

# Loops of Infinite Order and Toric Foliations

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**Abstract**—In 2005 Dullin et al. proved that the nonzero vector of Maslov indices is an eigenvector with eigenvalue 1 of the monodromy matrices of an integrable Hamiltonian system. We take a close look at the geometry behind this result and extend it to the more general context of possibly non-Hamiltonian systems. We construct a bundle morphism defined on the lattice bundle of an (general) integrable system, which can be seen as a generalization of the vector of Maslov indices. The nontriviality of this bundle morphism implies the existence of common eigenvectors with eigenvalue 1 of the monodromy matrices, and gives rise to a corank 1 toric foliation refining the original one induced by the integrable system. Furthermore, we show that, in the case where the system has 2 degrees of freedom, this implies the existence of a compatible free  $S^1$  action on the regular part of the system.

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## 1. BACKGROUND AND MOTIVATION

An integrable Hamiltonian system contains the following ingredients: a  $2n$ -dimensional symplectic manifold  $(M, \omega)$  as the phase space, an integral map  $F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$  which is regular almost everywhere, Hamiltonian vector fields  $\mathcal{X}_i$  for  $i = 1, \dots, n$  with  $f_i$  as the Hamiltonians and the commutativity condition:  $[\mathcal{X}_i, \mathcal{X}_j] = 0$ . The commutativity of the vector fields  $\mathcal{X}_i$  induces a Hamiltonian  $\mathbb{R}^n$  action  $\Phi$  on the phase space  $M$  given by  $\Phi^{(t_1, \dots, t_n)}(p) = \varphi_1^{t_1} \circ \dots \circ \varphi_n^{t_n}(p)$  with  $\varphi_i$  being the flow of  $\mathcal{X}_i$  for each  $i$ . The nondegeneracy of the symplectic form  $\omega$  implies that  $\mathcal{X}_1, \dots, \mathcal{X}_n$  are linearly independent wherever  $DF = (df_1, \dots, df_n)$  has full rank. Note that the level sets of the integral map are invariant under the action  $\Phi$ . For simplicity, we will refer to the domain consisting of regular level sets of the integral map together with the dynamics on it as the *regular part of the system*.

When a regular orbit of  $\Phi$ , i. e., an orbit consisting of regular points of  $F$ , is compact, it is an  $n$ -dimensional Lagrangian torus. The region  $M_0$  consisting of compact regular orbits of  $\Phi$  thereby admits a Lagrangian toric foliation  $\mathfrak{F}$  with these  $n$ -tori as the leaves. The Arnold–Liouville theorem shows that near each of the leaves the foliation is nice in the sense that local action-angle coordinates exist. However, topological and geometric obstructions prevent in general the global existence of action-angle coordinates. This has been investigated by Duistermaat in [7]. The nontriviality of the torus bundle  $F^{-1}(\mathcal{C})$  over some loop  $\mathcal{C}$  lying in the set of regular values of  $F$  serves as one of these obstructions, and it is characterized by the monodromy matrix  $\mathcal{M}_{\mathcal{C}}$  of the bundle. One should note that here, for simplicity, we have assumed each level set  $F^{-1}(c)$  with  $c \in \mathcal{C}$  to consist of a single

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orbit. However, in general,  $F^{-1}(c)$  can contain more than one orbit and hence  $F^{-1}(\mathcal{C}) \rightarrow \mathcal{C}$  is not necessarily a torus bundle. We will deal with this problem later in a more rigorous way.

Via the Bohr–Sommerfeld quantization of actions, their global nonexistence manifests itself in quantum mechanics as the nonexistence of global quantum numbers to assign the joint spectrum of commuting operators that are the quantum analogues of the functions  $f_i$ . Here Maslov indices determine whether the actions are required to be integer or half-integer multiples of Planck’s constant. From a geometric point of view, the Maslov index of a closed curve of Lagrangian planes counts the number of intersections of such a curve with a chosen Maslov cycle  $\Sigma$ . Usually  $\Sigma$  is induced by some Lagrangian vector subbundle  $\mathcal{V}$  of the tangent bundle  $TM$ , and the curve of Lagrangian planes is the one induced by a curve on some Lagrangian submanifold in the natural way. One can think of  $\mathcal{V}$  as a reference and then the Maslov index describes how the Lagrangian planes rotate with respect to the reference along the curve. In particular, when  $M$  is the cotangent bundle of another manifold  $N$ ,  $\mathcal{V}$  is usually taken as the vertical distribution of  $T^*N = M$ .

For a regular value  $c_0$  of  $F$ , a set of closed curves  $\{\lambda_1, \dots, \lambda_n\}$  on the Lagrangian torus  $F^{-1}(c_0)$  can be taken such that  $[\lambda_1], \dots, [\lambda_n]$  constitute a basis of  $H_1(F^{-1}(c_0)) \cong \mathbb{Z}^n$ . Let  $a_i$  be the Maslov index of  $\lambda_i$ . The main result in [8] shows that, when it is nonzero, the vector of Maslov indices  $(a_1, \dots, a_n)$  is an eigenvector with eigenvalue 1 of the monodromy matrices  $\mathcal{M}_{\mathcal{C}}$  for all loops  $\mathcal{C}$  of regular values of  $F$  based at  $c_0$ . As is pointed out in [8], one implication of the result is that a Hamiltonian  $S^1$  action exists on the space  $F^{-1}(\mathcal{C})$ . Moreover, it also imposes some restrictions on the forms of the monodromy matrices.

A close look at the result in [8] suggests that there should be a more general underlying structure. Namely, on the one hand, the monodromy matrices are determined solely by the foliation  $\mathfrak{F}$ . On the other hand, however, the Maslov indices depend on the choice of the Maslov cycle, or say, the Lagrangian subbundle  $\mathcal{V}$ . A change of the reference may result in a change of the triviality/nontriviality of the vector of Maslov indices, but it affects neither the foliation nor the monodromy matrices. It is then natural to ask whether we could restate the result in [8] in terms of the topology of  $M_0$  or  $\mathfrak{F}$ . This could then not only give a new interpretation in different terms, but also a generalization of the result in [8] to a broader context.

It turns out that we are able to work this out for (the regular parts of) systems which are integrable in a more general sense. These systems are called *integrable non-Hamiltonian* systems in the literature (see, e. g., [2, 16]) and the precise definition will be given in the next section. Note that the phrase “non-Hamiltonian” here means “not necessarily Hamiltonian”, and the notion also includes all integrable Hamiltonian systems. No symplectic structure will be required for the definition of such a system, and the  $M$  is not necessarily of even dimension.

## 2. BASIC SETTINGS AND LAYOUT OF THE PAPER

Since we are concerned with the toric foliation on the domain of compact regular orbits of  $\Phi$ , we make, for simplicity, some assumptions that lead to the following definition of a regular integrable non-Hamiltonian system with compact orbits.

**Definition 1.** A regular (non-Hamiltonian) integrable system of type  $(k, n)$  with compact orbits is a triple  $(M, F, \Phi)$  where  $M$  is a  $(k + n)$ -dimensional smooth manifold, the integral map

$$F = (f_1, \dots, f_k) : M \rightarrow \mathbb{R}^k$$

is a submersion and  $\Phi$  is an effective  $\mathbb{R}^n$  action with compact orbits such that

$$F \circ \Phi^{(t_1, \dots, t_n)}(p) = F(p).$$

**Remark 1.** By restricting a Hamiltonian integrable system to the set of regular values of the integral map we get a regular integrable Hamiltonian system.

**Remark 2.** Without loss of generality,  $M$  is assumed in this article to be connected unless otherwise stated.

Since the orbits of  $\Phi$  are compact, they are  $n$ -tori and thereby constitute a toric foliation  $\mathfrak{F}$  on  $M$ . As mentioned earlier, the map  $F : M \rightarrow \mathbb{R}^n$  is in general not a torus bundle since a level set of  $F$  may contain more than one orbit. In order to get a torus bundle, we introduce an orbit space  $\mathcal{O}_M$  whose elements are the orbits of  $\Phi$ . The bundle  $\mathcal{L}$  of period lattices will play a central role in our discussion. This is a bundle over the orbit space  $\mathcal{O}_M$  where the fibers are the isotropy subgroups of the action  $\Phi$ . Under Definition 1 both these spaces have nice structures and properties, and they will be discussed in detail in Sections 3 and 4.

The definitions of monodromy maps and monodromy matrices will be formally introduced in Section 5 along with their basic properties.

The main construction and results are given in Section 6. This comprises a bundle epimorphism  $\rho = (\rho_1, \dots, \rho_l)$  from  $\mathcal{L}$  to  $\mathbb{Z}^l$  for some integer  $l \leq n$ . This is a purely topological object associated with the system  $(M, F, \Phi)$ , and as we will see in Section 7,  $\rho$  can be seen as a generalization of the vector of Maslov indices. When  $l > 0$ , each component of  $\rho$  is a common eigenvector with eigenvalue 1 of the (transposed) monodromy matrices. This will have implications on the toric foliation  $\mathfrak{F}$ , and in the case of  $n = 2$ , it implies the existence of a free  $S^1$  action on  $M$  which commutes with  $\Phi$ . The precise statements of these results form the main results of this paper: Theorem 5 and Theorem 6 in Section 6.

In Section 7, we revisit the case of integrable Hamiltonian systems. We show that the vector of Maslov indices being nonzero implies the nontriviality of  $\rho$ , and then show how the result in [8] follows from our construction and results. Moreover, with Audin's conjecture (which has been proved), our results imply that, for any integrable Hamiltonian system with a proper integral map in  $\mathbb{R}^4$ , each of the connected components of the regular part of the system admits a compatible free  $S^1$  action. This is stated as Theorem 7.

The appendix is devoted to a technical proof of the local triviality of the lattice bundle  $\mathcal{L}$  and an illustration on how the monodromy matrix  $\mathcal{M}_{\mathcal{L}}$  determines the torus bundle  $F^{-1}(\mathcal{C}) \rightarrow \mathcal{C}$ .

### 3. THE ORBIT SPACE $\mathcal{O}_M$ AND THE LATTICE BUNDLE $\mathcal{L}$

In this section we study the orbit space  $\mathcal{O}_M$  and the lattice bundle  $\mathcal{L}$  associated to a regular integrable system  $(M, F, \Phi)$  of type  $(k, n)$  with compact orbits.

