34] using an analytic argument that the monodromy along γ is given by the matrix

$$M_{\gamma} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

Remark 3.10. Duistermaat’s proof is based on the computation of the action coordinates. To be more specific, observe that for the spherical pendulum, there are ‘natural’ actions coming from the separation of the system in spherical coordinates. One of these actions is simply given by the function J ; it is globally defined on the phase space T^*S^2 . The other one is an elliptic integral. One can deduce the monodromy from the derivatives of the second action when J approaches zero; see [34] for details. We note that this kind of approach can be used more generally; it reduces the computation of monodromy to studying certain limits of elliptic integrals.

We note that the above result can directly be obtained from the geometric monodromy theorem, [Theorem 3.4](#). Indeed, it can be shown that the isolated critical value is a focus–focus singularity of complexity 1 (there is one and only one unstable equilibrium of the pendulum).

Below, following the work [\[67\]](#), we shall give a different proof of [Eq. \(5\)](#), without computing the action coordinates or invoking the geometric monodromy theorem, but using only topological ideas.

The first step, is to observe that J generates a Hamiltonian circle action on T^*S^2 . It follows that any orbit of this action on $F^{-1}(\gamma(0))$ can be transported along γ . Let (a, b) be a basis of $H_1(F^{-1}(\gamma(0)))$, where b is given by the homology class of such an orbit. Then the corresponding Hamiltonian monodromy matrix along γ is given by

$$M_\gamma = \begin{pmatrix} 1 & m_\gamma \\ 0 & 1 \end{pmatrix}$$

for some integer m_γ . We now prove that the integer $m_\gamma \neq 0$; this argument is due to R. Cushman.

Proof. Observe that the points

$$P_{min} = \{p = 0, z = -1\} \text{ and } P_c = \{p = 0, z = 1\}$$

are the only critical points of H , and they are non-degenerate. We have $H(P_{min}) = -1$ and $H(P_c) = 1$. From the Morse lemma, for small $\varepsilon > 0$ (ε should be less than 2), the manifold $H^{-1}(1 - \varepsilon)$ is diffeomorphic to the 3-sphere S^3 . On the other hand, it can be shown that $H^{-1}(1 + \varepsilon)$ is diffeomorphic to the unit cotangent bundle $T_1^*S^2$. It follows readily that $m_\gamma \neq 0$, for otherwise the manifolds $F^{-1}(\gamma_1)$ and $F^{-1}(\gamma_2)$, where γ_1 and γ_2 are the curves shown in [Fig. 3](#), would be diffeomorphic. This is not the case since $F^{-1}(\gamma_1)$ and $F^{-1}(\gamma_2)$ are isotopic to $H^{-1}(1 - \varepsilon)$ and $H^{-1}(1 + \varepsilon)$, respectively. \square

The next step was made by Floris Takens [\[89\]](#), who proposed the idea of using Chern numbers of energy hyper-surfaces $H^{-1}(h)^4$ and classical Morse theory for the computation of monodromy. More specifically, he observed that in integrable systems with a Hamiltonian circle action (in particular, in the spherical pendulum), the Chern number of energy hyper-surfaces changes when the energy passes a simple non-degenerate critical value of the Hamiltonian function:

Theorem 3.11 (*Takens’s Index Theorem* [\[89\]](#)). *Let H be a proper Morse function on an oriented 4-manifold. Assume that H is invariant under a circle action that is free outside the critical points. Let h_c be a critical value of H containing exactly one critical point. Then the Chern numbers of the nearby levels satisfy*

$$c(h_c + \varepsilon) = c(h_c - \varepsilon) \pm 1.$$

Here the sign is plus if the circle action defines the anti-Hopf fibration near the critical point and minus for the Hopf fibration.

For the spherical pendulum, the circle action comes from rotational symmetry. The Chern number $c(1 + \varepsilon)$ of the energy level $H^{-1}(1 + \varepsilon) \simeq T_1^*S^2$ is equal to 2, and the Chern number $c(1 - \varepsilon)$ of $H^{-1}(1 - \varepsilon) \simeq S^3$ is equal to 1. Thus, to conclude the proof in this case, it is

⁴ To be precise, the Chern numbers are defined for the principal circle bundles $\rho: H^{-1}(h) \rightarrow H^{-1}(h)/\mathbb{S}^1$, where ρ is the reduction map for the Hamiltonian circle action; see [\[47,75\]](#) for a relevant background.

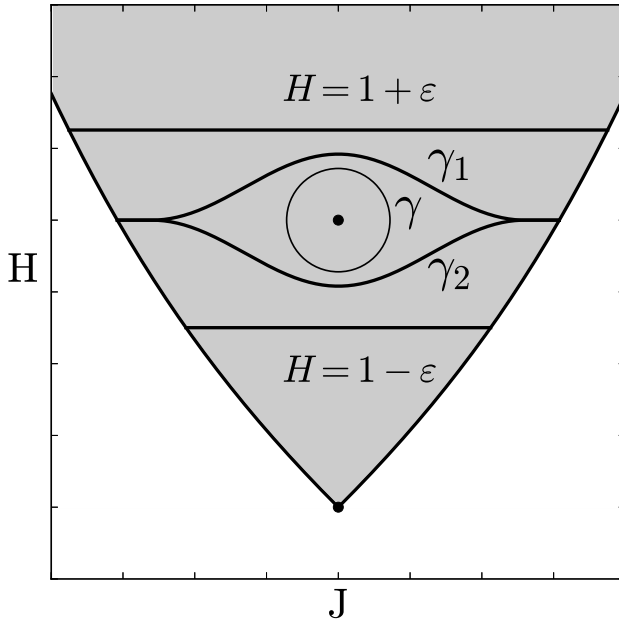


Fig. 3. Bifurcation diagram for the spherical pendulum, the energy levels, the curves γ_1 and γ_2 , and the loop γ around the focus–focus singularity.
 Source: The figure is taken from [67].

left to show that $m_\gamma = c(1 + \varepsilon) - c(1 - \varepsilon)$. This last step was made in [67], where it was observed that the monodromy of a two-degree of freedom system with a circle action is given by the difference of the Chern numbers of appropriately chosen energy levels. For the spherical pendulum, the proof is also based on Fig. 3. First, one observes that the Chern number of $F^{-1}(\gamma_1)$ is equal to $c(1 + \varepsilon)$ and the Chern number of $F^{-1}(\gamma_2)$ to $c(1 - \varepsilon)$. The manifolds $F^{-1}(\gamma_i)$ are obtained from two solid tori by gluing the boundary tori via

$$\begin{pmatrix} a_- \\ b_- \end{pmatrix} = \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_+ \\ b_+ \end{pmatrix},$$

where c_i is the Chern number of $F^{-1}(\gamma_i)$. (We note that this representation using gluing matrices is a very special case of Fomenko–Zieschang theory [11,48].) It follows that the monodromy matrix along γ is given by the product

$$M_\gamma = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Since $c_1 - c_2 = 1$, we conclude that the monodromy matrix

$$M_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We note that the above Morse-theoretic approach works for more general two-degree of freedom systems that have a global circle action. In particular, one can prove the geometric monodromy theorem using this point of view.