By  $\sim_{\Phi}$  we denote the equivalence relation on  $M$  such that  $p \sim_{\Phi} p'$  if and only if  $p$  and  $p'$  are on the same orbit of  $\Phi$ . The orbit space  $\mathcal{O}_M$  is the quotient space  $M / \sim_{\Phi}$ , that is, each element of  $\mathcal{O}_M$  represents an orbit of  $\Phi$ . Then  $F$  factors as  $F = \bar{F} \circ q_{\Phi}$  with  $q_{\Phi} : M \rightarrow \mathcal{O}_M = M / \sim_{\Phi}$  being the quotient map and  $\bar{F} : \mathcal{O}_M \rightarrow \mathbb{R}^k$ .

Note that, for any open set  $W$  in  $M$ ,  $q_{\Phi}^{-1}(q_{\Phi}(W)) = \Phi^{\mathbb{R}^n}(W)$  is open. This implies that  $q_{\Phi}(W)$  is open, and therefore  $q_{\Phi}$  is an open map. Since each orbit is compact,  $\mathcal{O}_M$  is Hausdorff. Since  $F$  is a submersion, for any  $o \in \mathcal{O}_M$  and  $p \in o$ , there exists a local section  $\sigma$  of  $F$  over some open neighborhood  $U$  of  $b = F(p)$  such that  $\sigma(b) = p$ . Then  $V = q_{\Phi} \circ \sigma(U) = q_{\Phi}(\Phi^{\mathbb{R}^n} \circ \sigma(U))$  is an open neighborhood of  $o$  and  $q_{\Phi} \circ \sigma : U \rightarrow V$  is the inverse of  $\bar{F}|_V : V \rightarrow U$ . Therefore,  $\bar{F}$  is a local homeomorphism. Choose an open covering  $\{V_{\alpha}\}$  of  $\mathcal{O}_M$  such that  $\bar{F}_{\alpha} = \bar{F}|_{V_{\alpha}} : V_{\alpha} \rightarrow U_{\alpha}$  is a homeomorphism for each  $\alpha$ . Then  $\{(V_{\alpha}, \bar{F}_{\alpha})\}$  is a smooth structure on  $\mathcal{O}_M$  and naturally it makes  $\bar{F}$  a local diffeomorphism. Since  $F = \bar{F} \circ q_{\Phi}$  and  $F$  is submersive, such a smooth structure makes  $q_{\Phi}$  a submersion. The discussion here yields the following proposition.

**Proposition 1.** *There exists a unique smooth structure on  $\mathcal{O}_M$  such that  $q_{\Phi}$  is a submersion and  $\bar{F}$  is a local diffeomorphism.*

As in the case of the regular points of Hamiltonian integrable systems, for any  $p \in M$  the isotropy group  $\mathcal{T}_p \subset \mathbb{R}^n$  of  $\Phi$  at  $p$  is a free Abelian group of rank  $n$  [5], and  $\mathcal{T}_p = \mathcal{T}_{p'}$  if  $p$  and  $p'$  are on the same orbit. Hence, for any orbit  $o \in \mathcal{O}_M$ , define its period lattice to be  $\mathcal{L}_o = \mathcal{T}_p$  with any  $p \in o$ . The lattice bundle is then defined as  $\mathcal{L} = \bigsqcup_{o \in \mathcal{O}_M} \mathcal{L}_o$  and then it holds naturally that  $\mathcal{L} \subset \mathcal{O}_M \times \mathbb{R}^n$ . Denote by  $\pi_{\mathcal{L}}$  the natural projection  $\mathcal{L} \rightarrow \mathcal{O}_M : (o, T) \rightarrow o$  and endow  $\mathcal{L}$  with the subspace topology inherited from  $\mathcal{O}_M \times \mathbb{R}^n$ . The following basic fact holds and its proof is inherent in the proof of the (Hamiltonian) Liouville theorem in [5]:

**Theorem 1.** *For any  $o \in \mathcal{O}_M$ , there exists a neighborhood  $U$  and smooth sections  $\tau_i : U \rightarrow U \times \mathbb{R}^n$  with  $i = 1, \dots, n$ , such that, for each  $o' \in U$ ,  $\tau_1(o'), \dots, \tau_n(o')$  constitute a basis of  $\mathcal{L}_{o'}$ . As a consequence, there exists an isomorphism  $\rho : \pi_{\mathcal{L}}^{-1}(U) \rightarrow U \times \mathbb{Z}^n$ .*

*Proof.* See Lemma 2.8 in [11] for a complete proof. In Appendix A, we give an argument with a similar idea, but different in technical details.

We call a neighborhood  $U$  such as in the theorem above a  $\mathcal{O}_M$ -neighborhood, and the sections  $\tau_1, \dots, \tau_n$  a smooth local basis of  $\mathcal{L}$  over  $U$ . Theorem 1 above actually shows that  $\mathcal{L}$  is a smooth submanifold of  $\mathcal{O}_M \times \mathbb{R}^n$ . As a corollary of Theorem 1, we have:

**Corollary 1.** *The map  $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow \mathcal{O}_M$  is a covering map of  $\mathcal{O}_M$ .*

Since for each  $o \in \mathcal{O}_M$  the group  $\mathcal{L}_o$  is a subgroup of the fiber  $\{o\} \times \mathbb{R}^n$ , we can define an equivalence relation  $\sim_{\mathcal{L}}$  on  $\mathcal{O}_M \times \mathbb{R}^n$  given by  $(o, T) \sim_{\mathcal{L}} (o', T')$  if and only if  $o = o'$  and  $(o, T - T') \in \mathcal{L}_o$ . We denote the quotient map of  $\sim_{\mathcal{L}}$  by  $q_{\mathcal{L}}$ .

The following non-Hamiltonian version of the Liouville theorem is a consequence of Theorem 1.

**Theorem 2 ([16]).** *For any  $o \in \mathcal{O}_M$ , there exists some neighborhood  $U$  of  $o$  in  $\mathcal{O}_M$  with a diffeomorphism  $\phi : (U \times \mathbb{R}^n) / \sim_{\mathcal{L}} \rightarrow q_{\Phi}^{-1}(U)$ . Moreover, the  $\mathbb{T}^n$  action on  $q_{\Phi}^{-1}(U)$  induced by the natural  $\mathbb{T}^n$  action on  $(U \times \mathbb{R}^n) / \sim_{\mathcal{L}} \cong U \times \mathbb{T}^n$  via this diffeomorphism commutes with the  $\mathbb{R}^n$  action  $\Phi$ .*

*Proof.* Choose a local section  $\sigma$  of  $q_{\Phi} : M \rightarrow \mathcal{O}_M$  over some  $\mathcal{O}_M$ -neighborhood  $U$  of  $o$ . Define  $\tilde{\phi} : U \times \mathbb{R}^n \rightarrow q_{\Phi}^{-1}(U)$  as  $\tilde{\phi}(c, T) = \Phi^T \circ \sigma(c)$ . Then  $\tilde{\phi}$  factors as  $\phi \circ q_{\mathcal{L}}$ . It can be checked that  $\phi$  is a homeomorphism from  $(U \times \mathbb{R}^n) / \sim_{\mathcal{L}}$  to  $q_{\Phi}^{-1}(U)$ .

To show that  $(U \times \mathbb{R}^n) / \sim_{\mathcal{L}}$  is homeomorphic to  $U \times \mathbb{T}^n$ , one can resort to a smooth local basis  $\tau_1, \dots, \tau_n$  of  $\mathcal{L}$  over  $U$  and a rectifying map

$$\begin{aligned} rec : U \times \mathbb{R}^n &\rightarrow U \times \mathbb{R}^n, \\ (c; t_1, \dots, t_n) &\rightarrow (c; t_1\tau_1(c), \dots, t_n\tau_n(c)). \end{aligned}$$

This descends to an isomorphism  $\overline{rec}$  from  $U \times \mathbb{T}^n$  to  $(U \times \mathbb{R}^n) / \sim_{\mathcal{L}}$ . It can be checked that  $\overline{rec}$  is a bijective submersion, and thereby is a diffeomorphism.

**Corollary 2.** *If the bundle  $q_{\Phi} : M \rightarrow \mathcal{O}_M$  admits a global section, then it is isomorphic to the bundle  $\bar{\pi}_{\mathcal{O}} : (\mathcal{O}_M \times \mathbb{R}^n) / \sim_{\mathcal{L}} \rightarrow \mathcal{O}_M$  with  $\bar{\pi}_{\mathcal{O}}([o, T]) = o$ .*

*Proof.* The proof of Theorem 2 actually guarantees that  $\bar{\pi}_{\mathcal{O}} : (\mathcal{O}_M \times \mathbb{R}^n) / \sim_{\mathcal{L}} \rightarrow \mathcal{O}_M$  with  $\bar{\pi}_{\mathcal{O}}([o, T]) = o$  being a locally trivial torus fibration. If  $q_{\Phi} : M \rightarrow \mathcal{O}_M$  admits a global section  $\sigma : \mathcal{O}_M \rightarrow M$ , then the map  $\tilde{\phi}_M : \mathcal{O}_M \times \mathbb{R}^n \rightarrow M$  with  $\tilde{\phi}_M(c, T) = \Phi^T \circ \sigma(c)$  for any  $c \in \mathcal{O}_M$  factors as  $\tilde{\phi}_M = \phi_M \circ q_{\mathcal{L}}$  with  $\phi_M$  being a bundle isomorphism.

#### 4. THE LATTICE BUNDLE $\mathcal{L}$ AND THE SHEAF $\mathcal{R}$ OF COMPATIBLE $S^1$ ACTIONS

The sheaf  $\mathcal{R}$  of compatible  $S^1$  actions over  $\mathcal{O}_M$  was first introduced in [14] in the context of Hamiltonian integrable systems and then in [15] for the non-Hamiltonian case. We explain the relation between the sheaf  $\mathcal{R}$  and the lattice bundle  $\mathcal{L}$  in this section.

**Definition 2.** For any open set  $U$  of  $\mathcal{O}_M$ , an  $S^1$  action  $\Theta : S^1 \times q_{\Phi}^{-1}(U) \rightarrow q_{\Phi}^{-1}(U)$  acting on  $q_{\Phi}^{-1}(U)$  is called a **compatible  $S^1$  action** over  $U$  if it commutes with the  $\mathbb{R}^n$  action  $\Phi$  and preserves  $q_{\Phi}$  in the sense that  $q_{\Phi} \circ \Theta(z, p) = q_{\Phi}(p)$  for all  $z \in S^1$  and  $p \in q_{\Phi}^{-1}(U)$ .

For any open set  $U$  of  $\mathcal{O}_M$ , denote by  $\mathcal{R}_U$  the set of all compatible  $S^1$  actions over  $U$ . From Proposition 2 it follows that, for any  $\Theta, \Theta' \in \mathcal{R}_U$  and  $z, z' \in S^1$ , the identity  $\Theta^z \circ \Theta'^{z'}(p) = \Theta'^{z'} \circ \Theta^z(p)$  holds, and thus there is a natural Abelian structure on  $\mathcal{R}_U$  with the addition  $\Theta + \Theta'$  given by  $(\Theta + \Theta')^z(p) = \Theta^z \circ \Theta'^z(p)$ .

**Definition 3.** [15] For any pair of open sets  $U, V$  with  $U \subset V$ , denote by  $\rho_U^V$  the restriction map from  $\mathcal{R}_V$  to  $\mathcal{R}_U$ . Then  $\mathcal{R} = (\{\mathcal{R}_U\}, \{\rho_U^V | U \subset V\})$  is a sheaf of Abelian groups and it is called *the sheaf of compatible  $S^1$  actions*.

The following proposition characterizes the compatible  $S^1$  actions.

**Proposition 2.** *Suppose  $\Theta : S^1 \times q_\Phi^{-1}(U) \rightarrow q_\Phi^{-1}(U)$  is a compatible  $S^1$  action. Then there exists a unique continuous section  $\sigma : U \rightarrow \mathcal{L}$  such that  $\Theta(e^{i \cdot 2\pi t}, p) = \Phi^{t \cdot \sigma \circ q_\Phi(p)}(p)$ . Conversely, if  $\sigma$  is a continuous section of  $\mathcal{L}$  over  $U$ , then  $\Theta(e^{i \cdot 2\pi t}, p) = \Phi^{t \cdot \sigma \circ q_\Phi(p)}(p)$  defines a compatible  $S^1$  action.*

*Proof.* Let  $X_\Theta$  be the infinitesimal generator of the flow  $\psi^t(p) = \Theta(e^{i \cdot 2\pi t}, p)$  on  $q_\Phi^{-1}(U)$ , i.e.,  $X_\Theta(p) = \frac{d}{dt} \Big|_{t=0} \Theta(e^{i \cdot 2\pi t}, p)$ . Let  $\frac{\partial}{\partial t_i} \Big|_p = \frac{d}{dt_i} \Phi^{(0, \dots, t_i, \dots, 0)}(p)$ . Since  $\Theta$  preserves  $q_\Phi$ , the following relation holds:

$$X_\Theta(p) \in \ker q_{\Phi*} = \text{span} \left\{ \frac{\partial}{\partial t_i} \Big|_p, i = 1, \dots, n \right\}.$$

Hence, there exist smooth functions  $a_1, \dots, a_n$  on  $q_\Phi^{-1}(U)$  such that  $X_\Theta = \sum_{i=1}^n a_i \frac{\partial}{\partial t_i}$ . Moreover, since  $\Theta$  commutes with  $\Phi$ , the functions  $a_i$  are invariant on each orbit, and hence they can be seen as functions on  $U$ . Note that the flow  $\varphi$  on  $q_\Phi^{-1}(U)$  with

$$\varphi^t(p) = \Phi^{t \cdot a_1 \circ q_\Phi(p), \dots, t \cdot a_n \circ q_\Phi(p)}(p)$$

has the same infinitesimal generator as  $\psi$ , and hence  $\varphi = \psi$ . Since

$$\Phi^{a_1 \circ q_\Phi(p), \dots, a_n \circ q_\Phi(p)}(p) = \Theta(e^{i \cdot 2\pi}, p) = p$$

for all  $p \in q_\Phi^{-1}(U)$ ,  $(q_\Phi(p); a_1 \circ q_\Phi(p), \dots, a_n \circ q_\Phi(p)) \in \mathcal{L}_{q_\Phi(p)}$ . Hence,  $\sigma : o \rightarrow (o; a_1(o), \dots, a_n(o))$  is the smooth section we are looking for. If there is another section  $\sigma'$  on  $U$  satisfying the relation  $\Theta(e^{i \cdot 2\pi t}, p) = \Phi^{t \cdot \sigma' \circ q_\Phi(p)}(p)$  with  $\sigma' : o \rightarrow (o; a'_1(o), \dots, a'_n(o))$ , then it holds that  $X_\Theta = \sum_{i=1}^n a'_i \frac{\partial}{\partial t_i}$ . As a consequence, we have  $a_i(o) = a'_i(o)$  for all  $i = 1, \dots, n$  and  $o \in U$ , and thereby  $\sigma = \sigma'$ .

The argument for the converse direction is straightforward.

Accordingly, compatible  $S^1$  actions over  $U$  are in one-one correspondence with the continuous sections of  $\mathcal{L}$  over  $U$ , and this implies that  $\mathcal{L}$  is the associated sheaf/etale space of  $\mathcal{R}$ . The discussion here amounts to the following corollary.

**Corollary 3.** *The etale space of the sheaf  $\mathcal{R}$  of compatible  $S^1$  actions is isomorphic to the lattice bundle  $\mathcal{L}$ .*

If  $\sigma : \mathcal{O}_M \rightarrow \mathcal{L}$  is a global continuous section of  $\mathcal{L}$ , then  $\sigma$  corresponds to a compatible  $S^1$  action  $\Theta$  on  $M$  by  $\Theta(e^{i \cdot 2\pi t}, p) = \Phi^{t \cdot \sigma \circ q_\Phi(p)}(p)$ . When it is nonzero, this is a nontrivial  $S^1$  action. Actually in this case the section  $\sigma$  is nonzero everywhere and the corresponding  $S^1$  action  $\Theta$  is thereby effective. To see this, first note that  $\mathcal{O}_M \times \{0\}$  is closed in  $\mathcal{O}_M \times \mathbb{R}^n$  and hence is closed in  $\mathcal{L}$ . Moreover, the fact that  $\mathcal{L}$  is locally isomorphic to  $U \times \mathbb{Z}^n$  implies that  $\mathcal{O}_M \times \{0\}$  is also open in  $\mathcal{L}$ . Since we assume  $M$  to be connected,  $\mathcal{O}_M \times \{0\}$  is exactly one connected component of  $\mathcal{L}$ . Hence, for any continuous section  $\sigma : \mathcal{O}_M \rightarrow \mathcal{L}$ , as long as  $\sigma(c_0) \in \mathcal{O}_M \times \{0\}$ , we have  $\sigma(\mathcal{O}_M) = \mathcal{O}_M \times \{0\}$ . As a consequence, if  $\sigma$  is nonzero somewhere, it is nonzero everywhere, and the induced  $S^1$  action  $\Theta$  is effective.

Note that  $\sigma$  being nonzero does not imply that  $\Theta$  is free. However, the existence of a nonzero section does imply the existence of a free  $S^1$  action compatible with  $\Phi$ .

**Proposition 3.** *The existence of a global nonzero continuous section of  $\mathcal{L}$  implies the existence of a free  $S^1$  action on  $M$ .*

*Proof.* First we show that, for any nonzero integer  $n$ , the map  $*n : (o, T) \mapsto (o, nT)$  defined on  $\mathcal{O}_M \times \mathbb{R}^n$  induces an open and closed map on  $\mathcal{L}$ . Note that  $*n$  is a diffeomorphism on  $\mathcal{O}_M \times \mathbb{R}^n$ , and  $\mathcal{L}$  is closed in  $\mathcal{O}_M \times \mathbb{R}^n$  with  $*n(\mathcal{L}) \subset \mathcal{L}$ . As a consequence,  $*n$  induces a closed map on  $\mathcal{L}$ , which is also denoted by  $*n$ . For the openness, first note that, for any continuous section  $\sigma$  of  $\mathcal{L}$  over some open set  $U$  in  $\mathcal{O}_M$ ,  $\sigma(U)$  is open in  $\mathcal{L}$ . Then, for any  $w \in \mathcal{L}$  and its neighborhood  $W$ , there exists some continuous section  $\sigma$  over  $U \ni c$  with  $c = q_\Phi(w)$  such that  $\sigma(U) \subset W$  and  $\sigma(c) = w$ . Then  $*n \circ \sigma$  is also a continuous section of  $\mathcal{L}$  over the open set  $U$ , and  $*n \circ \sigma(U) \subset *n(W)$ . Therefore,  $*n(W)$  is a neighborhood of  $*n(w)$  and this implies  $*n$  is an open map on  $\mathcal{L}$ .

Now suppose that  $\sigma$  is a nonzero section. Then  $\sigma(\mathcal{O}_M)$  is a connected component of  $\mathcal{L}$ . Fix a point  $c_0$  in  $\mathcal{O}_M$ . Choose  $w_0 \in \mathcal{L}_{c_0}$  such that  $*n(w_0) = \sigma(c_0)$  for some  $n \in \mathbb{Z}$ , and  $\mathbb{R} \cdot w_0 \cap \mathcal{L} = \mathbb{Z} \cdot w_0$ . Suppose that  $S_{w_0} \subset \mathcal{L}$  is the connected component of  $\mathcal{L}$  containing  $w_0$ . Then  $*n(S_{w_0})$  is another component of  $\mathcal{L}$  since  $*n$  is both an open and a closed map on  $\mathcal{L}$ . Since  $\sigma(c_0) = *n(w_0) \in *n(S_{w_0})$ , it yields  $\sigma(\mathcal{O}_M) = *n(S_{w_0})$ , and therefore  $\sigma_1 = \frac{1}{n}\sigma$  is also a continuous section of  $\mathcal{L}$  with  $\sigma_1(\mathcal{O}_M) = S_{w_0}$ . It remains to show that, for any  $c \in \mathcal{O}_M$ , the identity  $\mathbb{R} \cdot \sigma_1(c) \cap \mathcal{L} = \mathbb{Z} \cdot \sigma_1(c)$  holds. That is,  $\sigma_1(c)$  is a generator of the subgroup  $\mathbb{R} \cdot \sigma_1(c) \cap \mathcal{L}$  for each  $c \in \mathcal{O}_M$ .