3.4. Several remarks

There are various cases (systems with many degrees of freedom, non-compact energy levels) when Morse theory cannot be used directly for the computation of monodromy. Nonetheless, as was shown in [43,69], even in such cases, one can effectively compute the monodromy for integrable systems that are invariant under a global circle action (or a complexity 1 torus action).

The first observation, which is the starting point of the work [43], is that in the case of a global circle action, the monodromy of a torus bundle $F : F^{-1}(\gamma) \rightarrow \gamma$ is given by the Chern number of $F^{-1}(\gamma)$ (the Chern number comes from the circle action). Specifically, there is the following result.

Theorem 3.12 ([11, §4.3.2], [43]). *Assume that the energy–momentum map F is proper and invariant under a Hamiltonian circle action. Let $\gamma \subset \text{image}(F)$ be a simple closed curve in the set of the regular values of the map F . Then the Hamiltonian monodromy of the 2-torus bundle $F : F^{-1}(\gamma) \rightarrow \gamma$ is given by*

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),$$

where m is the Chern number of the principal circle bundle $\rho : F^{-1}(\gamma) \rightarrow F^{-1}(\gamma)/\mathbb{S}^1$, which is defined by reducing the circle action.

The second observation is that in the case when the curve γ bounds a disk $D \subset \text{image}(F)$, the Chern number m can be computed from the singularities of the circle action that project into D . More specifically, there is the following result, which can be proven by applying Stokes’s theorem to the Chern class of $\rho : U \rightarrow U/\mathbb{S}^1$, where U is the subset of $F^{-1}(D)$ on which the circle action is free. (Note that according to Theorem 3.12, the monodromy index m is equal to the integral of the Chern class over the 2-torus $F^{-1}(\gamma)/\mathbb{S}^1$.)

Theorem 3.13 ([43]). *Let F and γ be as in Theorem 3.12. Assume that γ bounds a 2-disk $D \subset \text{image}(F)$ and that the circle action is free in $F^{-1}(D)$ outside isolated fixed points. Then the Hamiltonian monodromy of $F : F^{-1}(\gamma) \rightarrow \gamma$ is given by the number of positive⁵ fixed points minus the number of negative fixed points in $F^{-1}(D)$.*

We note that Theorems 3.12 and 3.13 were generalized to a much more general setting of fractional monodromy and Seifert fibrations; see the work [69]. Such a generalization allows one, in particular, to define a notion of monodromy for circle bundles over 2-dimensional surfaces of genus $g \geq 1$; in the standard case, the genus $g = 1$. We will come back to fractional monodromy and Seifert manifolds in Section 5. For an introduction to Seifert manifolds, we refer the reader to [47,53].

The works [43,69] essentially settle the monodromy question in the case when the 2 degree of freedom system admits a circle action (or, in the case of many degrees of freedom, a complexity 1 torus action). The case when no such action exists is much less understood. In view of the above Morse theory approach, the following problem seems natural.

⁵ The sign of a fixed point depends on whether the circle action defines the anti-Hopf or the Hopf fibration near this point.

Problem 3.14. Is it possible to generalize Cushman–Takens approach to the case when there is no Hamiltonian circle action?

We note that there are examples of integrable systems with focus–focus fibers and no global circle action; see for example [64,87,90,97]. The Hamiltonian monodromy around several such fibers does not have to be of the form

$$M_\gamma = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

In fact, it can be any $SL(2, \mathbb{Z})$ matrix (this follows from properties of the group $SL(2, \mathbb{Z})$); see [30,31].

In this connection, we mention the class of integrable geodesic flows on *Sol*-manifolds that was constructed in [10]. This class comes from a deep problem of non-integrability in classical mechanics [16,60,61]. In this case, the monodromy is associated to a degenerate singular fiber, and a 2×2 block of the Hamiltonian monodromy matrix is given by an integer hyperbolic matrix. One particular example is

$$M_\gamma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that cases of such general $SL(n, \mathbb{Z})$ monodromy matrices (in $n = 2$ or 3 degree of freedom systems) are not yet understood and new examples are currently missing.

Problem 3.15 (*A. Bolsinov*). Construct new examples of integrable systems with a prescribed monodromy around a (possibly degenerate) singular fiber.

3.5. Monodromy in nearly integrable systems

Let $F : M \rightarrow \mathbb{R}^n$ be a proper integral map of an integrable Hamiltonian system on M . Assume that the Hamiltonian H is real-analytic and Kolmogorov nondegenerate. Then, according to the Kolmogorov–Arnol’d–Moser theory [1,59,74], there are invariant Liouville tori $F^{-1}(f)$, forming a set of measure $1 - O(\sqrt{\varepsilon})$, which survive small perturbations $H + \varepsilon P$ of H . This leads to the following natural question, which was addressed in [18,20,81], cf. [102]: can one extend geometric invariants of integrable systems (like monodromy) to the nearly-integrable case? It turns out that this is indeed possible, at least in the topological setting. More specifically, one can ‘smoothly interpolate’ the invariant tori given by the KAM theorem in a global way. Such an interpolation results in a torus bundle for the perturbed system which is diffeomorphic to the original torus bundle associated to H . This implies that the topology of the original torus bundle, given by the non-singular part of F , is preserved under the perturbation. In particular, Hamiltonian monodromy can be extended to nearly-integrable systems. Below we discuss this idea in more detail, following mainly [18].

Consider the product $D^n \times \mathbb{T}^n$ of an n -disk and an n -torus with the standard symplectic structure $dI \wedge d\varphi$. Suppose that H is a non-degenerate Hamiltonian of the integral map $F = \text{Pr} : D^n \times \mathbb{T}^n \rightarrow D^n$. This means that the frequency map

$$\omega_i = \frac{\partial H}{\partial I_i} : D^n \rightarrow \mathbb{R}^n$$

is a diffeomorphism onto its image. For $\tau \geq n$ and $\gamma > 0$, let

$$D_{\tau,\gamma} = \{\omega \in \mathbb{R}^n \mid \langle \omega, k \rangle \geq \gamma |k|^{-\tau}, \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}\}$$

be the set of Diophantine frequency vectors. We also let

$$A_{\tau,\gamma} = \{I \in D^n \mid \omega(I) \in D_{\tau,\gamma} \text{ and } \text{dist}(\omega(I), \partial\omega(D^n)) > \gamma\}.$$

A main ingredient in the proofs of the monodromy invariance under perturbations is the following (semi-)local theorem of Pöschel [80].