Suppose that  $w_1 \in \mathcal{L}_c$  is a generator of  $\mathbb{R} \cdot \sigma_1(c) \cap \mathcal{L}$ . Then there exists some integer  $n_1$  such that  $n_1 w_1 = \sigma_1(c)$ . Repeating the argument above yields another section  $\sigma_2 = \frac{1}{n_1} \sigma_1$ . Since  $\sigma_1(c_0) = w_0$  and  $\mathbb{R} \cdot w_0 \cap \mathcal{L} = \mathbb{Z} \cdot w_0$ ,  $\sigma_2(c_0) = \frac{1}{n_1} \sigma_1(c_0) = \frac{1}{n_1} w_0 \in \mathbb{R} \cdot w_0 \cap \mathcal{L} = \mathbb{Z} \cdot w_0$ . This implies  $n_1 = \pm 1$ . Hence, for any  $c \in \mathcal{O}_M$ ,  $\mathbb{R} \cdot \sigma_1(c) \cap \mathcal{L} = \mathbb{Z} \cdot \sigma_1(c)$ , and therefore the  $S^1$  action  $\Theta$  on  $M$  defined by  $\Theta(e^{i \cdot 2\pi t}, p) = \Phi^{t \cdot \sigma_1 \circ q_\Phi(p)}(p)$  is free.

5. MONODROMY MAPS

Consider a loop  $\mathcal{C}$  in the orbit space  $\mathcal{O}_M$  with a point  $c_0 \in \mathcal{C}$ . For convenience, we view  $\mathcal{C}$  both as a subset  $\mathcal{C} \subset \mathcal{O}_M$  and a fixed parametrization  $\mathcal{C} : [0, 1] \rightarrow \mathcal{O}_M$  with  $\mathcal{C}(0) = \mathcal{C}(1) = c_0$ . Let  $M_{\mathcal{C}} = q_\Phi^{-1}(\mathcal{C})$  and let  $\mathcal{L}_{\mathcal{C}} = \pi_{\mathcal{L}}^{-1}(\mathcal{C})$ . Then  $q_{\mathcal{C}} = q_\Phi|_{\mathcal{C}} : M_{\mathcal{C}} \rightarrow \mathcal{C}$  is a locally trivial torus fibration over  $\mathcal{C}$ . Since  $q_{\mathcal{C}}$  is a locally trivial fibration over a loop with connected fibers, it always admits a section over  $\mathcal{C}$ .

By replacing  $\mathcal{O}_M$  with  $\mathcal{C}$  and  $M$  with  $M_{\mathcal{C}}$ , the same argument for Corollary 2 shows that there is an isomorphism from  $\bar{\pi}_{\mathcal{C}} : (\mathcal{C} \times \mathbb{R}^n) / \sim_{\mathcal{L}} \rightarrow \mathcal{C}$  to  $q_{\mathcal{C}} : M_{\mathcal{C}} \rightarrow \mathcal{C}$ . Recall that  $\mathcal{L} \rightarrow \mathcal{O}_M$  is a covering, and denote by  $\mathcal{M}_{\mathcal{C}}$  the monodromy action of  $\mathcal{C}$  on  $\mathcal{L}_{c_0}$ .  $\mathcal{M}_{\mathcal{C}}$  is actually an isomorphism on the lattice  $\mathcal{L}_{c_0}$ . To see this, suppose  $\gamma_1$  and  $\gamma_2$  are the lifts of  $\mathcal{C}$  with base points  $v_1$  and  $v_2$ , with  $v_1, v_2 \in \mathcal{L}_{c_0}$ .  $\gamma_1 + \gamma_2$  is the lift of  $\mathcal{C}$  at  $v_1 + v_2$ , which implies that  $\mathcal{M}_{\mathcal{C}}$  is a group homomorphism. It can be checked that  $\mathcal{M}_{\mathcal{C}}$  is bijective. By fixing a basis  $\bar{w} = (u_1, \dots, u_n)$  of  $\mathcal{L}_{c_0}$ ,  $\mathcal{M}_{\mathcal{C}}$  is represented by some element  $\mathcal{M}_{\mathcal{C}, \bar{w}} \in SL(n, \mathbb{Z})$ . Note that  $\mathcal{M}_{\mathcal{C}, \bar{w}} \in GL(n, \mathbb{Z})$ . To see that  $\det \mathcal{M}_{\mathcal{C}, \bar{w}} = 1$ , we only need to show that  $\det \mathcal{M}_{\mathcal{C}, \bar{w}} > 0$ . For each  $i \in \{1, \dots, n\}$ , let  $\tau_i : [0, 1] \rightarrow \mathcal{L} \subset \mathcal{O}_M \times \mathbb{R}^n$  be the lift of  $\mathcal{C}$  with  $\tau_i(0) = u_i$ . It holds that  $\tau_i(s) = (\mathcal{C}_t, \tau'_i(s))$  with  $\tau'_i : [0, 1] \rightarrow \mathbb{R}^n$  being continuous for each  $i \in \{1, \dots, n\}$ . Consequently,  $\det[\tau'_1(s), \dots, \tau'_n(s)]$  is continuous and nonzero everywhere with respect to  $s$  and hence its sign does not change. By the definition of  $\mathcal{M}_{\mathcal{C}, \bar{w}}$ , it holds that  $[\tau'_1(1), \dots, \tau'_n(1)] = [\tau'_1(0), \dots, \tau'_n(0)] \cdot \mathcal{M}_{\mathcal{C}, \bar{w}}$  and thereby  $\det \mathcal{M}_{\mathcal{C}, \bar{w}} > 0$ .

We call  $\mathcal{M}_{\mathcal{C}}$  the monodromy map associated to  $\mathcal{C}$  (or, of the fibration  $q_{\mathcal{C}} : M_{\mathcal{C}} \rightarrow \mathcal{C}$ ), and  $\mathcal{M}_{\mathcal{C}, \bar{w}}$  the monodromy matrix with respect to  $w$ . The monodromy map  $\mathcal{M}_{\mathcal{C}}$  determines the structure of  $q_{\mathcal{C}} : M_{\mathcal{C}} \rightarrow \mathcal{C}$  (see Appendix B).

6. THE MAIN CONSTRUCTION

In this section we construct a bundle morphism  $\rho : \mathcal{L} \rightarrow \mathbb{Z}^l$  with  $l$  being an integer and show the main results of this article.

Let  $o$  be an element in  $\mathcal{O}_M$ . For  $T \in \mathcal{L}_o$  and  $x \in q_\Phi^{-1}(o)$ , let  $\lambda_{T,x} : [0, 1] \rightarrow q_\Phi^{-1}(o)$  be the closed path on the torus  $q_\Phi^{-1}(o)$  with  $\lambda_{T,x}(s) = \Phi^{s \cdot T}(x)$ . Denote by  $Tor_{H_1(M)}$  the torsion subgroup of  $H_1(M)$ . Define  $\rho_o : \mathcal{L}_o \rightarrow H_1(M) / Tor_{H_1(M)}$  by assigning to each  $T \in \mathcal{L}_o$  the element  $[\lambda_{T,x}]$  in  $H_1(M) / Tor_{H_1(M)}$ . Note that such an assignment is independent of  $x$  since, for any  $x, x'$  on  $q_\Phi^{-1}(o)$ ,  $\lambda_{T,x}$  and  $\lambda_{T,x'}$  are homotopic on  $q_\Phi^{-1}(o)$ . The map  $\rho_o$  is a homomorphism between Abelian groups.

**Definition 4.** The bundle morphism  $\rho : \mathcal{L} \rightarrow H_1(M)/\text{Tor}_{H_1(M)}$  is the one satisfying the identity  $\rho|_{\mathcal{L}_o} = \rho_o$  for each  $o \in \mathcal{O}_M$ .

We show that the value of  $\rho$  is invariant under parallel translation on  $\mathcal{L}$  and, as a consequence,  $\rho$  can be seen as a continuous bundle epimorphism from  $\mathcal{L}$  to  $\mathbb{Z}^l$  with  $l \leq n$ .

**Theorem 3.** For any path  $\tilde{\gamma} : [0, 1] \rightarrow \mathcal{L}$ ,  $\rho \circ \tilde{\gamma}(t) = \rho \circ \tilde{\gamma}(0)$  for all  $t \in [0, 1]$ . As a consequence,  $\rho$  is locally constant, and its image  $\text{Im } \rho = \text{Im } \rho_o$  is a free Abelian group with rank  $l \leq n$ .

*Proof.* Let  $\gamma$  be the path in  $\mathcal{O}_M$  defined as  $\gamma = \pi_{\mathcal{L}} \circ \tilde{\gamma}$ . Take a section  $\sigma$  of  $q_{\Phi} : M \rightarrow \mathcal{O}_M$  over  $\gamma$ . Then  $(t, \cdot) \mapsto \lambda_{\tilde{\gamma}(t), \sigma \circ \gamma(t)}(\cdot)$  gives a homotopy in  $M$  and hence  $\rho \circ \tilde{\gamma}(t) = [\lambda_{\tilde{\gamma}(t), \sigma \circ \gamma(t)}]$  remains invariant in  $H_1(M)/\text{Tor}_{H_1(M)}$ . Since  $\mathcal{O}_M$  is path-connected, for any  $o, o' \in \mathcal{O}_M$ ,  $\mathcal{L}_{o'}$  can be obtained via the parallel translation of  $\mathcal{L}_o$  along some path connecting  $o$  and  $o'$ . Therefore,  $\text{Im } \rho_o = \text{Im } \rho_{o'}$  and it is a finitely generated subgroup in  $H_1(M)/\text{Tor}_{H_1(M)}$  and thereby a free Abelian group with rank no larger than that of  $\mathcal{L}_o$ . Finally,  $\rho$  is locally constant since  $\mathcal{L}$  is locally path-connected.