Theorem 3.16 (*Semi-local KAM Theorem [80]*). *Consider the product $D^n \times \mathbb{T}^n$ with the standard symplectic structure. Suppose that H is a non-degenerate integral of $F = \text{Pr}: D^n \times \mathbb{T}^n \rightarrow D^n$. Let P be a smooth function on $D^n \times \mathbb{T}^n$. Then for all sufficiently small ε , there exists a diffeomorphism $\Phi_\varepsilon: D^n \times \mathbb{T}^n \rightarrow D^n \times \mathbb{T}^n$ such that*

- (i) Φ_ε is close to the identity;
- (ii) the restriction of Φ_ε to $A_{\tau,\gamma} \times \mathbb{T}^n$ conjugates the Hamiltonian flows of H and $H + \varepsilon P$.

We note that in integrable systems, the product $D^n \times \mathbb{T}^n$ appearing in [Theorem 3.16](#) comes from semi-local action–angle coordinates. This is why this theorem is semi-local. In [18], by using a partition of unity and a convexity argument, this result was extended to the global setting of (possibly non-trivial) Lagrangian torus bundles. More specifically, there is the following result.

Theorem 3.17 ([18]). *Let $F: M \rightarrow \mathbb{R}^n$ be the integral map of an integrable system such that all of the fibers $F^{-1}(f)$ are compact and connected. Suppose that H is a non-degenerate integral of F , and let P be a smooth function on M . Finally, consider the non-singular part of F over a relatively compact set $R \subset \mathbb{R}^n$: the n -torus bundle*

$$F: F^{-1}(R) \rightarrow R.$$

Then for all sufficiently small ε , there exist a subset $R'_\varepsilon \subset R$ and a diffeomorphism $\Phi_\varepsilon: F^{-1}(R) \rightarrow F^{-1}(R)$ such that

- (i) Φ_ε is close to the identity;
- (ii) R'_ε is nowhere dense in \mathbb{R}^n and the measure of $R \setminus R'_\varepsilon$ tends to zero when ε tends to zero;
- (iii) the restriction of Φ_ε to $F^{-1}(R'_\varepsilon)$ conjugates the Hamiltonian flows of H and $H + \varepsilon P$.

Remark 3.18. The construction of the global diffeomorphism Φ_ε is based heavily on the Whitney extension theorem [98] and a unicity theorem [20], stating that the local KAM conjugacies provided by [Theorem 3.16](#) are unique up to a torus translation on the set of Diophantine tori corresponding to the density points of $A_{\tau,\gamma}$.

Remark 3.19. In the two degree of freedom case of a focus–focus singularity, the important condition of nondegeneracy of H is fulfilled in a small neighborhood of the focus–focus fiber; [102].

From this theorem it readily follows that the notion of Hamiltonian monodromy (as well as Duistermaat’s Chern class [34]) can be extended to sufficiently small perturbations $H + \varepsilon P$ of H .

We note that in the two-degree of freedom case of monodromy around a focus–focus singularity, it is essentially sufficient to apply only the semi-local theorem of Pöschel by assuming the interpolation diffeomorphism Φ_ε to be the identity outside a suitably chosen action–angle chart; for details see [81].

4. Quantum monodromy

Consider an integrable system $F = (f_1, \dots, f_n)$ on a cotangent bundle T^*N , for instance, the spherical pendulum. Assume for simplicity, that all of the fibers of F are compact and connected. Since the symplectic form is exact, one can construct semi-local action coordinates via the formula

$$I_i = \frac{1}{2\pi} \int_{\alpha_i} pdq,$$

where $\alpha_1, \dots, \alpha_n$ is a family of (bases of) homology cycles on Liouville tori. Different choices of such cycles result in different sets of (semi-local) action coordinates. These sets of semi-local action coordinates are related by a $SL(n, \mathbb{Z})$ transformation⁶

$$(I_1, \dots, I_n) = M(I'_1, \dots, I'_n), \quad M \in SL(n, \mathbb{Z}). \tag{6}$$

Recall that each of the actions I_i is a function of $F = (F_1, \dots, F_n)$. Equating

$$I_i = \hbar(n_i + \mu_i), \quad i = 1, \dots, n, \tag{7}$$

the actions I_i to integer multiples of the reduced Planck constant (up to the addition of Maslov’s correction μ_i), gives a set of points in the (F_1, \dots, F_n) -space. This set of points is called a *semi-classical spectrum* and Eq. (7) is the so-called *Bohr–Sommerfeld* or *action quantization*. We note that the semi-classical spectrum does not depend on the specific choice of the cycles α_i because of Eq. (6). In fact, this set locally looks like a regular \mathbb{Z}^n lattice by the Arnol’d–Liouville theorem. However, due to a possibly non-trivial Hamiltonian monodromy, this does not have to be the case globally. The global lattice may exhibit a non-trivial *lattice defect* [99,100].

A model example of a lattice defect can be constructed as follows; see [99,103] for more details. Let $n = 2$ and consider the standard \mathbb{Z}^2 lattice in \mathbb{R}^2 . Remove the open solid angle from \mathbb{R}^2 that is spanned by the vectors $(-1, 0)$ and $(-1, 1)$. Identify the edges of the solid angle via a vertical shift, that is, by gluing pairs of boundary points of the angle having the same first coordinate. Note that this transformation identifies the corresponding pairs of \mathbb{Z}^2 lattice points. After the identification, we get a new lattice on the quotient space (the quotient space is again an \mathbb{R}^2). This lattice has a non-trivial monodromy defect, which can be revealed through a transport of an elementary cell, defined by adjacent points of the lattice, around the origin; see Fig. 4.

We remark that a similar type of a lattice defect is always present when there is a focus–focus singularity of the system. In particular, it is present in the spherical pendulum [26]; compare with Fig. 2, where the transport of an elementary cell of the quantum spectrum is shown.

This is the basic idea behind the *quantum monodromy*. One can call the monodromy based on action quantization *semi-classical*, since it is constructed out of the underlying classical integrable system.

To get to a purely quantum case, one considers a set of commuting pseudodifferential operators⁷ $\hat{F}_1, \dots, \hat{F}_n$ whose principal symbols define a classical integrable system on T^*M as above; see [92] for more details. For instance, for the spherical pendulum,

$$\hat{F}_1 = \hat{H} = -\frac{1}{2}\hbar^2 \Delta + V$$

⁶ In general, different sets of action coordinates are related by a $SL(n, \mathbb{Z}) \times \mathbb{R}^n$ transformation; note that in our case, we have the canonical one-form pdq .