**Corollary 4.** The map  $\rho$  is a bundle epimorphism from  $\mathcal{L}$  to  $\mathbb{Z}^l$  with  $l \leq n$ . For any path  $\gamma$  in  $\mathcal{O}_M$  with  $\mathcal{M}_{\gamma}$  being the associated monodromy map,  $\rho \circ \mathcal{M}_{\gamma} = \rho$  on  $\mathcal{L}_{\gamma(0)}$ .

Suppose  $l > 0$ , i. e.,  $\rho$  is nontrivial. Then  $\rho = (\rho_1, \dots, \rho_l)$  with  $\rho_i : \mathcal{L} \rightarrow \mathbb{Z}$  being locally constant and  $\rho_i \circ \mathcal{M}_{\gamma} = \rho_i$ . This means:

**Corollary 5.** For  $o \in \mathcal{O}_M$ , the linear functionals  $\rho_i|_{\mathcal{L}_o}$  are common eigenvectors with eigenvalue 1 of the transposes of the monodromy matrices  $\mathcal{M}_{\gamma}$  for all paths  $\gamma$  in  $\mathcal{O}_M$  with  $\gamma(0) = o$ .

Such a structure gives rise to a descending chain of lattice subbundles

$$\mathcal{L} \supset \ker \rho_1 \supset \ker(\rho_1, \rho_2) \supset \dots \supset \ker(\rho_1, \dots, \rho_l).$$

Since  $\mathbb{Z}$  is free, on each fiber  $\mathcal{L}_o$  the following short exact sequence splits:

$$0 \rightarrow \ker \rho_1|_{\mathcal{L}_o} \rightarrow \mathcal{L}_o \xrightarrow{\rho_1} \mathbb{Z} \rightarrow 0,$$

and then  $\ker \rho_1|_{\mathcal{L}_o}$  is isomorphic to  $\mathbb{Z}^{n-1}$ .

**Theorem 4.** The sublattice bundle  $\ker \rho_1$  is a smooth lattice subbundle of  $\mathcal{L}$ . Moreover,  $\ker \rho_1$  locally splits  $\mathcal{L}$ . To be precise, for any point  $o \in \mathcal{O}_M$ , there exists some neighborhood  $U$  such that  $\mathcal{L}|_U = \ker \rho_1|_U \oplus \mathcal{L}''_U$  with  $\mathcal{L}''_U$  being some sublattice bundle of  $\mathcal{L}|_U$  over  $U$ .

*Proof.* Since  $\rho_1$  is locally constant on  $\mathcal{L}$ , it is constant on each of the connected components. Hence,  $\ker \rho_1$  consists of several connected components of  $\mathcal{L}$  and is a submanifold of  $\mathcal{L}$ .

For any  $c \in \mathcal{O}_M$ , we construct a local trivialization of  $\ker \rho_1$  in the vicinity of  $c$ . Let  $U \ni c$  be a connected neighborhood over which  $\mathcal{L}$  admits a local trivialization. Then for each connected component  $\mathcal{S}$  of  $\mathcal{L}|_U$  there exists a section  $\sigma : U \rightarrow \mathcal{L}$  such that  $\mathcal{S} = \sigma(U)$ . Due to the connectedness of  $U$ , we have  $\sigma(U) \subset \ker \rho_1$  if and only if  $\sigma(U) \cap \ker \rho_1$  is nonempty. Fix a basis  $w_1, \dots, w_{n-1}$  of  $\ker \rho_1|_{\mathcal{L}_c}$  with local sections  $\sigma_1, \dots, \sigma_{n-1}$  over  $U$  such that  $\sigma_i(c) = w_i$ . For each  $c' \in U$ , the linear independence of  $\sigma_1(c'), \dots, \sigma_{n-1}(c')$  follows from the fact that the zero section  $U \times \{\mathbf{0}\}$  is a component of  $\ker \mathcal{L}|_U$ .

Now we show that for any  $w' \in \ker \rho_1|_{\mathcal{L}_{c'}}$  there exist integers  $k_1, \dots, k_{n-1}$  such that  $\sum_i k_i \sigma_i(c') = w'$ . Let  $\sigma' : U \rightarrow \mathcal{L}$  be the section with  $w' = \sigma'(c')$ . Then it holds that  $\sigma'(U) \subset \ker \rho_1$  due to the connectness of  $U$ , and in particular,  $\sigma'(c) \in \ker \rho_1|_{\mathcal{L}_c}$ . Then there exist integers  $k_1, \dots, k_{n-1}$  such that  $\sum_{i=1}^n k_i \sigma_i(c) = \sigma'(c) \in \mathcal{L}'_c$ . Since  $\tau = \sum_i k_i \sigma_i$  is also a continuous section of  $\mathcal{L}$  over  $U$ , it yields  $\tau(U) \subset \ker \rho_1$  and thereby  $\tau = \sigma'$ . Then  $w' = \sigma'(c') = \sum_i k_i \sigma_i(c')$  with  $k_i$  integers. Hence,  $\{\sigma_1, \dots, \sigma_{n-1}\}$  gives rise to a local trivialization of  $\ker \rho_1$  over  $U$ .

To see that  $\ker \rho_1$  locally splits  $\mathcal{L}$ , first note that, for every  $c \in \mathcal{O}_M$ ,  $\ker \rho_1|_{\mathcal{L}_c}$  splits  $\mathcal{L}_c$ . Then it holds that  $\mathcal{L}_c = \mathcal{L}'_c \oplus \mathbb{Z} \cdot v_c$  for some  $v_c \in \mathcal{L}_c$ . Let  $\sigma$  be a section over  $U$  such that  $\sigma(c) = v_c$ . For any  $c' \in U$  and  $w' \in \mathcal{L}_{c'}$ , there is a section  $\sigma'$  of  $\mathcal{L}$  over  $U$  such that  $\sigma'(c') = w'$ . Since  $\sigma'(c) \in \mathcal{L}_c$  and  $\sigma_1(c), \dots, \sigma_{n-1}(c), \sigma(c)$  constitute a basis of  $\mathcal{L}_c$ , there exist integers  $b_1, \dots, b_{n-1}, b$  such that  $\sigma'(c) = \sum_{i=1}^{n-1} b_i \sigma_i(c) + b \sigma(c)$ . It then holds that  $(\sum_{i=1}^n b_i \sigma_i)(U) = \sigma'(U)$  since both of them are connected components of  $\mathcal{L}|_U$ . Hence,  $\sum_{i=1}^{n-1} b_i \sigma_i(c') + b \sigma(c') = \sigma'(c') = w'$ . As a result,  $\mathcal{L}|_U = \ker \rho_1|_U \oplus \mathcal{L}''_U$  with  $\mathcal{L}''_U = \mathbb{Z} \cdot \sigma(U)$ .

**Corollary 6.**  *$\ker \rho_1$  gives rise to an  $(n - 1)$ -toric foliation  $\mathfrak{F}^{(1)}$  refining  $\mathfrak{F}$ . More precisely, for any  $b \in \mathcal{O}_M$ , there exists a neighborhood  $U$  of  $b$  with a local trivialization  $\bar{\psi} : U \times S^1 \times \mathbb{T}^{n-1} \rightarrow q_{\Phi}^{-1}(U)$  such that, for any  $(u, z) \in U \times S^1$ ,  $\bar{\psi}((u, z) \times \mathbb{T}^{n-1})$  is a leaf of  $\mathfrak{F}^{(1)}$ .*

*Proof.* Denote by  $V_o$  the  $(n - 1)$ -dimensional subspace of  $\{o\} \times \mathbb{R}^n$  spanned by  $\ker \rho_1|_{\mathcal{L}_o}$ . Then, for each  $p \in q_{\Phi}^{-1}(o)$ ,  $\mathfrak{T}_p = \Phi^{V_o}(p)$  is an  $(n - 1)$ -torus. To see that  $\mathfrak{F}^{(1)} = \{\mathfrak{T}_p | p \in M\}$  is a foliation, it only needs to be checked that there are local flat charts everywhere on  $M$  [10].

For any  $p \in M$  with  $o = q_{\Phi}(p)$ , choose a neighbourhood  $U$  over which it admits a section  $\tilde{\sigma}$  of  $q_{\Phi}$  with  $\tilde{\sigma}(o) = p$  and trivializations of  $\mathcal{L}_U$ . Note that each component of  $\mathcal{L}_U$  takes the form  $\sigma(U)$  with  $\sigma$  being a continuous section of  $\mathcal{L}$  over  $U$ . Choose  $w_1, \dots, w_{n-1} \in \ker \rho_1|_{\mathcal{L}_o}$  and  $v \in \mathcal{L}_o$  such that  $\{w_1, \dots, w_{n-1}, v\}$  forms a basis of  $\mathcal{L}_o$  and  $\{w_1, \dots, w_{n-1}\}$  forms a basis of  $\ker \rho_1|_{\mathcal{L}_o}$ . Let  $\sigma_1, \dots, \sigma_{n-1}, \tau$  be sections of  $\mathcal{L}$  over  $U$  such that  $\sigma_i(o) = w_i$  and  $\tau(o) = v$ . Then, for each  $c \in U$ ,  $\{\sigma_1(c), \dots, \sigma_{n-1}(c), \tau(c)\}$  is a basis for  $\mathcal{L}_c$  with  $\{\sigma_1(c), \dots, \sigma_{n-1}(c)\}$  being a basis for  $\ker \rho_1|_{\mathcal{L}_c}$ .