⁷ For compact prequantizable Kähler manifolds, one can use Berezin–Toeplitz operators [17,22,78].

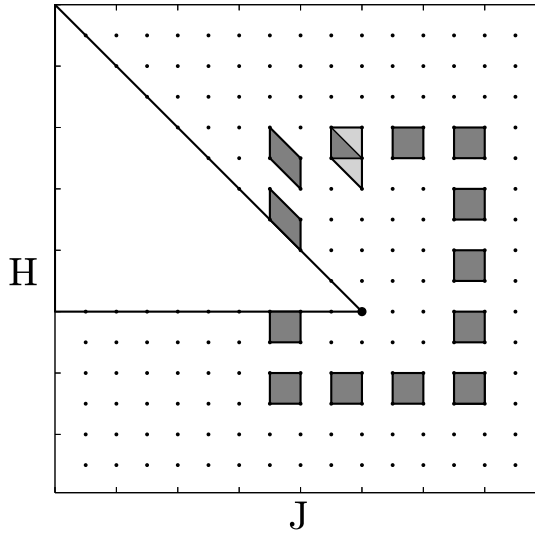


Fig. 4. Model example of a monodromy defect.

is the corresponding Schrödinger operator on S^2 and

$$\hat{F}_2 = \hat{J} = -i\hbar(x\partial_y - y\partial_x).$$

The main paradigm is that the semi-classical spectrum obtained from the action quantization gives an approximation (in terms of \hbar) to the *joint spectrum*

$$\sigma(\hat{F}_1, \dots, \hat{F}_n) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \bigcap_{i=1}^n \text{Ker}(\hat{F}_i - \lambda_i I) \neq 0\}$$

of the commuting operators $\hat{F}_1, \dots, \hat{F}_n$. In particular, one can observe a lattice defect also in the quantum problem; see Fig. 2.

These ideas were originally introduced by Cushman–Duistermaat [26] and Guillemin–Uribe [50] for the spherical pendulum. They were made precise by S. Vũ Ngọc in [92]; see also [21,24]. There are now various examples of quantum systems with a non-trivial quantum monodromy, such as the quantum versions of integrable systems arising from classical mechanics, and having a non-trivial Hamiltonian monodromy, as well as certain (simplified models of) quantum physical systems, such as the hydrogen atom in crossed fields [29], coupled angular momenta [83], the water molecule [23,101], the LiNC/NCLi molecule [42,55], trapped Bose condensates [96], etc. For more information on quantum monodromy and the spectral theory of integrable systems, we refer the reader to [22,62,78,79,83,92,100].

5. Fractional monodromy

As we have seen in the previous sections, Hamiltonian monodromy is intimately related to the singularities of a given integrable system. However, this invariant is defined for the non-singular part

$$F: F^{-1}(R) \rightarrow R$$

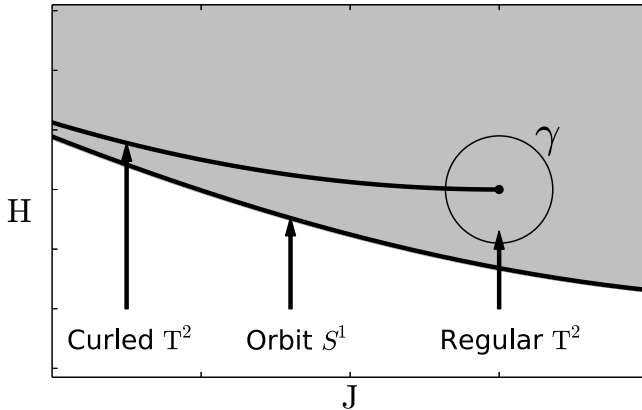


Fig. 5. The bifurcation diagram of the integrable 1: (-2) resonance. The closed curve γ around the origin intersects the critical hyperbolic branch.

of the possibly singular torus fibration $F: M \rightarrow \mathbb{R}^n$ that comes with the system. An invariant that generalizes Hamiltonian monodromy to singular torus fibrations was introduced by Nekhoroshev, Sadovskii and Zhilinskii in [77] and it is called *fractional monodromy*.

5.1. 1: (-2) resonant system

Fractional monodromy has up until now been discussed mainly for so-called $m: (-n)$ resonances; see [39,40,84,88]. We shall only focus here on the special case of 1: (-2) resonance, which is the simplest and historically the first example of an integrable Hamiltonian system with fractional monodromy introduced in the work [77].

Consider \mathbb{R}^4 with the standard symplectic structure $\omega = dq \wedge dp$. Let the integral map $F = (H, J): \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be defined by the Hamiltonian function

$$H = 2q_1 p_1 q_2 + (q_1^2 - p_1^2) p_2 + R^2,$$

where $R = \frac{1}{2}(q_1^2 + p_1^2) + (q_2^2 + p_2^2)$, and the ‘momentum’

$$J = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2).$$

We note that the functions H and J are involution, so that F is indeed the integral map of an integrable Hamiltonian system. We also note that the function J defines a Hamiltonian circle action on \mathbb{R}^4 which preserves the fibration given by F .

The bifurcation diagram of the integral map F is shown in Fig. 5. From the structure of the diagram, we observe that the Hamiltonian monodromy is trivial. Indeed, the set

$$R = \{f \in \text{image}(F) \mid f \text{ is a regular value of } F\}$$

is contractible. In particular, every closed path in R can be deformed to a constant path within R . Non-triviality appears if one considers the closed curve γ that is shown in Fig. 5.

More specifically, consider a non-singular point $\gamma(t_0)$ and a basis (a_0, b_0) of the integer homology group $H_1(F^{-1}(\gamma(t_0))) \simeq \mathbb{Z}^2$. Then one can try to ‘parallel transport’ these cycles along γ such that at each regular point $\gamma(t)$ they form a basis of $H_1(F^{-1}(\gamma(t)))$ and such that the resulting family of cycles is (locally) continuous, also at the critical fiber, corresponding