Define a map  $\psi : U \times \mathbb{R}^n \rightarrow M$  by

$$(u; s, t_1, \dots, t_{n-1}) \rightarrow \Phi^{s \cdot \tau(c) + t_1 \cdot \sigma_1(c) + \dots + t_{n-1} \cdot \sigma_{n-1}(c)} \circ \tilde{\sigma}(c).$$

It factors as  $\psi = \bar{\psi} \circ q_{\mathbb{Z}^n}$  with  $q_{\mathbb{Z}^n} : U \times \mathbb{R}^n \rightarrow U \times \mathbb{T}^n \cong U \times (\mathbb{R}^n / \mathbb{Z}^n)$  the quotient map and  $\bar{\psi} = U \times \mathbb{T}^n \rightarrow W = \psi(U \times \mathbb{R}^n)$  a diffeomorphism. For any  $p' \in \psi(u', s', t'_1, \dots, t'_{n-1})$ ,

$$\bar{\psi}^{-1}(\mathfrak{T}_{p'} \cap W) = \bar{\psi}^{-1}(\mathfrak{T}_{p'}) = \{(u, \bar{s}, \bar{t}_1, \dots, \bar{t}_{n-1}) \in U \times \mathbb{T}^n | u = u', \bar{s} = e^{i2\pi \cdot s'}\}.$$

Note that  $\rho$  is an epimorphism onto  $\mathbb{Z}^l$  and therefore  $\ker(\rho_1, \rho_2)$  is a lattice subbundle of  $\ker \rho_1$  and has corank 1 in  $\ker \rho_1$ . Consequently, it induces an  $(n - 2)$ -toric foliation  $\mathfrak{F}^{(2)}$  on  $M$  that refines  $\mathfrak{F}^{(1)}$ . This process can be iterated for all  $i \leq l$ . Note that, when  $l$  is nonzero, the image of  $\rho$  is nontrivial, which is equivalent to saying that there exists at least one loop on  $q_{\Phi}^{-1}(c_0)$  which is of infinite order in  $H_1(M)$ , and then the discussion here amounts to the following theorem.

**Theorem 5.** *Let  $(M, F, \Phi)$  be a regular integrable system of type  $(k, n)$  with compact orbits. Suppose that  $M$  (or equivalently, its orbit space  $\mathcal{O}_M$ ) is connected. If for some point  $c_0 \in \mathcal{O}_M$  there exists a loop  $\lambda$  on  $q_{\Phi}^{-1}(c_0)$  such that  $[\lambda]$  has infinite order in  $H_1(M)$ , then the lattice bundle  $\mathcal{L}$  of the system has a series of sublattice bundles  $\mathcal{L}^{(1)} \supset \dots \supset \mathcal{L}^{(l)}$  for some positive integer  $l$  no more than  $n$  such that  $\mathcal{L}^{(i)} = \ker(\rho_1, \dots, \rho_i)$  has rank  $n - i$  for each  $i \in \{1, \dots, l\}$ ,  $\mathcal{L}^{(i+1)}$  locally splits  $\mathcal{L}^{(i)}$  for each  $i \in \{1, \dots, l - 1\}$  and  $\mathcal{L}^{(1)}$  locally splits  $\mathcal{L}$ . As a consequence, such a sequence of sublattice bundles gives rise to a sequence of toric foliations  $\mathfrak{F}^{(1)}, \dots, \mathfrak{F}^{(l)}$  such that  $\mathfrak{F}^{(i+1)}$  refines  $\mathfrak{F}^{(i)}$  for  $i = 1, \dots, l - 1$  and  $\mathfrak{F}^{(1)}$  refines the fibration  $\mathfrak{F}$ . For each  $i \in \{1, \dots, l\}$ , the leaves of  $\mathfrak{F}^{(i)}$  are  $(n - i)$ -tori.*

In the case  $n = 2$ , when  $\rho : \mathcal{L} \rightarrow H_1(M)/\text{Tor}_{H_1(M)}$  is nontrivial,  $\text{Im} \rho$  is isomorphic to either  $\mathbb{Z}$  or  $\mathbb{Z}^2$ , and then we at least get one nonzero linear functional  $\rho_1 : \mathcal{L} \rightarrow \mathbb{Z}$ . Then  $\ker \rho_1$  is a lattice subbundle with rank 1. Actually,  $\ker \rho_1 \cong \mathcal{O}_M \times \mathbb{Z}$  and thus by Proposition 3 there is a free  $S^1$  action on  $M$ .



**Theorem 6.** *In the case  $n = 2$ , if  $\rho : \mathcal{L} \rightarrow H_1(M)/\text{Tor}_{H_1(M)}$  is nontrivial, there exists a compatible free  $S^1$  action on  $M$ .*

*Proof.* We only need to show that  $\ker \rho_1 \cong \mathcal{O}_M \times \mathbb{Z}$ . Fix a point  $c_0$  in  $\mathcal{O}_M$ . For any other point  $o \in \mathcal{O}_M$ ,  $\ker \rho_1|_{\mathcal{L}_o}$  can be obtained via the parallel transport of  $\ker \rho_1|_{\mathcal{L}_{c_0}}$  along any path joining  $c_0$  and  $o$ . Choose a(n) (ordered) basis  $w = (u, v)$  of  $\mathcal{L}_{c_0}$  and let  $z \in \mathbb{Z}^2$  with  $z = (\rho_1(u), \rho_1(v))$ . For any closed path  $\gamma$  in  $\mathcal{O}_M$  with  $\gamma(0) = \gamma(1) = c_0$ ,  $z \cdot \mathcal{M}_{\gamma, w} = z$  with  $\mathcal{M}_{\gamma, w}$  being the monodromy matrices. Hence, both of the eigenvalues of  $\mathcal{M}_{\gamma, w}$  are equal to 1. Since  $\ker \rho_1|_{\mathcal{L}_{c_0}}$  is a one-dimensional invariant space of the monodromy map  $\mathcal{M}_\gamma$ , it holds that  $\mathcal{M}_\gamma \cdot T = T$  for all  $T \in \ker \rho_1|_{\mathcal{L}_{c_0}}$ . Due to the arbitrariness of  $\gamma$ , this implies the triviality of the bundle  $\ker \rho_1 \rightarrow \mathcal{O}_M$ .

### 7. MASLOV INDICES, MONODROMY MATRICES AND TORIC FOLIATIONS WITH CORANK 1

Now we restrict to the Hamiltonian context and show how the results obtained in the previous section are related to the work in [8].

Consider a regular integrable Hamiltonian system  $(M, \omega, F)$  with  $F = (f_1, \dots, f_n)$  being the integral map. Let  $\mathcal{X}_i$  be the Hamiltonian vector fields with  $df_i(\cdot) = \omega(\mathcal{X}_i, \cdot)$ . We assume that the orbits of the Hamiltonian  $\mathbb{R}^n$  action  $\Phi$  are all compact. Denote by  $\Lambda_M$  the bundle of Lagrangian Grassmannians of  $M$ .

Fix an almost complex structure  $\mathcal{J}$  compatible with the symplectic form  $\omega$ , and let  $g_{\mathcal{J}}(\cdot, \cdot) = \omega(\mathcal{J}\cdot, \cdot)$  be the compatible Riemannian structure. Recall that the Hamiltonian vector fields  $\mathcal{X}_1, \dots, \mathcal{X}_n$  are independent everywhere by the assumption we made and  $\text{span}_{\mathbb{R}}\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$  is a Lagrangian distribution. Apply the Gram-Schmidt process to obtain  $\mathcal{X}'_1, \dots, \mathcal{X}'_n$  that are orthonormal with respect to  $g_{\mathcal{J}}$ . Note that  $\text{span}_{\mathbb{R}}\{\mathcal{X}'_1, \dots, \mathcal{X}'_n\} = \text{span}_{\mathbb{R}}\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$  and it is Lagrangian. Then

$$\{\mathcal{X}'_1, \dots, \mathcal{X}'_n, \mathcal{J}(\mathcal{X}'_1), \dots, \mathcal{J}(\mathcal{X}'_n)\}$$

is a globally defined symplectic frame for the tangent bundle  $TM$ .

Recall that, for a symplectic manifold  $(M, \omega)$ , the symplectic form  $\omega$  is a symplectic bilinear form on its tangent bundle, and  $(TM, \omega, M)$ , as a vector bundle endowed with a symplectic bilinear form, is a symplectic vector bundle (see page 79 in [12]). Then the argument above shows that the symplectic vector bundle  $(TM, \omega, M)$  is isomorphic to the trivial symplectic vector bundle  $(M \times \mathbb{R}^{2n}, \omega_0, M)$  with  $\omega_0|_{\{p\} \times \mathbb{R}^{2n}} = dx_i \wedge dy_i$  being the standard symplectic bilinear form on  $\{p\} \times \mathbb{R}^{2n}$ . Therefore, there exists an isomorphism  $\Lambda_M \cong M \times \mathbb{U}(n)/\mathbb{O}(n)$  associated to such a trivialization. Let  $\Lambda_{M_C} \rightarrow M_C$  denote the restriction of  $\Lambda_M$  to  $M_C$  and then  $\Lambda_{M_C} \cong M_C \times \mathbb{U}(n)/\mathbb{O}(n)$ . Let  $E : M \rightarrow \Lambda_M$  be a section, i. e., a Lagrangian vector bundle over  $M$ . Then, for any  $p \in M$ ,  $E(p) = [E_p] \in \mathbb{U}(n)/\mathbb{O}(n)$  with  $E_p \in \mathbb{U}(n)$  being a representative.

For any section  $E$ , define  $m_E : \Lambda_M \cong M \times \mathbb{U}(n)/\mathbb{O}(n) \rightarrow S^1$  with  $m_E(p, [A]) = (\det_{\mathbb{C}}(A \cdot E_p^{-1}))^2$ . Note that, for any  $p \in M$ ,  $\text{span}_{\mathbb{R}}(\mathcal{X}_1(p), \dots, \mathcal{X}_n(p)) \in \Lambda_M|_p$  and hence there is a natural embedding  $\tilde{l}$  of  $M$  into  $\Lambda_M$  by  $p \mapsto (p, \text{span}_{\mathbb{R}}(\mathcal{X}_1(p), \dots, \mathcal{X}_n(p)))$ .

**Definition 5.** The *Maslov map* for the integrable system  $(M, F, \Phi)$  is the map  $\tilde{m}_E = m_E \circ \tilde{l}$ .

**Definition 6.** The *Maslov index*  $\mu_\lambda$  of a loop  $\lambda : S^1 \rightarrow M$  in the integrable system  $(M, F, \Phi)$  with respect to  $E$  is the degree of the map  $\tilde{m}_E \circ \lambda$ .

Definition 6 is slightly different from the usual definition of the Maslov indices in that we do not require the loop  $\lambda$  to be on some Lagrangian submanifold, since the integrable system already prescribes to each point in  $M$  a Lagrangian subspace. Yet this definition is consistent with the usual one when  $\lambda$  lies on a Lagrangian submanifold.

Note that, for any point  $x_0 \in M$ , the homomorphism  $\pi_1(M, x_0) \ni [\lambda] \mapsto \mu_\lambda \in \mathbb{Z}$  factors through  $H_1(M)/\text{Tor}_{H_1(M)}$  since  $\mathbb{Z}$  is Abelian and torsionless. That is, there is a homomorphism  $\mu : H_1(M)/\text{Tor}_{H_1(M)} \rightarrow \mathbb{Z}$  such that, for any loop  $\lambda$  in  $M$ ,  $\mu([\lambda]) = \mu_\lambda$ . The composition  $\mu \circ \rho : \mathcal{L} \rightarrow \mathbb{Z}$  is then a bundle morphism. Note that, for  $c \in \mathcal{O}_M$ ,  $\mu \circ \rho(\mathcal{L}_c) = n_c \cdot \mathbb{Z}$  for some nonnegative integer  $n_c$ .  $n_c$  is called the minimal Maslov number of the Lagrangian torus  $q^{-1}(c)$ . If the minimal Maslov number on some Lagrangian torus  $q_\Phi^{-1}(c_0)$  is nonzero, then  $\mu \circ \rho$  is nonzero, implying that  $\rho$  is nontrivial. Then  $\mu \circ \rho|_{\mathcal{L}_{c_0}}$  is a common eigenvector with eigenvalue 1 for the transpose of the monodromy maps  $\mathcal{M}_\gamma$  of all the loops  $\gamma$  in  $\mathcal{O}_M$  with  $c_0 \in \gamma$ . For a basis  $(u_1, \dots, u_n)$  of  $\mathcal{L}_{c_0}$ ,  $(\mu \circ \rho(u_1), \dots, \mu \circ \rho(u_n))$  gives the corresponding vector of Maslov indices. In this way, it yields Theorem 1 of [8].

We conclude this article with a result for integrable Hamiltonian systems in  $\mathbb{R}^4$ .

Audin’s conjecture [1] asserts that, for any Lagrangian torus in  $(\mathbb{R}^{2n}, dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n)$ , the minimal Maslov number is 2 and this has been confirmed in [4] following a series of partial results [3, 6, 9, 13] (see page 118 in [12] for a brief introduction to the results obtained in these papers). Combining this result with Theorem 6 yields:

**Theorem 7.** *Suppose that  $(\mathbb{R}^4, \omega, F)$  is a Hamiltonian integrable system (not necessarily regular) with integral map  $F : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ . Denote by  $\mathbb{R}_{reg}^4$  the set of regular points of  $F$ . If  $M_0$  is a connected domain of  $\mathbb{R}_{reg}^4$  within which the orbits of the Hamiltonian  $\mathbb{R}^2$  action are compact, then there exists a free Hamiltonian  $S^1$  action on  $M_0$  which is compatible with the system.*

A direct implication of Theorem 6 is that, for a proper integral map  $F$ , there is a compatible free  $S^1$  action on each connected component of the regular part of the system. In particular, this is the case when  $(\mathbb{R}^4, \omega, F = (f_1, f_2))$  is a polynomial system with the leading term of  $f_1$  (or  $f_2$ ) taking the form of  $(x_1^2 + y_1^2 + x_2^2 + y_2^2)^k$  for some positive integer  $k$ .

APPENDIX A

Let  $(M, F, \Phi)$  be a regular integrable system of type  $(k, n)$  with compact  $\Phi$ -orbits. In this appendix, we give a detailed argument for Theorem 1 in Section 3. For convenience, we state Theorem 1 again.

**Theorem 1.** For any  $o \in \mathcal{O}_M$ , there exists some neighborhood  $U_o$  over which there are smooth sections  $\tau_i : U \rightarrow U \times \mathbb{R}^n$  with  $i = 1, \dots, n$ , such that, for each  $c \in U$ ,  $\tau_1(c), \dots, \tau_n(c)$  constitute a basis of  $\mathcal{L}_c$ . As a consequence, there exists an isomorphism (algebraically and topologically)  $\beta : \pi_{\mathcal{L}}^{-1}(U) \rightarrow U \times \mathbb{Z}^n$ .

*Proof.* As is shown in Proposition 1,  $q_\Phi : M \rightarrow \mathcal{O}_M$  is a submersion. Hence, there exists some smooth section  $\sigma : U \rightarrow M$  over some open neighborhood  $U$  of  $o$ . Define  $\Psi : U \times \mathbb{R}^n \rightarrow M$  as  $\Psi(b, T) \rightarrow \Phi^T \circ \sigma(b)$ . According to Definition 1,  $\Psi$  is transversal to the submanifold  $\sigma(U)$  and is a local diffeomorphism. Hence,  $\mathcal{S} = \Psi^{-1}(\sigma(U))$  is an embedded submanifold of  $U \times \mathbb{R}^n$  with the corank equal to that of  $\sigma(U)$ , which is  $n$ . Note that  $\mathcal{S}$  is also a subspace of  $\mathcal{L}$  (and therefore  $\mathcal{L}$  is a submanifold of  $\mathcal{O}_M \times \mathbb{R}^n$  due to the arbitrariness of  $U$ ). Moreover,  $\mathcal{S}$  is closed in  $U \times \mathbb{R}^n$ . To see this, suppose that  $(u_i, T_i) \rightarrow (u, T)$  with  $(u_i, T_i) \in \mathcal{S}$  and  $(u, T) \in U \times \mathbb{R}^n$ . Then  $\Psi(u, T) = \lim \Psi(u_i, T) = \lim \sigma(u_i) = \sigma(u) \in \mathcal{S}$ .

For any  $x \in \mathcal{S}$ , the tangent map  $\Psi_*$  maps  $T_x \mathcal{S}$  to  $T_{\Psi(x)} \sigma(U)$  and therefore  $T_x \mathcal{S} \cap T_x \mathbb{R}^n = \{0\}$ , implying  $T_x \mathcal{S} \oplus T_x \mathbb{R}^n = T_x(U \times \mathbb{R}^n)$ . As a result,  $\pi_{\mathcal{S}} = pr_U|_{\mathcal{S}} : \mathcal{S} \rightarrow U$  is a local diffeomorphism with  $pr_U : U \times \mathbb{R}^n \rightarrow U$  being the canonical projection. Note that  $U \times \{0\}$  is closed in  $U \times \mathbb{R}^n$  and therefore is closed in  $\mathcal{S}$ , while it is also open in  $\mathcal{S}$ . Hence,  $U \times \{0\}$  is a connected component of  $\mathcal{S}$ .

Let  $z_1, \dots, z_n$  be a basis of the the lattice  $\mathcal{L}_o$ . Shrink  $U$  if necessary. Then there exist local sections  $\tau_i : U \rightarrow \mathcal{S}$  for  $i = 1, \dots, n$  such that  $\tau_i(o) = z_i$ . Note that  $\{z_1, \dots, z_n\}$  is also a basis of the linear space  $\{o\} \times \mathbb{R}^n$ , and that, for any  $w \in \{o\} \times \mathbb{R}^n$ ,  $w \in \mathcal{L}_o$  if and only if  $w = a_1 z_1 + \dots + a_n z_n$  with  $a_i$  integers. Shrinking  $U$  again if necessary, one can make the determinant of the matrix

$\{\tau_1(c), \dots, \tau_n(c)\}$  nonzero for each  $c \in U$ , and then  $\{\tau_1(c), \dots, \tau_n(c)\}$  is a basis for  $\{c\} \times \mathbb{R}^n$ . Note that this implies that, for each  $c \in U$ ,  $\mathcal{L}_c$  is of rank  $n$ .

Now we show that, for each  $c \in U$ ,  $\{\tau_1(c), \dots, \tau_n(c)\}$  is also a basis of the  $\mathbb{Z}^n$  lattice  $\mathcal{L}_c$ . Let  $\{z_1^c, \dots, z_n^c\}$  be a basis of  $\mathcal{L}_c$ . Then there exist intergers  $k_1^j, \dots, k_n^j$  such that  $\tau_j(c) = k_1^j z_1^c + \dots + k_n^j z_n^c$ . Taking the inverse of the matrix  $[k_i^j]$  shows that, for each  $w \in \mathcal{L}_c$ , there exist  $p_1, \dots, p_n \in \mathbb{Q}$  such that  $w = p_1 \tau_1(c) + \dots + p_n \tau_n(c)$ . Hence, it remains to show that, for each  $w$ , the corresponding coefficients  $p_i = \frac{k_i}{m_i}$  are integers.

Denote by  $\mathcal{S}_w$  the component of  $\mathcal{S}$  that contains  $w$ . Note that there exists some nonzero integer  $m$  such that  $m \cdot p_i$  are integers for all  $i$ . By  $*m$  denote the map  $(c, T) \mapsto (c, m \cdot T)$ . Then  $*m$  is a diffeomorphism on  $\mathcal{O}_M \times \mathbb{R}^n$  and maps  $\mathcal{S}$  to  $\mathcal{S}$ . Hence, it is a local diffeomorphism on  $\mathcal{S}$ . Moreover, as shown above,  $\mathcal{S}$  is closed in  $U \times \mathbb{R}^n$  and therefore  $*m$  is a closed map on  $\mathcal{S}$ . Hence,  $*m(\mathcal{S}_w)$  is another connected component of  $\mathcal{S}$  which contains  $m \cdot w$ . Meanwhile,  $\tau : U \rightarrow U \times \mathbb{R}^n : u \rightarrow m \cdot (p_1 \tau_1(u) + \dots + p_n \tau_n(u))$  is also a section of  $\mathcal{S} \rightarrow U$  and hence  $\tau(U)$  is a component of  $\mathcal{S}$  which also contains  $m \cdot w$ . As a result,  $\tau(U) = *m(\mathcal{S}_w)$ . In particular,  $m \cdot p_1 \tau_1(o) + \dots + m \cdot p_n \tau_n(o) = \tau(o) = m \cdot w_0 \in \mathcal{L}_o$  for some  $w_0 \in \mathcal{S}_w \cap \mathcal{L}_o$  and hence  $w_0 = p_1 \tau_1(o) + \dots + p_n \tau_n(o) = p_1 z_1 + \dots + p_n z_n$ , implying that the  $p_i$  are integers.