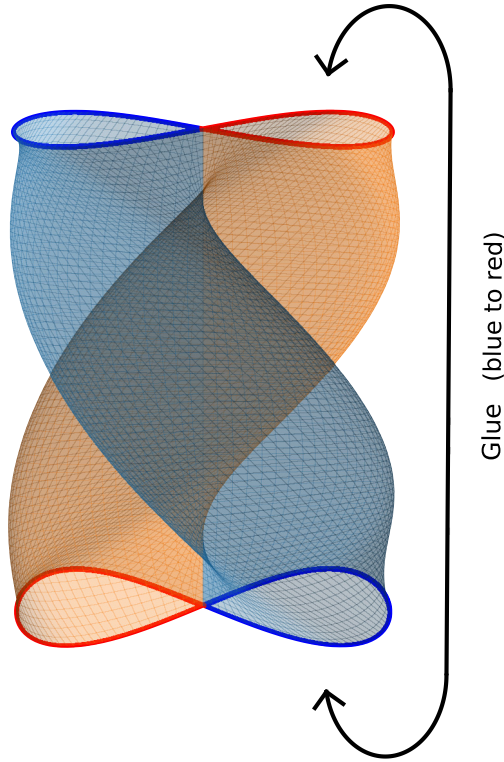


Fig. 6. Curled torus.

to the intersection of γ with the critical hyperbolic branch.⁸ We note that in the case of Hamiltonian monodromy, when we are moving along regular Liouville tori, such a parallel transport is always possible [34]. In this fractional monodromy case, it turns out that only a subgroup of $H_1(F^{-1}(\gamma(t_0)))$ can be transported through the critical fiber. Specifically, there is the following result.

Theorem 5.1 ([77]). *Let (a_0, b_0) be an integer basis of $H_1(F^{-1}(\gamma(t_0)))$, where $\gamma(t_0) \in R$ and b_0 is an orbit of the circle action. The parallel transport (fractional monodromy) along the curve γ is given by*

$$2a_0 \mapsto 2a_0 + b_0, \quad b_0 \mapsto b_0.$$

Remark 5.2. When written formally in the integer basis (a_0, b_0) , the parallel transport has the form of the *rational* matrix

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Q}),$$

called the matrix of *fractional* monodromy.

⁸ This critical fiber is a so-called curled torus, which can be obtained as follows. Take the direct product of a figure eight and a segment. Identify the upper and the lower boundary components of this product after making a rotation (of the upper component) by the angle π . The result is schematically shown in Fig. 6.

Remark 5.3. [Theorem 5.1](#) is closely related to Fomenko-Zieschang theory. More specifically, to the curve γ one can associate a so-called loop molecule, which consists of one atom A^* , corresponding to a neighborhood of the curled torus, and the marks $r = \infty$, $\varepsilon = 1$, $n = 1$ (see [\[11\]](#) for relevant definitions). Fractional monodromy is a function of these invariants, in this case determined by the atom A^* and the n -mark; see [\[13,66\]](#) for more details.

Since the pioneering work [\[77\]](#), various proofs of [Theorem 5.1](#) appeared; see [\[19,39,40,49,69,88,91\]](#). A natural approach, which was pursued in [\[19,49,91\]](#), is to separate the problem into two parts: the computation of fractional monodromy in a neighborhood U of the curled torus and the computation of (essentially) the usual monodromy outside of this neighborhood U . We note that the Liouville fibration inside U is topologically standard (that is, does not depend on the specific system, but only on the singularity). Another approach, which was pursued in the work [\[88\]](#), is to complexify the system to bypass the hyperbolic branch and compute the variation of the rotation number in the complexified domain; cf. [\[5\]](#). We note that this approach works also for higher order resonances. Below we sketch a different proof of [Theorem 5.1](#), following the point of view of Seifert manifolds, developed in the work [\[69\]](#).

Proof of Theorem 5.1. Consider again the curve γ shown in [Fig. 5](#). The key observation, which was already made in [\[13\]](#), is that $F^{-1}(\gamma)$ is a Seifert 3-manifold. The structure of a Seifert fibration comes from the circle action given by the momentum J . In complex coordinates $z = p_1 + iq_1$ and $w = p_2 + iq_2$, this circle action has the form

$$(t, z, w) \mapsto (e^{it}z, e^{-2it}w), \quad t \in \mathbb{S}^1. \tag{8}$$

We observe that the origin is fixed under this action and that the set

$$P = \{(q, p) \mid q_1 = p_1 = 0 \text{ and } q_2^2 + p_2^2 \neq 0\}$$

consists of points with \mathbb{Z}_2 isotropy group. This implies that the Euler number⁹ of the Seifert manifold $F^{-1}(\gamma)$ equals $1/2 \neq 0$. Indeed, Stokes’ theorem implies that the Euler number of $F^{-1}(\gamma)$ coincides with the Euler number of a small 3-sphere around the origin $z = w = 0$. The latter Euler number equals $1/2$ because of [\(8\)](#). From this and [Theorem 3.12](#), we get the following.

Lemma 5.4 ([\[69\]](#)). *The quotient space $F^{-1}(\gamma)/\mathbb{Z}_2$ is the total space of a torus bundle over γ . Its monodromy is given by*

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

From [Lemma 5.4](#) we infer that the parallel transport along the curve γ in the \mathbb{Z}_2 -quotient space has the form

$$a_0^r \mapsto a_0^r + b_0^r, \quad b_0^r \mapsto b_0^r,$$

where the cycles $a_0^r = a_0/\mathbb{Z}_2$ and $b_0^r = b_0/\mathbb{Z}_2$ form the induced basis of the group $H_1(F^{-1}(\gamma(t_0))/\mathbb{Z}_2)$. Observe that a_0 is not affected by the quotient map and the orbit b_0

⁹ For a definition of the Euler number of a Seifert fibration, see, for instance, [\[47,53\]](#). Note that, in general, this is a rational number. The integer Chern number, considered in [Section 3](#), appears as a special case when the Seifert fibration is a principal circle bundle over a 2-surface. When we specialize to unit tangent bundles, the Chern number coincides with the Euler characteristic of the 2-surface.

becomes ‘shorter’: $2b'_0 \simeq b_0$. It follows that the parallel transport in the original space has the form

$$2a_0 \mapsto 2a_0 + b_0, \quad b_0 \mapsto b_0.$$

This concludes the proof of [Theorem 5.1](#). \square

We note that the idea of computing fractional monodromy using a covering map appeared in the work [39], where the authors computed fractional monodromy for a large class of integrable systems with an m : $(-n)$ resonance. There an *uncovering* map was used to lift the (possibly singular) Lagrangian fibers to a union of tori. Here we used a covering map instead. Moreover, we focused not on the fibers of the energy–momentum map, but rather on the global topology of an associated Seifert fibration. This approach, which was developed in the work [69], turned out to be very effective and allowed one to define fractional monodromy over an arbitrary Seifert manifold with an orientable base of genus $g \geq 1$. (We note that the known examples appeared as a special case of this construction when the genus $g = 1$ and there are at most two singular fibers of the Seifert fibration.) The precise results can be stated as follows; cf. [Theorems 3.12](#) and [3.13](#).

Theorem 5.5 ([69]). *Let X be the total space of a Seifert fibration with an orientable base such that the boundary of X consists of two tori. Let X_f be the closed Seifert manifold obtained by gluing these tori via a fiber-preserving diffeomorphism f . Take bases of these tori (a_0, b_0) and (a_1, b_1) such that b_0, b_1 correspond to non-singular fibers of the Seifert fibration. Let N denote the least common multiple of the orders of exceptional fibers. Then only linear combinations of Na_0 and b_0 can be parallel transported along X and under the parallel transport*

$$\begin{aligned} Na_0 &\mapsto Na_1 + kb_1 \\ b_0 &\mapsto b_1 \end{aligned}$$

for some integer $k = k(f)$ which depends only on the isotopy class of the diffeomorphism f . Moreover, the Euler number of $X(f)$ is given by $e(f) = k(f)/N$.

Remark 5.6. We note that, in this case, the matrix of fractional monodromy is given by

$$M_X = \begin{pmatrix} 1 & e(f) \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Q}).$$

Remark 5.7. In [Theorem 5.5](#), we use the notion of parallel transport introduced in [39]. Specifically, let $\partial X = \mathbb{T}_1^2 \sqcup \mathbb{T}_0^2$. By definition, a cycle $\alpha_1 \in H_1(\mathbb{T}_1^2)$ is a parallel transport of $\alpha_0 \in H_1(\mathbb{T}_0^2)$ if these cycles are of the same integer homology class in X . We note that this definition of parallel transport can be used for abstract manifolds with boundary, without an explicit connection to integrability. However, such a parallel transport is not always well defined: one can construct examples of 3-manifolds where parallel transport is not unique or does not give rise to a well-defined automorphism [66]. According to [Theorem 5.5](#), this notion of parallel transport is well defined for Seifert manifolds with an orientable base and results in an automorphism of an index- N subgroup of $H_1(\mathbb{T}_0^2 \simeq_f \mathbb{T}_1^2)$.

[Theorem 5.5](#) implies that in order to compute fractional monodromy for a specific integrable system, it is sufficient to compute the orders of exceptional orbits and the Euler number of the corresponding Seifert fibration. We note that in concrete examples of integrable systems, the orders of exceptional orbits are often known from the circle action. To compute the Euler number, one can use the following result.

Theorem 5.8 ([69]). *Let M be a compact oriented 4-manifold that admits an effective circle action. Assume that the action is fixed-point free on the boundary ∂M and has only finitely many fixed points p_1, \dots, p_ℓ in the interior. Then*

$$e(\partial M) = \sum_{k=1}^{\ell} \frac{1}{m_k n_k},$$

where (m_k, n_k) are isotropy weights of the fixed points p_k .

We note that the idea of using Seifert fibration in the context of integrable systems goes back to A. T. Fomenko and H. Zieschang. In their molecule theory [11,48], atoms and Seifert manifolds appear as the basic building blocks. However, not every loop molecule admits the structure of a global Seifert fibration.

Problem 5.9 (A.T. Fomenko). *Suppose that X corresponds to a loop molecule of an integrable and non-degenerate two-degree of freedom system. Then X admits a decomposition into Seifert-fibered pieces. Can one construct an algorithm that computes fractional monodromy of X , when it exists?*

A related problem is the following.

Problem 5.10. *Suppose X is a graph-manifold (a loop molecule). Under which geometric conditions does fractional monodromy exist along X ?*

5.2. Towards quantum fractional monodromy

Let us come back to the example of a system with 1: (-2) resonance. Consider the (semi-local) action coordinates

$$I_1 = \frac{1}{2\pi} \int_{\alpha_1} pdq \quad \text{and}$$

$$I_2 = \frac{1}{2\pi} \int_{\alpha_2} pdq,$$

where the cycle α_2 corresponds to the circle action and α_1 is such that (α_1, α_2) form a basis in the first homology group of a Liouville torus. Note that $I_2 = J$.

As in the case of the usual quantum monodromy, one can consider the quantization condition

$$I_1 = \hbar(n_1 + \mu_1),$$

$$I_2 = \hbar(n_2 + \mu_2),$$

which gives a semi-classical spectrum locally outside the hyperbolic branch. However, for this spectrum one cannot transport an elementary cell around the singularity in a continuous way. The novel idea that was introduced in [77] is to consider not an elementary cell, but a double cell in this case. Let us explain this idea on the level of the actions. Observe that by [Theorem 5.1](#), it is possible to define $2I_1$ and I_2 also in a neighborhood of the curled torus. Therefore, the action quantization

$$I_1 = \hbar(n_1 + \mu_1),$$

$$I_2 = \hbar(2n_2 + \mu_2)$$

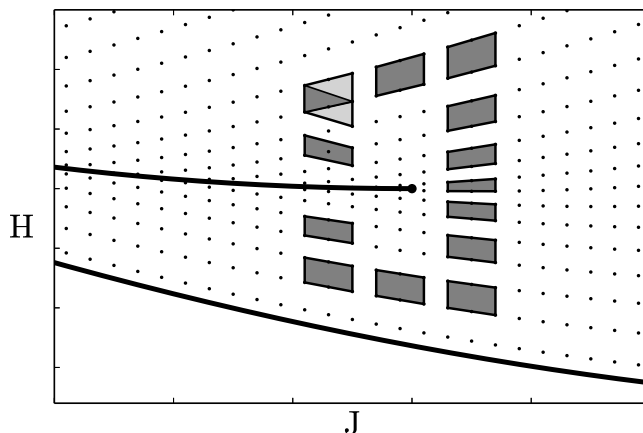


Fig. 7. The joint spectrum of the quantum 1: (-2) resonance ($\hbar = 0.004$), and the transport of a double cell around the singularity.

will result in a globally defined lattice which is contained in the original semi-classical spectrum and for which one can transport an elementary cell around the origin. By the construction, an elementary cell for this lattice is a double cell for the original spectrum. After the transport around the origin, this double cell will not come back to its initial position, but will be transformed to another double cell, related to the initial one by the quantum monodromy transformation

$$M_{quant} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

see Fig. 7, where the joint spectrum of (\hat{H}, \hat{J}) for the quantum 1: (-2) resonance system and the transport of a double cell are shown.

We note that here we suppress the question of a continuous transport of an elementary cell in the joint spectrum of (\hat{H}, \hat{J}) for the quantum 1: (-2) resonance system near the hyperbolic branch.

The idea of considering a double or an n -cell leads to the notion of *quantum fractional monodromy* [77]. We refer the reader to [77] for more details.

6. Scattering monodromy

Up until now we considered integrable Hamiltonian systems such that the corresponding integral map F has compact invariant fibers $F^{-1}(f)$, $f \in \mathbb{R}^n$. In this section, we mainly discuss the non-compact case. In particular, we discuss the so-called *scattering monodromy* in the context of classical potential scattering theory.