Define  $\beta : U \times \mathbb{Z}^n \rightarrow \pi_{\mathcal{L}}^{-1}(U)$  with  $\rho^{-1}(u; a_1, \dots, a_n) = a_1 \tau_1(u) + \dots + a_n \tau_n(u)$ . Then  $\beta$  is a local trivialization of  $\mathcal{L}$  over  $U$ .

**Corollary 7.** *The lattice bundle  $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow \mathcal{O}_M$  is a locally trivial smooth  $\mathbb{Z}^n$  bundle with the transition group  $SL(n, \mathbb{Z})$ .*

*Proof.* Choose a covering  $\{U_\alpha\}$  of  $\mathcal{O}_M$  with sections  $\tau_i^\alpha : U_\alpha \rightarrow \mathcal{L}$  forming a basis of  $\mathcal{L}|_{U_\alpha}$ . Reorder  $\{\tau_i^\alpha | i = 1, \dots, n\}$  if necessary to fit the orientation of the vector bundle  $\mathcal{O}_M \times \mathbb{R}^n \rightarrow \mathcal{O}_M$ , and then for any pair  $\alpha, \alpha'$  with  $U_\alpha \cap U_{\alpha'}$  being nonempty,  $\rho_{\alpha'} \circ \rho_\alpha^{-1}$  should be a linear map on  $U_\alpha \cap U_{\alpha'} \times \mathbb{Z}^n$  preserving the orientation and hence  $\rho_{\alpha'} \circ \rho_\alpha^{-1} \in SL(n, \mathbb{Z})$ .

APPENDIX B

In the following we give an illustration of how the monodromy map  $\mathcal{M}$  determines the torus fibration  $M_{\mathcal{C}} = q_{\mathbb{F}}^{-1}(\mathcal{C}) \rightarrow \mathcal{C}$ .

Recall that  $\mathcal{C} \subset \mathcal{O}_M$  is a loop with  $M_{\mathcal{C}} = q_{\mathbb{F}}^{-1}(\mathcal{C}) \rightarrow \mathcal{C}$  being a torus fibration. Let  $c_0$  be a point in  $\mathcal{C}$ . The lift of  $\mathcal{C}$  to  $\mathcal{L}$  gives rise to a group isomorphism  $\mathcal{M}_{\mathcal{C}}$  of  $\mathcal{L}_{c_0}$ , which is called the monodromy map. Note that it extends canonically and uniquely to an isomorphism of  $\{c_0\} \times \mathbb{R}^n$ , which we also denote by  $\mathcal{M}_{\mathcal{C}}$ . Fix a parametrization  $\gamma : [0, 1] \rightarrow \mathcal{C}$  with  $\gamma(0) = \gamma(1) = c_0$ .

According to Corollary 2,  $M_{\mathcal{C}} = q_{\mathbb{F}}^{-1}(\mathcal{C}) \rightarrow \mathcal{C}$  is isomorphic to  $\bar{\pi}_{\mathcal{C}} : (\mathcal{C} \times \mathbb{R}^n) / \sim_{\mathcal{L}} \rightarrow \mathcal{C}$  with  $\bar{\pi}_{\mathcal{C}}([c, T]) = c$ . Choose an ordered basis  $\bar{w} = (u_1, \dots, u_n)$  of  $\mathcal{L}_{c_0}$ , and let  $\tau_i : [0, 1] \rightarrow \mathcal{L}$  be the lift of  $\gamma$  at  $u_i$  for each  $i = 1, \dots, n$  and  $\tau'_i : [0, 1] \rightarrow \mathbb{R}^n$  be the map such that  $\tau_i(s) = (\gamma(s), \tau'_i(s))$ . Then  $\mathcal{M}_{\mathcal{C}}$  is the map mapping  $u_i = \tau_i(0)$  to  $\tau_i(1)$ . Denote by  $\mathcal{M}_{\mathcal{C}, \bar{w}}$  the matrix representation of  $\mathcal{M}_{\mathcal{C}}$  with respect

to  $\bar{w}$ , i. e.,  $\mathcal{M}_{\mathcal{C}}(t_i u_i) = (u_1, \dots, u_n) \cdot \mathcal{M}_{\mathcal{C}, \bar{w}} \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ . In other words,  $\mathcal{M}_{\mathcal{C}}(u_j) = (u_1, \dots, u_n) \cdot \mathcal{M}_{\mathcal{C}, \bar{w}}^j$

with  $\mathcal{M}_{\mathcal{C}, \bar{w}}^j$  the  $j$ th column of  $\mathcal{M}_{\mathcal{C}, \bar{w}}$ , and hence  $(\tau'_1(1), \dots, \tau'_n(1)) = (\tau'_1(0), \dots, \tau'_n(0)) \cdot \mathcal{M}_{\mathcal{C}, \bar{w}}$ .

Denote by  $\sim_\gamma$  the equivalence relation on  $[0, 1] \times \mathbb{R}^n$  that identifies  $(0, T)$  with  $(1, T)$  and by  $q_\gamma$  the corresponding quotient map from  $[0, 1] \times \mathbb{R}^n$  to  $\mathcal{C} \times \mathbb{R}^n$ . Define  $r_{\bar{w}} : [0, 1] \times \mathbb{R}^n \rightarrow [0, 1] \times \mathbb{R}^n$  by

$$r_{\bar{w}}(s; \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}) = (s, t_1 \tau'_1(s) + \dots + t_n \tau'_n(s)). \tag{B.1}$$

This is an isomorphism mapping  $[0, 1] \times \mathbb{Z}^n \rightarrow \gamma^*(\mathcal{L})$ . Hence, the map  $h_{\bar{w}} = q_\gamma \circ r_{\bar{w}} : [0, 1] \times \mathbb{R}^n \rightarrow \mathcal{C} \times \mathbb{R}^n$  is a bundle epimorphism mapping  $[0, 1] \times \mathbb{Z}^n$  to  $\mathcal{L}|_{\mathcal{C}}$  and it is a fiberwise isomorphism.

Moreover,  $h_{\bar{w}}$  identifies  $\{0\} \times \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$  with  $\{1\} \times \mathcal{M}_{\mathcal{C}, \bar{w}}^{-1} \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ . To see this, first note that, since  $q_\gamma$  identifies  $(0, T)$  with  $(1, T)$  for every  $T \in \mathbb{R}^n$ ,  $q_\gamma \circ r_{\bar{w}}$  identifies  $(0, T)$  with  $(1, T')$  if and only if  $r_{\bar{w}}(0, T) = (0, T'')$  and  $r_{\bar{w}}(1, T) = (1, T'')$ . Then with  $T = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$  and  $T' = \begin{bmatrix} t'_1 \\ \vdots \\ t'_n \end{bmatrix}$ , according to

Eq. (B.1), it holds that

$$t_1 \tau'_1(0) + \dots + t_n \tau'_n(0) = t'_1 \tau'_1(1) + \dots + t'_n \tau'_n(1) = (\tau'_1(0), \dots, \tau'_n(0)) \cdot \mathcal{M}_{\mathcal{C}, \bar{w}} \cdot \begin{bmatrix} t'_1 \\ \vdots \\ t'_n \end{bmatrix}.$$

Hence,  $\begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = \mathcal{M}_{\mathcal{C}, \bar{w}} \cdot \begin{bmatrix} t'_1 \\ \vdots \\ t'_n \end{bmatrix}$  and thus  $\begin{bmatrix} t'_1 \\ \vdots \\ t'_n \end{bmatrix} = \mathcal{M}_{\mathcal{C}, \bar{w}}^{-1} \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ .

Note that  $quo = q_{\mathcal{L}} \circ h_{\bar{w}}$  is a quotient map from  $[0, 1] \times \mathbb{R}^n$  to  $M_{\mathcal{C}}$  and it characterizes the structure of the fibration  $M_{\mathcal{C}} \rightarrow \mathcal{C}$ . Denote by  $\sim$  the equivalence relation induced by  $quo$  on  $[0, 1] \times \mathbb{R}^n$ .  $M_{\mathcal{C}} \rightarrow \mathcal{C}$  is isomorphic to  $\pi_{\sim} : ([0, 1] \times \mathbb{R}^n) / \sim \rightarrow \mathcal{C}$  with  $\pi_{\sim}([s, T]) = \gamma(s)$ . It can be checked that, for each  $s \in [0, 1]$ ,  $(s, T) \sim (s, T')$  if and only if  $T - T' \in \mathbb{Z}^n$ . Then  $quo$  (re) factors as  $q_{\bar{\eta}} \circ q_{\mathbb{Z}^n}$  with  $q_{\mathbb{Z}^n} : [0, 1] \times \mathbb{R}^n \rightarrow [0, 1] \times \mathbb{T}^n$  being the quotient map which sends  $(s; t_1, \dots, t_n)$  to  $(s; e^{i2\pi t_1}, \dots, e^{i2\pi t_n})$ , and with  $q_{\bar{\eta}} : [0, 1] \times \mathbb{T}^n \rightarrow M_{\mathcal{C}}$  being a bundle morphism which is a fiberwise isomorphism. Denote by  $\eta$  the isomorphism  $(1, T) \rightarrow (h_{\bar{w}}|_{\{0\} \times \mathbb{R}^n})^{-1} \circ (h_{\bar{w}}|_{\{1\} \times \mathbb{R}^n})(1, T) = (0, \mathcal{M}_{\mathcal{C}, \bar{w}} \cdot T)$  and by  $\bar{\eta}$  the isomorphism  $(1, z) \rightarrow (q_{\bar{\eta}}|_{\{0\} \times \mathbb{T}^n})^{-1} \circ (q_{\bar{\eta}}|_{\{1\} \times \mathbb{T}^n})(1, z)$ .

Then  $M_{\mathcal{C}} \cong ([0, 1] \times \mathbb{T}^n) / \sim_{\bar{\eta}}$ . Moreover, the following identity holds:

$$\bar{\eta} \circ q_{\mathbb{Z}^n} = q_{\mathbb{Z}^n} \circ \eta.$$

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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