6.1. Preliminaries

A notion of scattering monodromy was originally introduced by L. M. Bates and R. H. Cushman in [7] for a two degree of freedom hyperbolic oscillator.¹⁰ At about the same time,

¹⁰ The hyperbolic oscillator is not a scattering system in the sense of, for instance, [57], since the potential of this system is unbounded at infinity and is not decaying to zero. Nonetheless, the system shares some of the properties of scattering systems, such as the existence of the so-called *deflection angle*; see below.

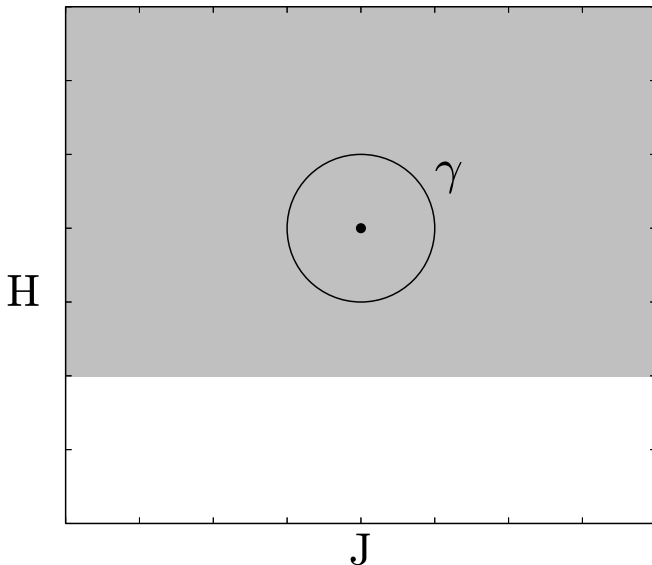


Fig. 8. Bifurcation diagram for the integral map $F = (H, J)$ of a planar scattering integrable system with a repulsive potential.

scattering monodromy was introduced by H. R. Dullin and H. Waalkens in [37] for planar scattering systems with a repulsive rotationally symmetric potential, both in the classical and quantum settings. The idea behind the works [7,37] is as follows.

Consider a Hamiltonian system on $T^*\mathbb{R}^2$ with canonical coordinates (q_1, q_2, p_1, p_2) defined by the Hamiltonian function

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(r),$$

where V is a radially symmetric potential $r^2 = q_1^2 + q_2^2$. This system describes the motion of a particle on the plane \mathbb{R}^2 with coordinates (q_1, q_2) under the influence of the potential function V . We observe that the system is Liouville integrable since the momentum $J = q_1 p_2 - q_2 p_1$ is conserved.

We shall assume, for simplicity, that the potential V is a smooth, positive, and monotone function, decaying at infinity sufficiently fast. The bifurcation diagram of $F = (H, J)$ is shown in Fig. 8. It consists of a single critical value, corresponding the maximum of V . This is a focus–focus singularity if the maximum is non-degenerate. In particular, the set R of the regular values of F is not simply-connected. Nonetheless, it can be shown that global action–angle coordinates exist for this system; see [7]. Topologically, the bundle $F^{-1}(\gamma) \rightarrow \gamma$ is a trivial cylinder $S^1 \times \mathbb{R}$ -bundle. Moreover, the energy levels $H^{-1}(h_{max} \pm \varepsilon)$ below and above $h_{max} = \max V$ are topologically the same.

To get a non-trivial invariant, the authors of [7,37] considered the so-called deflection angle of a trajectory. Specifically, observe that under the Hamiltonian dynamics, a particle in the plane gets deflected by V . It proceeds to spatial infinity in both forward and backward time, unless it approaches the maximum of the potential. To any such scattering trajectory, one can

associate the deflection angle

$$\Phi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\varphi(q(t))}{dt} dt,$$

where φ is the polar angle in the configuration q_1q_2 -plane. Due to rotational symmetry, the deflection angle is a function of $F = (H, J)$. Hence, one can consider its variation along γ .

Theorem 6.1 ([7,37]). *In the above setting, the variation of the deflection angle Φ along γ is equal to -1 .*

The above approach to scattering monodromy is based on the notion of the deflection angle, which is very close to the notion of the rotation number for compact systems. We note that one can approach scattering monodromy also from other (related) perspectives. For instance, in [37] the authors used radial actions for the pair of integrable systems: the original system given by V and a reference system with the zero potential (the free flow). These radial actions

$$I = \frac{1}{\pi} \int_{r_0}^{\infty} p_r dr \quad \text{and} \quad I_{\text{ref}} = \frac{1}{\pi} \int_{r'_0}^{\infty} p_r^{\text{ref}} dr$$

do not exist individually. However, if the potential V decays sufficiently fast, their difference exists. More specifically, the limit

$$\lim_{r \rightarrow \infty} \frac{1}{\pi} \int_{r_0}^R p_r dr - \frac{1}{\pi} \int_{r'_0}^R p_r^{\text{ref}} dr$$

exists and behaves like the usual radial action of a compact system with a rotationally-symmetric potential. In particular, transporting this radial action and the action J along γ , one gets a monodromy automorphism of the usual form:

$$M_\gamma = \begin{pmatrix} 1 & m_\gamma \\ 0 & 1 \end{pmatrix},$$

where $m_\gamma = -1$ is the variation of the deflection angle.

Related to this is a ‘billiard’ approach, which is also based on the action coordinates. It is applicable whenever a given integrable system with non-compact fibers is separable. We refer the reader to the works [32,73,82].

We also mention the work [41], where the notion of *non-compact monodromy* was introduced. Here the idea is that for a non-compact integrable system with the integral map F and a global circle action, one can compactify the fibers of F near a focus–focus fiber preserving the circle action. Then one gets a compact fibration with the usual monodromy around the focus–focus fiber. In [41], this monodromy is called non-compact. It coincides with the scattering monodromy for the above two-degree of freedom systems.

Finally, we mention the work [68], where the authors follow the point of view of classical potential scattering theory; see, in particular, [57]. The novelty of this work is that it is applicable to possibly many degrees of freedom scattering and integrable systems that are not necessarily rotationally symmetric. This approach generalizes the above approaches to scattering monodromy. We discuss it in some more detail below.

6.2. Classical scattering theory

Below we briefly review classical potential scattering theory, following mainly A. Knauf [57,58] and J. Dereziński and C. Gerard [33]; see also [66,68].

Consider a pair of Hamiltonians on $T^*\mathbb{R}^n$ given by

$$H = \frac{1}{2}\|p\|^2 + V(q) \text{ and } H_r = \frac{1}{2}\|p\|^2 + V_r(q),$$

where the (singular) potentials V and V_r are assumed to decay sufficiently fast. Let g_H^t denote the Hamiltonian flow. Define the invariant set s of scattering states by

$$s = \{(q, p) \in T^*\mathbb{R}^n \mid H(q, p) > 0, \sup_{t \in \mathbb{R}^\pm} \|g_H^t(q, p)\| = \infty\}.$$

If the potential V decays at infinity sufficiently fast (for example, is of short range [33,57]), then the trajectories are asymptotic to straight lines. Moreover, for any $x \in s$, the following functions, usually called the *asymptotic direction* and the *impact parameter* of $g_H^t(x)$,

$$\hat{p}^\pm(x) = \lim_{t \rightarrow \pm\infty} p(t, x) \text{ and } q_\perp^\pm(x) = \lim_{t \rightarrow \pm\infty} q(t, x) - \langle q(t, x), \hat{p}^\pm(x) \rangle \frac{\hat{p}^\pm(x)}{2h},$$

are defined and depend continuously on $x \in s$. (Here h is the energy of g_H^t .) In other words, the space of trajectories s/g_H^t , that is, the quotient space of s with respect to the Hamiltonian flow g_H^t , gets parametrized by the trajectories of the free Hamiltonian $H = \frac{1}{2}\|p\|^2$. Due to the g_H^t -invariance, we get the maps

$$A^\pm = (\hat{p}^\pm, q_\perp^\pm): s/g_H^t \rightarrow AS$$

from s/g_H^t to a subset $AS \subset \mathbb{R}^n \times \mathbb{R}^n$ of the ‘asymptotic states’.

Similarly, one can construct the maps

$$A_r^\pm = (\hat{p}^\pm, q_\perp^\pm): s_r/g_{H_r}^t \rightarrow AS$$

for the Hamiltonian $H_r = \frac{1}{2}p^2 + V_r(q)$.

Definition 6.2 ([57,68]). Let M be a g_H^t -invariant submanifold of s . Assume that the composition map

$$S = (A^-)^{-1} \circ A_r^- \circ (A_r^+)^{-1} \circ A^+$$

is well defined and maps the set $B = M/g_H^t$ to itself. The map S is then called the *scattering map* with respect to H, H_r and B .

6.3. Monodromy in scattering systems

To define scattering monodromy, we need to restrict the class of possible reference systems to those for which the corresponding scattering map preserves the integral fibration at infinity.

Definition 6.3 ([68]). Consider a Hamiltonian H which gives rise to a scattering integrable system with the integral map F . A Hamiltonian H_r will be called a *reference* Hamiltonian for this system if

$$F \left(\lim_{t \rightarrow +\infty} g_{H_r}^t(x) \right) = F \left(\lim_{t \rightarrow -\infty} g_{H_r}^t(x) \right) \tag{9}$$

for every scattering trajectory $t \mapsto g_{H_r}^t(x)$.

Remark 6.4. We note that Eq. (9) appeared in a related context in the work [56].

Consider the Liouville fibration $F : s \rightarrow \mathbb{R}^n$. Let H_r be a reference Hamiltonian for F such that $A^\pm(s) \subset A^\pm(s_r)$ holds. Then we have the scattering map

$$S : B \rightarrow B, \quad B = s/g_H^t,$$

which allows us to identify the asymptotic states of s at $t = +\infty$ and $t = -\infty$. This results in a new total space s_c and a new fibration

$$F_c : s_c \rightarrow \mathbb{R}^n.$$

Definition 6.5 ([68]). Assume that the fibration

$$F_c : s_c \rightarrow \mathbb{R}^n$$

is a torus bundle. The Hamiltonian monodromy of this bundle is called *scattering monodromy* of F with respect to H_r .

One distinctive property of scattering monodromy in the sense of Definition 6.5 is its relative form (dependence on the choice of H_r). For instance, if we choose H_r to coincide with the original Hamiltonian H , Duistermaat’s Hamiltonian monodromy is recovered.

Another property that we mention here is that using an appropriately chosen scattering map, one can define scattering monodromy for certain scattering systems that are not necessarily Liouville integrable or even nearly-integrable on the whole phase space $T^*\mathbb{R}^n$. This is similar to the case of another scattering invariant (the so-called *scattering degree*) introduced by A. Knauf in [57] outside the context of integrability; cf. also the work [70].

6.4. Example

Let us come back to the example considered at the beginning of this section: a Hamiltonian system on $T^*\mathbb{R}^2$ given by the Hamiltonian function

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(r),$$

where V is a radially symmetric, positive, monotone decaying potential. Let $J = q_1 p_2 - q_2 p_1$ denote the angular momentum. Consider the curve γ around the focus–focus fiber shown in Fig. 8. Setting $H_r = \frac{1}{2}(p_1^2 + p_2^2)$ and $M = F^{-1}(\gamma)$, we get the scattering map

$$S : B \rightarrow B, \quad B = M/g_H^t.$$

Note that the manifold B is a two-torus in this case.

Theorem 6.6 ([68]). In the first homology group of $B = F^{-1}/g_H^t$, the scattering map S is given by the matrix

$$M_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This scattering monodromy along γ (w.r.t. H and $H_r = \frac{1}{2}(p_1^2 + p_2^2)$) is given by the same matrix M_γ .

Another interesting example, where a natural choice of H_r is not given by the free flow, is the (spatial) Euler two-center problem. We refer to the work [68] for details.

6.5. Quantum scattering monodromy

We have already noted that for a scattering system on $T^*\mathbb{R}^2$ with a decaying rotationally symmetric potential $V(r)$, one can define a notion of scattering monodromy using the difference of the radial actions

$$I_{\text{diff}} = \lim_{r \rightarrow \infty} \frac{1}{\pi} \int_{r_0}^R p_r dr - \frac{1}{\pi} \int_{r'_0}^R p_r^{\text{ref}} dr, \quad (10)$$

for the original system and the reference system with zero potential (the free flow); see [37]. Using this idea, it was shown in the same work [37] that for scattering systems in the plane, one can define a quantum analogue of scattering monodromy. The non-triviality of this invariant also leads to a lattice defect, similarly to the compact case.

We note, however, that in quantum scattering (and even in the case of quantum scattering in the plane), there is an additional difficulty related to the decay of the potential function: if the potential V is of long range, then the corresponding action difference given in Eq. (10) diverges. This is not a problem for the classical scattering monodromy (in the sense of Definition 6.5). Another interesting and related problem is to define quantum scattering monodromy for scattering integrable systems with many degrees of freedom. For a discussion of these problems, we refer the reader to [66].

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